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More on Inequalities for Weaving Frames in Hilbert Spaces

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Abstract: In this paper, we present several new inequalities for weaving frames in Hilbert spaces from the point of view of operator theory, which are related to a linear bounded operator induced by three Bessel sequences and a scalar in the set of real numbers. It is indicated that our results are more general and cover the corresponding results recently obtained by Li and Leng. We also give a triangle inequality for weaving frames in Hilbert spaces, which is structurally different from previous ones.

Keywords: frame; weaving frame; weaving frame operator; alternate dual frame; Hilbert space

MSC: 42C15; 47B40

1. Introduction

Throughout this paper, $\mathbb{H}$ is a separable Hilbert space, and $\mathrm{Id}_{\mathbb{H}}$ is the identity operator on $\mathbb{H}$. The notations $\mathbb{J}$, $\mathbb{R}$, and $B(\mathbb{H})$ denote, respectively, an index set which is finite or countable, the real number set, and the family of all linear bounded operators on $\mathbb{H}$.

A sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ of vectors in $\mathbb{H}$ is a frame (classical frame) if there are constants $A, B > 0$ such that

$$A\|x\|^2 \le \sum_{j\in\mathbb{J}} |\langle x, f_j\rangle|^2 \le B\|x\|^2, \quad \forall x \in \mathbb{H}. \tag{1}$$

The frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ is said to be Parseval if $A = B = 1$. If $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ satisfies the inequality to the right in Equation (1) we say that $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ is a Bessel sequence.

The appearance of frames can be tracked back to the early 1950s when they were used in the work on nonharmonic Fourier series owing to Duffin and Schaeffer [1]. We refer to [2–16] for more information on general frame theory. It should be pointed out that frames have played an important role such as in signal processing [17,18], sigma-delta quantization [19], quantum information [20], coding theory [21], and sampling theory [22], due to their nice properties.

Motivated by a problem deriving from distributed signal processing, Bemrose et al. [23] put forward the notion of (discrete) weaving frames for Hilbert spaces. The theory may be applied to deal with wireless sensor networks that require distributed processing under different frames, which could also be used in the pre-processing of signals by means of Gabor frames. Recently, weaving frames have attracted many scholars' attention, please refer to [24–30] for more information.

Balan et al. [31] discovered an interesting inequality when further discussing the remarkable Parseval frames identity arising in their work on effective algorithms for computing the reconstructions of signals, which was then extended to general frames and alternate dual frames [32], and based on the work in [31,32], some inequalities for generalized frames associated with a scalar are also established (see [33–35]). Borrowing the ideas from [34,35], Li and Leng [36] have generalized the inequalities for frames to weaving frames with a more general form. In this paper, we present several new inequalities for weaving frames and we show that our results can lead to the corresponding results in [36]. We also obtain a triangle inequality for weaving frames, which differs from previous ones in the structure.

One calls two frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ woven, if there exist universal constants C and D such that for each partition $\sigma \subset \mathbb{J}$, the family $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is a frame for $\mathbb{H}$ with frame bounds C and D and, in this case, we say that $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is a weaving frame.

Suppose that $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ are woven, then associated with every weaving frame $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ there is a positive, self-adjoint and invertible operator, called the weaving frame operator, given below

$$S_W : \mathbb{H} \to \mathbb{H}, \quad S_W x = \sum_{j\in\sigma}\langle x, f_j\rangle f_j + \sum_{j\in\sigma^c}\langle x, g_j\rangle g_j.$$

We recall that a frame $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ is said to be an alternate dual frame of $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ if

$$x = \sum_{j\in\sigma}\langle x, f_j\rangle h_j + \sum_{j\in\sigma^c}\langle x, g_j\rangle h_j \tag{2}$$

is valid for every $x \in \mathbb{H}$.

For each $\sigma \subset \mathbb{J}$, let $S^\sigma_{\mathcal{F}}$ be the positive and self-adjoint operator induced by σ and a given frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ of $\mathbb{H}$, defined by

$$S^\sigma_{\mathcal{F}} : \mathbb{H} \to \mathbb{H}, \quad S^\sigma_{\mathcal{F}} x = \sum_{j\in\sigma}\langle x, f_j\rangle f_j.$$

Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$, $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$, and $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ be Bessel sequences for $\mathbb{H}$, then it is easy to check that the operators

$$S_{\mathcal{FGH}} : \mathbb{H} \to \mathbb{H}, \quad S_{\mathcal{FGH}} x = \sum_{j\in\sigma}\langle x, f_j\rangle h_j + \sum_{j\in\sigma^c}\langle x, g_j\rangle h_j \tag{3}$$

and

$$S_{\mathcal{HFG}} : \mathbb{H} \to \mathbb{H}, \quad S_{\mathcal{HFG}} x = \sum_{j\in\sigma}\langle x, h_j\rangle f_j + \sum_{j\in\sigma^c}\langle x, h_j\rangle g_j \tag{4}$$

are well-defined and, further, $S_{\mathcal{FGH}}, S_{\mathcal{HFG}} \in B(\mathbb{H})$.

2. Main Results and Their Proofs

We start with the following result on operators, which will be used to prove Theorem 1.

Lemma 1. *If $P, Q, L \in B(\mathbb{H})$ satisfy $P + Q = L$, then for any $\lambda \in \mathbb{R}$,*

$$P^*P + \frac{\lambda}{2}(Q^*L + L^*Q) = Q^*Q + (1 - \frac{\lambda}{2})(P^*L + L^*P) + (\lambda - 1)L^*L \geq (\lambda - \frac{\lambda^2}{4})L^*L.$$

Proof. We have

$$P^*P + \frac{\lambda}{2}(Q^*L + L^*Q) = P^*P - \frac{\lambda}{2}(P^*L + L^*P) + \lambda L^*L,$$

and

$$\begin{aligned} Q^*Q + (1 - \frac{\lambda}{2})(P^*L + L^*P) + (\lambda - 1)L^*L &= P^*P - \frac{\lambda}{2}(P^*L + L^*P) + \lambda L^*L \\ &= (P - \frac{\lambda}{2}L)^*(P - \frac{\lambda}{2}L) + (\lambda - \frac{\lambda^2}{4})L^*L \geq (\lambda - \frac{\lambda^2}{4})L^*L. \end{aligned}$$

Thus the result holds. □

Taking 2λ instead of λ in Lemma 1 yields an immediate consequence as follows.

Corollary 1. *If $P, Q, L \in B(\mathbb{H})$ satisfy $P + Q = L$, then for any $\lambda \in \mathbb{R}$,*

$$P^*P + \lambda(Q^*L + L^*Q) = Q^*Q + (1-\lambda)(P^*L + L^*P) + (2\lambda - 1)L^*L \geq (2\lambda - \lambda^2)L^*L.$$

Theorem 1. *Suppose that two frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven, and that $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ is a Bessel sequences for $\mathbb{H}$. Then for any $\sigma \subset \mathbb{J}$, for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{H}$, we have*

$$\begin{aligned} \Big\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle &= \Big\|\sum_{j\in\sigma^c}\langle x, g_j\rangle h_j\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle \\ &\geq (\lambda - \tfrac{\lambda^2}{4})\mathrm{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle + (1 - \tfrac{\lambda^2}{4})\mathrm{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Big\|\sum_{j\in\sigma}\langle x, h_j\rangle f_j\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma^c}\langle x, h_j\rangle\langle g_j, S_{\mathcal{HFG}}x\rangle &= \Big\|\sum_{j\in\sigma^c}\langle x, h_j\rangle g_j\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma}\langle x, h_j\rangle\langle f_j, S_{\mathcal{HFG}}x\rangle \\ &\geq (2\lambda - \lambda^2)\mathrm{Re}\sum_{j\in\sigma}\langle x, h_j\rangle\langle f_j, S_{\mathcal{HFG}}x\rangle + (1 - \lambda^2)\mathrm{Re}\sum_{j\in\sigma^c}\langle x, h_j\rangle\langle g_j, S_{\mathcal{HFG}}x\rangle, \end{aligned} \tag{6}$$

where $S_{\mathcal{FGH}}$ and $S_{\mathcal{HFG}}$ are defined respectively in Equations (3) and (4).

Proof. For any $\sigma \subset \mathbb{J}$, we define

$$Px = \sum_{j\in\sigma}\langle x, f_j\rangle h_j \quad \text{and} \quad Qx = \sum_{j\in\sigma^c}\langle x, g_j\rangle h_j, \quad \forall x \in \mathbb{H}. \tag{7}$$

Then $P, Q \in B(\mathbb{H})$, and a simple calculation gives

$$Px + Qx = \sum_{j\in\sigma}\langle x, f_j\rangle h_j + \sum_{j\in\sigma^c}\langle x, g_j\rangle h_j = S_{\mathcal{FGH}}x.$$

By Lemma 1 we obtain

$$\|Px\|^2 + \lambda\mathrm{Re}\langle S^*_{\mathcal{FGH}}Qx, x\rangle = \|Qx\|^2 + 2(1 - \frac{\lambda}{2})\mathrm{Re}\langle S^*_{\mathcal{FGH}}Px, x\rangle + (\lambda - 1)\|S_{\mathcal{FGH}}x\|^2.$$

Therefore,

$$\begin{aligned} \|Px\|^2 &= \|Qx\|^2 + 2(1 - \frac{\lambda}{2})\mathrm{Re}\langle S^*_{\mathcal{FGH}}Px, x\rangle + (\lambda - 1)\mathrm{Re}\langle S_{\mathcal{FGH}}x, S_{\mathcal{FGH}}x\rangle - \lambda\mathrm{Re}\langle S^*_{\mathcal{FGH}}Qx, x\rangle \\ &= \|Qx\|^2 + 2\mathrm{Re}\langle S^*_{\mathcal{FGH}}Px, x\rangle - \lambda\mathrm{Re}\langle (P+Q)x, S_{\mathcal{FGH}}x\rangle + (\lambda - 1)\mathrm{Re}\langle S_{\mathcal{FGH}}x, S_{\mathcal{FGH}}x\rangle \\ &= \|Qx\|^2 + 2\mathrm{Re}\langle S^*_{\mathcal{FGH}}Px, x\rangle - \mathrm{Re}\langle S_{\mathcal{FGH}}x, S_{\mathcal{FGH}}x\rangle \\ &= \|Qx\|^2 + 2\mathrm{Re}\langle Px, S_{\mathcal{FGH}}x\rangle - \mathrm{Re}\langle Px, S_{\mathcal{FGH}}x\rangle - \mathrm{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle \\ &= \|Qx\|^2 + \mathrm{Re}\langle Px, S_{\mathcal{FGH}}x\rangle - \mathrm{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle, \end{aligned}$$

from which we conclude that

$$\begin{aligned} &\Big\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle \\ &\quad = \|Px\|^2 + \mathrm{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle = \|Qx\|^2 + \mathrm{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\ &\quad = \Big\|\sum_{j\in\sigma^c}\langle x, g_j\rangle h_j\Big\|^2 + \mathrm{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle. \end{aligned} \tag{8}$$

For the inequality in Equation (5), we apply Lemma 1 again,

$$\|Px\|^2 + \lambda \mathrm{Re}\langle S^*_{\mathcal{FGH}} Qx, x\rangle \geq (\lambda - \frac{\lambda^2}{4})\langle S^*_{\mathcal{FGH}} S_{\mathcal{FGH}} x, x\rangle$$

for any $x \in \mathbb{H}$. Hence

$$\begin{aligned}\|Px\|^2 &\geq (\lambda - \frac{\lambda^2}{4})\langle S^*_{\mathcal{FGH}} S_{\mathcal{FGH}} x, x\rangle - \lambda \mathrm{Re}\langle Qx, S_{\mathcal{FGH}} x\rangle \\ &= (\lambda - \frac{\lambda^2}{4} - \lambda)\mathrm{Re}\langle Qx, S_{\mathcal{FGH}} x\rangle + (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\langle Px, S_{\mathcal{FGH}} x\rangle \\ &= (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\langle Px, S_{\mathcal{FGH}} x\rangle - \frac{\lambda^2}{4}\mathrm{Re}\langle Qx, S_{\mathcal{FGH}} x\rangle,\end{aligned} \tag{9}$$

and consequently,

$$\begin{aligned}&\left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2 + \mathrm{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}} x\rangle = \|Px\|^2 + \mathrm{Re}\langle Qx, S_{\mathcal{FGH}} x\rangle \\ &\geq (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\langle Px, S_{\mathcal{FGH}} x\rangle + (1 - \frac{\lambda^2}{4})\mathrm{Re}\langle Qx, S_{\mathcal{FGH}} x\rangle \\ &= (\lambda - \frac{\lambda^2}{4})\mathrm{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}} x\rangle + (1 - \frac{\lambda^2}{4})\mathrm{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}} x\rangle.\end{aligned}$$

Similar arguments hold for Equation (6), by using Corollary 1. □

Corollary 2. *Let two frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ be woven. Then for any $\sigma \subset \mathbb{J}$, for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{H}$, we have*

$$\begin{aligned}&\sum_{j\in\sigma}|\langle S_W^{-1} S_{\mathcal{F}}^{\sigma} x, f_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_W^{-1} S_{\mathcal{F}}^{\sigma} x, g_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle x, g_j\rangle|^2 \\ &= \sum_{j\in\sigma}|\langle S_W^{-1} S_{\mathcal{G}}^{\sigma^c} x, f_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_W^{-1} S_{\mathcal{G}}^{\sigma^c} x, g_j\rangle|^2 + \sum_{j\in\sigma}|\langle x, f_j\rangle|^2 \\ &\geq (\lambda - \frac{\lambda^2}{4})\sum_{j\in\sigma}|\langle x, f_j\rangle|^2 + (1 - \frac{\lambda^2}{4})\sum_{j\in\sigma^c}|\langle x, g_j\rangle|^2.\end{aligned}$$

Proof. For each $j \in \mathbb{J}$, taking

$$h_j = \begin{cases} S_W^{-\frac{1}{2}} f_j, & j \in \sigma, \\ S_W^{-\frac{1}{2}} g_j, & j \in \sigma^c. \end{cases}$$

Then, clearly, $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ is a Bessel sequence for $\mathbb{H}$. Since for any $x \in \mathbb{H}$, $S_{\mathcal{FGH}} x = \sum_{j\in\sigma}\langle x, f_j\rangle S_W^{-\frac{1}{2}} f_j + \sum_{j\in\sigma^c}\langle x, g_j\rangle S_W^{-\frac{1}{2}} g_j = S_W^{-\frac{1}{2}} S_W x = S_W^{\frac{1}{2}} x$, we have $S_{\mathcal{FGH}} = S_W^{\frac{1}{2}}$. Now

$$\begin{aligned}\left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2 &= \left\|\sum_{j\in\sigma}\langle x, f_j\rangle S_W^{-\frac{1}{2}} f_j\right\|^2 = \left\|S_W^{-\frac{1}{2}}\sum_{j\in\sigma}\langle x, f_j\rangle f_j\right\|^2 \\ &= \|S_W^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} x\|^2 = \langle S_W^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} x, S_W^{-\frac{1}{2}} S_{\mathcal{F}}^{\sigma} x\rangle \\ &= \sum_{j\in\sigma}\langle S_W^{-1} S_{\mathcal{F}}^{\sigma} x, f_j\rangle\langle f_j, S_W^{-1} S_{\mathcal{F}}^{\sigma} x\rangle + \sum_{j\in\sigma^c}\langle S_W^{-1} S_{\mathcal{F}}^{\sigma} x, g_j\rangle\langle g_j, S_W^{-1} S_{\mathcal{F}}^{\sigma} x\rangle \\ &= \sum_{j\in\sigma}|\langle S_W^{-1} S_{\mathcal{F}}^{\sigma} x, f_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_W^{-1} S_{\mathcal{F}}^{\sigma} x, g_j\rangle|^2.\end{aligned} \tag{10}$$

A similar discussion leads to

$$\Big\|\sum_{j\in\sigma^c}\langle x,g_j\rangle h_j\Big\|^2=\sum_{j\in\sigma}|\langle S_W^{-1}S_{\mathcal{G}}^{\sigma^c}x,f_j\rangle|^2+\sum_{j\in\sigma^c}|\langle S_W^{-1}S_{\mathcal{G}}^{\sigma^c}x,g_j\rangle|^2. \tag{11}$$

We also get

$$\mathrm{Re}\sum_{j\in\sigma}\langle x,f_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle=\mathrm{Re}\sum_{j\in\sigma}\langle x,f_j\rangle\langle S_W^{-\frac{1}{2}}f_j,S_W^{\frac{1}{2}}x\rangle=\sum_{j\in\sigma}|\langle x,f_j\rangle|^2, \tag{12}$$

and

$$\mathrm{Re}\sum_{j\in\sigma^c}\langle x,g_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle=\mathrm{Re}\sum_{j\in\sigma^c}\langle x,g_j\rangle\langle S_W^{-\frac{1}{2}}g_j,S_W^{\frac{1}{2}}x\rangle=\sum_{j\in\sigma^c}|\langle x,g_j\rangle|^2. \tag{13}$$

Thus the result follows from Theorem 1. □

Corollary 3. *Suppose that two frames $\mathcal{F}=\{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G}=\{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then for any $\sigma\subset\mathbb{J}$, for all $\lambda\in\mathbb{R}$ and all $x\in\mathbb{H}$,*

$$\begin{aligned}\mathrm{Re}\Big(\sum_{j\in\sigma}\langle x,h_j\rangle\langle f_j,x\rangle\Big)+\Big\|\sum_{j\in\sigma^c}\langle x,h_j\rangle g_j\Big\|^2&=\mathrm{Re}\Big(\sum_{j\in\sigma^c}\langle x,h_j\rangle\langle g_j,x\rangle\Big)+\Big\|\sum_{j\in\sigma}\langle x,h_j\rangle f_j\Big\|^2\\&\geq(2\lambda-\lambda^2)\mathrm{Re}\Big(\sum_{j\in\sigma}\langle x,h_j\rangle\langle f_j,x\rangle\Big)+(1-\lambda^2)\mathrm{Re}\Big(\sum_{j\in\sigma^c}\langle x,h_j\rangle\langle g_j,x\rangle\Big),\end{aligned}$$

where $\mathcal{H}=\{h_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of the weaving frame $\{f_j\}_{j\in\sigma}\cup\{g_j\}_{j\in\sigma^c}$.

Proof. For any $\sigma\subset\mathbb{J}$, since $\mathcal{H}=\{h_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of the weaving frame $\{f_j\}_{j\in\sigma}\cup\{g_j\}_{j\in\sigma^c}$, Equation (2) gives

$$x=\sum_{j\in\sigma}\langle x,h_j\rangle f_j+\sum_{j\in\sigma^c}\langle x,h_j\rangle g_j$$

for any $x\in\mathbb{H}$ and thus, $S_{\mathcal{HFG}}=\mathrm{Id}_{\mathbb{H}}$. By Theorem 1 we obtain the relation shown in the corollary. □

Remark 1. *Corollaries 2 and 3 are respectively Theorems 7 and 9 in [36].*

Theorem 2. *Suppose that two frames $\mathcal{F}=\{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G}=\{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven, and that $\mathcal{H}=\{h_j\}_{j\in\mathbb{J}}$ is a Bessel sequences for $\mathbb{H}$. Then for any $\sigma\subset\mathbb{J}$, for all $\lambda\in\mathbb{R}$ and all $x\in\mathbb{H}$, we have*

$$\begin{aligned}&\mathrm{Re}\sum_{j\in\sigma}\langle x,f_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle-\Big\|\sum_{j\in\sigma}\langle x,f_j\rangle h_j\Big\|^2\\&\quad\leq\frac{\lambda^2}{4}\mathrm{Re}\sum_{j\in\sigma^c}\langle x,g_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle+(1-\frac{\lambda}{2})^2\mathrm{Re}\sum_{j\in\sigma}\langle x,f_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle,\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\Big\|\sum_{j\in\sigma}\langle x,f_j\rangle h_j\Big\|^2+\Big\|\sum_{j\in\sigma^c}\langle x,g_j\rangle h_j\Big\|^2\\&\quad\geq(2\lambda-\frac{\lambda^2}{2}-1)\mathrm{Re}\sum_{j\in\sigma}\langle x,f_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle+(1-\frac{\lambda^2}{2})\mathrm{Re}\sum_{j\in\sigma^c}\langle x,g_j\rangle\langle h_j,S_{\mathcal{FGH}}x\rangle,\end{aligned} \tag{15}$$

where $S_{\mathcal{FGH}}$ is defined in Equation (3).

Moreover, if the operators P and Q given in Equation (7) satisfy the condition that P^*Q is positive, then

$$0 \le \operatorname{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle - \left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2,$$

and

$$\left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2 + \left\|\sum_{j\in\sigma^c}\langle x, g_j\rangle h_j\right\|^2 \le \|S_{\mathcal{FGH}}x\|^2.$$

Proof. For any $\sigma \subset \mathbb{J}$, let P and Q be defined in Equation (7). Then all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{H}$, we see from Equation (9) that

$$\begin{aligned}
\operatorname{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle - \left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2 &= \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle - \|Px\|^2 \\
&\le \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle + \frac{\lambda^2}{4}\operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle - (\lambda - \frac{\lambda^2}{4})\operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= \frac{\lambda^2}{4}\operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle + (1 - \lambda + \frac{\lambda^2}{4})\operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= \frac{\lambda^2}{4}\operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle + (1 - \frac{\lambda}{2})^2\operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= \frac{\lambda^2}{4}\operatorname{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle + (1 - \frac{\lambda}{2})^2\operatorname{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle.
\end{aligned}$$

We next prove Equation (15). By combining Equation (8) with Equation (9) we conclude that

$$\begin{aligned}
&\left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2 + \left\|\sum_{j\in\sigma^c}\langle x, g_j\rangle h_j\right\|^2 \\
&= \|Px\|^2 + \|Qx\|^2 = 2\|Px\|^2 + \operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle - \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&\ge (2\lambda - \frac{\lambda^2}{2})\operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle - \frac{\lambda^2}{2}\operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle + \operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle - \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= (2\lambda - \frac{\lambda^2}{2} - 1)\operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle + (1 - \frac{\lambda^2}{2})\operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle \\
&= (2\lambda - \frac{\lambda^2}{2} - 1)\operatorname{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle + (1 - \frac{\lambda^2}{2})\operatorname{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle, \quad \forall x \in \mathbb{H}.
\end{aligned}$$

Suppose now that P^*Q is positive, then for any $x \in \mathbb{H}$,

$$\begin{aligned}
\operatorname{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle - \left\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\right\|^2 &= \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle - \operatorname{Re}\langle Px, Px\rangle \\
&= \operatorname{Re}\langle Px, Qx\rangle = \operatorname{Re}\langle P^*Qx, x\rangle \ge 0.
\end{aligned}$$

Noting that

$$\begin{aligned}
\|Px\|^2 &= \|Qx\|^2 - \operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle + \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= \operatorname{Re}\langle Qx, Qx\rangle - \operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle + \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= -(\operatorname{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle - \operatorname{Re}\langle Qx, Qx\rangle) + \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \\
&= -\operatorname{Re}\langle Qx, Px\rangle + \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle \le \operatorname{Re}\langle Px, S_{\mathcal{FGH}}x\rangle,
\end{aligned}$$

and similarly,

$$\|Qx\|^2 \le \mathrm{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle,$$

we obtain

$$\begin{aligned}\Big\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\Big\|^2 + \Big\|\sum_{j\in\sigma^c}\langle x, g_j\rangle h_j\Big\|^2 &= \|Px\|^2 + \|Qx\|^2 \\ &\le \mathrm{Re}\langle Px, S_{\mathcal{FGH}}x\rangle + \mathrm{Re}\langle Qx, S_{\mathcal{FGH}}x\rangle \\ &= \mathrm{Re}\langle Px + Qx, S_{\mathcal{FGH}}x\rangle = \|S_{\mathcal{FGH}}x\|^2,\end{aligned}$$

and the proof is completed. □

Remark 2. *Suppose that the weaving frame $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$ is Parseval for each $\sigma \subset \mathbb{J}$, and letting $h_j = f_j$ if $j \in \sigma$ and $h_j = g_j$ if $j \in \sigma^c$, then it is easy to check that the operator P^*Q is positive.*

Corollary 4. *Suppose that two frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then for any $\sigma \subset \mathbb{J}$, for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{H}$, we have*

$$\begin{aligned}0 &\le \sum_{j\in\sigma}|\langle x, f_j\rangle|^2 - \sum_{j\in\sigma}|\langle S_W^{-1}S_{\mathcal{F}}^{\sigma}x, f_j\rangle|^2 - \sum_{j\in\sigma^c}|\langle S_W^{-1}S_{\mathcal{F}}^{\sigma}x, g_j\rangle|^2 \\ &\le \frac{\lambda^2}{4}\sum_{j\in\sigma^c}|\langle x, g_j\rangle|^2 + (1-\frac{\lambda}{2})^2\sum_{j\in\sigma}|\langle x, f_j\rangle|^2.\end{aligned} \tag{16}$$

$$\begin{aligned}&(2\lambda - \frac{\lambda^2}{2} - 1)\sum_{j\in\sigma}|\langle x, f_j\rangle|^2 + (1-\frac{\lambda^2}{2})\sum_{j\in\sigma^c}|\langle x, g_j\rangle|^2 \\ &\quad\le \sum_{j\in\sigma}|\langle S_W^{-1}S_{\mathcal{F}}^{\sigma}x, f_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_W^{-1}S_{\mathcal{F}}^{\sigma}x, g_j\rangle|^2 \\ &\qquad + \sum_{j\in\sigma}|\langle S_W^{-1}S_{\mathcal{G}}^{\sigma^c}x, f_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle S_W^{-1}S_{\mathcal{G}}^{\sigma^c}x, g_j\rangle|^2 \\ &\quad\le \sum_{j\in\sigma}|\langle x, f_j\rangle|^2 + \sum_{j\in\sigma^c}|\langle x, g_j\rangle|^2.\end{aligned} \tag{17}$$

Proof. Let $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ be the same as in the proof of Corollary 2. By combining Equations (10) and (12), and Theorem 2 we arrive at

$$\begin{aligned}&\sum_{j\in\sigma}|\langle x, f_j\rangle|^2 - \sum_{j\in\sigma}|\langle S_W^{-1}S_{\mathcal{F}}^{\sigma}x, f_j\rangle|^2 - \sum_{j\in\sigma^c}|\langle S_W^{-1}S_{\mathcal{F}}^{\sigma}x, g_j\rangle|^2 \\ &\quad= \mathrm{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle - \Big\|\sum_{j\in\sigma}\langle x, f_j\rangle h_j\Big\|^2 \\ &\quad\le \frac{\lambda^2}{4}\mathrm{Re}\sum_{j\in\sigma^c}\langle x, g_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle + (1-\frac{\lambda}{2})^2\mathrm{Re}\sum_{j\in\sigma}\langle x, f_j\rangle\langle h_j, S_{\mathcal{FGH}}x\rangle \\ &\quad= \frac{\lambda^2}{4}\sum_{j\in\sigma^c}|\langle x, g_j\rangle|^2 + (1-\frac{\lambda}{2})^2\sum_{j\in\sigma}|\langle x, f_j\rangle|^2\end{aligned}$$

for each $x \in \mathbb{H}$. Let P and Q be given in Equation (7). Then a direct calculation shows that $P = S_W^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}$ and $Q = S_W^{-\frac{1}{2}}S_{\mathcal{G}}^{\sigma^c}$ and, $P^*Q = S_{\mathcal{F}}^{\sigma}S_W^{-1}S_{\mathcal{G}}^{\sigma^c}$ as a consequence. Since $S_W^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}S_W^{-\frac{1}{2}}$ and $S_W^{-\frac{1}{2}}S_{\mathcal{G}}^{\sigma^c}S_W^{-\frac{1}{2}}$ are positive and commutative,

$$0 \le S_W^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}S_W^{-\frac{1}{2}}S_W^{-\frac{1}{2}}S_{\mathcal{G}}^{\sigma^c}S_W^{-\frac{1}{2}} = S_W^{-\frac{1}{2}}S_{\mathcal{F}}^{\sigma}S_W^{-1}S_{\mathcal{G}}^{\sigma^c}S_W^{-\frac{1}{2}},$$

implying that $S^{\sigma}_{\mathcal{F}} S_W^{-1} S^{\sigma^c}_{\mathcal{G}} = P^* Q \geq 0$. Again by Theorem 2,

$$\begin{aligned} 0 &\leq \operatorname{Re} \sum_{j\in\sigma} \langle x, f_j\rangle \langle h_j, S_{\mathcal{FGH}} x\rangle - \left\| \sum_{j\in\sigma} \langle x, f_j\rangle h_j \right\|^2 \\ &= \sum_{j\in\sigma} |\langle x, f_j\rangle|^2 - \sum_{j\in\sigma} |\langle S_W^{-1} S^{\sigma}_{\mathcal{F}} x, f_j\rangle|^2 - \sum_{j\in\sigma^c} |\langle S_W^{-1} S^{\sigma}_{\mathcal{F}} x, g_j\rangle|^2. \end{aligned}$$

We are now in a position to prove Equation (17). By Equations (10) and (11) we have

$$\begin{aligned} &\left\| \sum_{j\in\sigma} \langle x, f_j\rangle h_j \right\|^2 + \left\| \sum_{j\in\sigma^c} \langle x, g_j\rangle h_j \right\|^2 \\ &= \sum_{j\in\sigma} |\langle S_W^{-1} S^{\sigma}_{\mathcal{F}} x, f_j\rangle|^2 + \sum_{j\in\sigma^c} |\langle S_W^{-1} S^{\sigma}_{\mathcal{F}} x, g_j\rangle|^2 + \sum_{j\in\sigma} |\langle S_W^{-1} S^{\sigma^c}_{\mathcal{G}} x, f_j\rangle|^2 + \sum_{j\in\sigma^c} |\langle S_W^{-1} S^{\sigma^c}_{\mathcal{G}} x, g_j\rangle|^2 \end{aligned} \tag{18}$$

for any $x \in \mathbb{H}$. We also have

$$\|S_{\mathcal{FGH}} x\|^2 = \|S_W^{\frac{1}{2}} x\|^2 = \langle S_W x, x\rangle = \sum_{j\in\sigma} |\langle x, f_j\rangle|^2 + \sum_{j\in\sigma^c} |\langle x, g_j\rangle|^2.$$

This together with Equations (12), (13) and (18), and Theorem 2 gives Equation (17). □

Remark 3. *Inequalities (16) and (17) in Corollary 4 are respectively inequalities in Theorems 14 and 15 shown in [36].*

Suppose that $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$, $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$, and $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ are Bessel sequences for $\mathbb{H}$, and that $\{\alpha_j\}_{j\in\mathbb{J}}$ is a bounded sequence of complex numbers. For any $\sigma \subset \mathbb{J}$ and any $x \in \mathbb{H}$, we define linear bounded operators E^{σ}, E^{σ^c}, F^{σ} and F^{σ^c} respectively by

$$E^{\sigma} x = \sum_{j\in\sigma} (1-\alpha_j)\langle x, h_j\rangle f_j, \quad E^{\sigma^c} x = \sum_{j\in\sigma^c} (1-\alpha_j)\langle x, h_j\rangle g_j,$$

and

$$F^{\sigma} x = \sum_{j\in\sigma} \alpha_j \langle x, h_j\rangle f_j, \quad F^{\sigma^c} x = \sum_{j\in\sigma^c} \alpha_j \langle x, h_j\rangle g_j.$$

We are now ready to present a new triangle inequality for weaving frames.

Theorem 3. *Suppose that two frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then for any bounded sequence $\{\alpha_j\}_{j\in\mathbb{J}}$, for all $\sigma \subset \mathbb{J}$ and all $x \in \mathbb{H}$, we have*

$$\begin{aligned} \frac{3}{4}\|x\|^2 &\leq \left\| \sum_{j\in\sigma^c} \alpha_j \langle x, h_j\rangle g_j + \sum_{j\in\sigma} \alpha_j \langle x, h_j\rangle f_j \right\|^2 + \operatorname{Re}\left(\sum_{j\in\sigma} (1-\alpha_j)\langle x, h_j\rangle \langle f_j, x\rangle + \sum_{j\in\sigma^c} (1-\alpha_j)\langle x, h_j\rangle \langle g_j, x\rangle \right) \\ &\leq \frac{3 + \|(E^{\sigma} + E^{\sigma^c}) - (F^{\sigma} + F^{\sigma^c})\|^2}{4} \|x\|^2, \end{aligned} \tag{19}$$

where $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of the weaving frame $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$.

Proof. For any $\sigma \subset \mathbb{J}$, since $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of the weaving frame $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$, $E^\sigma + E^{\sigma^c} + F^\sigma + F^{\sigma^c} = \mathrm{Id}_{\mathbb{H}}$. For any $x \in \mathbb{H}$ we obtain

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma^c}\alpha_j\langle x,h_j\rangle g_j+\sum_{j\in\sigma}\alpha_j\langle x,h_j\rangle f_j\Big\|^2+\mathrm{Re}\Big(\sum_{j\in\sigma}(1-\alpha_j)\langle x,h_j\rangle\langle f_j,x\rangle+\sum_{j\in\sigma^c}(1-\alpha_j)\langle x,h_j\rangle\langle g_j,x\rangle\Big)\\
&=\langle (F^\sigma+F^{\sigma^c})^*(F^\sigma+F^{\sigma^c})x,x\rangle+\mathrm{Re}(\langle E^\sigma x,x\rangle+\langle E^{\sigma^c}x,x\rangle)\\
&=\frac{1}{2}\langle (E^\sigma+E^{\sigma^c}+(E^\sigma)^*+(E^{\sigma^c})^*)x,x\rangle+\langle (\mathrm{Id}_{\mathbb{H}}-(E^\sigma+E^{\sigma^c}))^*(\mathrm{Id}_{\mathbb{H}}-(E^\sigma+E^{\sigma^c}))x,x\rangle\\
&=\Big\langle\Big(\mathrm{Id}_{\mathbb{H}}-\frac{1}{2}(E^\sigma+E^{\sigma^c}+(E^\sigma)^*+(E^{\sigma^c})^*)+(E^\sigma+E^{\sigma^c})^*(E^\sigma+E^{\sigma^c})\Big)x,x\Big\rangle\\
&=\Big\langle\Big(\Big((E^\sigma+E^{\sigma^c})-\frac{1}{2}\mathrm{Id}_{\mathbb{H}}\Big)^*\Big((E^\sigma+E^{\sigma^c})-\frac{1}{2}\mathrm{Id}_{\mathbb{H}}\Big)+\frac{3}{4}\mathrm{Id}_{\mathbb{H}}\Big)x,x\Big\rangle\\
&=\Big\|\Big((E^\sigma+E^{\sigma^c})-\frac{1}{2}\mathrm{Id}_{\mathbb{H}}\Big)x\Big\|^2+\frac{3}{4}\|x\|^2\\
&\geq\frac{3}{4}\|x\|^2.
\end{aligned}\tag{20}$$

On the other hand we get

$$\begin{aligned}
&\Big\|\sum_{j\in\sigma^c}\alpha_j\langle x,h_j\rangle g_j+\sum_{j\in\sigma}\alpha_j\langle x,h_j\rangle f_j\Big\|^2+\mathrm{Re}\Big(\sum_{j\in\sigma}(1-\alpha_j)\langle x,h_j\rangle\langle f_j,x\rangle+\sum_{j\in\sigma^c}(1-\alpha_j)\langle x,h_j\rangle\langle g_j,x\rangle\Big)\\
&=\langle (F^\sigma+F^{\sigma^c})x,(F^\sigma+F^{\sigma^c})x\rangle+\mathrm{Re}\langle (E^\sigma+E^{\sigma^c})x,x\rangle\\
&=\langle (F^\sigma+F^{\sigma^c})x,(F^\sigma+F^{\sigma^c})x\rangle+\mathrm{Re}(\langle x,x\rangle-\langle (F^\sigma+F^{\sigma^c})x,x\rangle)\\
&=\langle x,x\rangle-\mathrm{Re}\langle (F^\sigma+F^{\sigma^c})x,x\rangle+\langle (F^\sigma+F^{\sigma^c})x,(F^\sigma+F^{\sigma^c})x\rangle\\
&=\langle x,x\rangle-\mathrm{Re}\langle (F^\sigma+F^{\sigma^c})x,(E^\sigma+E^{\sigma^c})x\rangle\\
&=\langle x,x\rangle-\frac{1}{2}\langle (F^\sigma+F^{\sigma^c})x,(E^\sigma+E^{\sigma^c})x\rangle-\frac{1}{2}\langle (E^\sigma+E^{\sigma^c})x,(F^\sigma+F^{\sigma^c})x\rangle\\
&=\frac{3}{4}\|x\|^2+\frac{1}{4}\langle ((E^\sigma+E^{\sigma^c})+(F^\sigma+F^{\sigma^c}))x,((E^\sigma+E^{\sigma^c})+(F^\sigma+F^{\sigma^c}))x\rangle\\
&\quad-\frac{1}{2}\langle (F^\sigma+F^{\sigma^c})x,(E^\sigma+E^{\sigma^c})x\rangle-\frac{1}{2}\langle (E^\sigma+E^{\sigma^c})x,(F^\sigma+F^{\sigma^c})x\rangle\\
&=\frac{3}{4}\|x\|^2+\frac{1}{4}\langle ((E^\sigma+E^{\sigma^c})-(F^\sigma+F^{\sigma^c}))x,((E^\sigma+E^{\sigma^c})-(F^\sigma+F^{\sigma^c}))x\rangle\\
&\leq\frac{3}{4}\|x\|^2+\frac{1}{4}\|(E^\sigma+E^{\sigma^c})-(F^\sigma+F^{\sigma^c})\|^2\|x\|^2\\
&=\frac{3+\|(E^\sigma+E^{\sigma^c})-(F^\sigma+F^{\sigma^c})\|^2}{4}\|x\|^2.
\end{aligned}\tag{21}$$

This along with Equation (20) yields Equation (19). □

Corollary 5. *Suppose that two frames $\mathcal{F} = \{f_j\}_{j\in\mathbb{J}}$ and $\mathcal{G} = \{g_j\}_{j\in\mathbb{J}}$ in $\mathbb{H}$ are woven. Then for all $\sigma \subset \mathbb{J}$ and all $x \in \mathbb{H}$, we have*

$$\frac{3}{4}\|x\|^2\leq\Big\|\sum_{j\in\sigma}\langle x,h_j\rangle f_j\Big\|^2+\mathrm{Re}\sum_{j\in\sigma^c}\langle x,h_j\rangle\langle g_j,x\rangle\leq\frac{3+\|S^{\sigma^c}_{\mathcal{HG}}-S^{\sigma}_{\mathcal{HF}}\|^2}{4}\|x\|^2,$$

where $S^{\sigma^c}_{\mathcal{HG}}, S^{\sigma}_{\mathcal{HF}} \in B(\mathbb{H})$ are defined respectively by

$$S^{\sigma^c}_{\mathcal{HG}}x=\sum_{j\in\sigma^c}\langle x,h_j\rangle g_j\quad\textit{and}\quad S^{\sigma}_{\mathcal{HF}}x=\sum_{j\in\sigma}\langle x,h_j\rangle f_j,$$

and $\mathcal{H} = \{h_j\}_{j\in\mathbb{J}}$ is an alternate dual frame of the weaving frame $\{f_j\}_{j\in\sigma} \cup \{g_j\}_{j\in\sigma^c}$.

Proof. The conclusion follows by Theorem 3 if we take

$$\alpha_j = \begin{cases} 1, & j \in \sigma, \\ 0, & j \in \sigma^c. \end{cases}$$

□

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References

1. Duffin, R.J.; Schaeffer, A.C. A class of nonharmonic Fourier series. *Trans. Am. Math. Soc.* **1952**, *72*, 341–366. [CrossRef]
2. Balan, R.; Wang, Y. Invertibility and robustness of phaseless reconstruction. *Appl. Comput. Harmonic Anal.* **2015**, *38*, 469–488. [CrossRef]
3. Botelho-Andrade, S.; Casazza, P.G.; Van Nguyen, H.; Tremain, J.C. Phase retrieval versus phaseless reconstruction. *J. Math. Anal. Appl.* **2016**, *436*, 131–137. [CrossRef]
4. Casazza, P.G. The art of frame theory. *Taiwan. J. Math.* **2000**, *4*, 129–201. [CrossRef]
5. Casazza, P.G.; Kutyniok, G. *Finite Frames: Theory and Applications*; Birkhäuser: Basel, Switzerland, 2013.
6. Casazza, P.G.; Ghoreishi, D.; Jose, S.; Tremain, J.C. Norm retrieval and phase retrieval by projections. *Axioms* **2017**, *6*, 6. [CrossRef]
7. Christensen, O. *An Introduction to Frames and Riesz Bases*; Birkhäuser: Boston, MA, USA, 2000.
8. Christensen, O.; Hasannasab, M. Operator representations of frames: Boundedness, duality, and stability. *Integral Equ. Oper. Theory* **2017**, *88*, 483–499. [CrossRef]
9. Christensen, O.; Hasannasab, M.; Rashidi, E. Dynamical sampling and frame representations with bounded operators. *J. Math. Anal. Appl.* **2018**, *463*, 634–644. [CrossRef]
10. Daubechies, I.; Grossmann, A.; Meyer, Y. Painless nonorthogonal expansions. *J. Math. Phys.* **1986**, *27*, 1271–1283. [CrossRef]
11. Găvruţa, P. On the Feichtinger conjecture. *Electron. J. Linear Algebra* **2013**, *26*, 546–552. [CrossRef]
12. Hasankhani Fard, M.A. Norm retrievable frames in $\mathbb{R}^n$. *Electron. J. Linear Algebra* **2016**, *31*, 425–432. [CrossRef]
13. Pehlivan, S.; Han, D.; Mohapatra, R.N. Spectrally two-uniform frames for erasures. *Oper. Matrices* **2015**, *9*, 383–399. [CrossRef]
14. Rahimi, A.; Seddighi, N. Finite equal norm Parseval wavelet frames over prime fields. *Int. J. Wavel. Multiresolut. Inf. Process.* **2017**, *15*, 1750040. [CrossRef]
15. Sahu, N.K.; Mohapatra, R.N. Frames in semi-inner product spaces. In *Mathematical Analysis and its Applicationsl*; Agrawal, P., Mohapatra, R., Singh, U., Srivastava, H., Eds.; Springer: New Delhi, India, 2015; Volume 143, pp. 149–158, ISBN 978-81-322-2484-6.
16. Xiao, X.C.; Zhou, G.R.; Zhu, Y.C. Uniform excess frames in Hilbert spaces. *Results Math.* **2018**, *73*, 108. [CrossRef]
17. Balan, R.; Casazza, P.G.; Edidin, D. On signal reconstruction without phase. *Appl. Comput. Harmonic Anal.* **2006**, *20*, 345–356. [CrossRef]
18. Han, D.; Sun, W. Reconstruction of signals from frame coefficients with erasures at unknown locations. *IEEE Trans. Inf. Theory* **2014**, *60*, 4013–4025. [CrossRef]
19. Benedetto, J.; Powell, A.; Yilmaz, O. Sigma-Delta (ΣΔ) quantization and finite frames. *IEEE Trans. Inf. Theory* **2006**, *52*, 1990–2005. [CrossRef]
20. Jivulescu, M.A.; Găvruţa, P. Indices of sharpness for Parseval frames, quantum effects and observables. *Sci. Bull. Politeh. Univ. Timiş. Trans. Math. Phys.* **2015**, *60*, 17–29.
21. Strohmer, T.; Heath, R. Grassmannian frames with applications to coding and communication. *Appl. Comput. Harmonic Anal.* **2003**, *14*, 257–275. [CrossRef]

22. Sun, W. Asymptotic properties of Gabor frame operators as sampling density tends to infinity. *J. Funct. Anal.* **2010**, *258*, 913–932. [CrossRef]
23. Bemrose, T.; Casazza, P.G.; Gröchenig, K.; Lammers, M.C.; Lynch, R.G. Weaving frames. *Oper. Matrices* **2016**, *10*, 1093–1116. [CrossRef]
24. Casazza, P.G.; Freeman, D.; Lynch, R.G. Weaving Schauder frames. *J. Approx. Theory* **2016**, *211*, 42–60. [CrossRef]
25. Deepshikha; Vashisht, L.K. On weaving frames. *Houston J. Math.* **2018**, *44*, 887–915.
26. Deepshikha; Vashisht, L.K. Weaving K-frames in Hilbert spaces. *Results Math.* **2018**, *73*, 81. [CrossRef]
27. Khosravi, A.; Banyarani, J.S. Weaving g-frames and weaving fusion frames. *Bull. Malays. Math. Sci. Soc.* **2018**. [CrossRef]
28. Rahimi, A.; Samadzadeh, Z.; Daraby, B. Frame related operators for woven frames. *Int. J. Wavel. Multiresolut. Inf. Process.* **2018**. [CrossRef]
29. Vashisht, L.K.; Garg, S.; Deepshikha; Das, P.K. On generalized weaving frames in Hilbert spaces. *Rocky Mt. J. Math.* **2018**, *48*, 661–685. [CrossRef]
30. Vashisht, L.K.; Deepshikha. Weaving properties of generalized continuous frames generated by an iterated function system. *J. Geom. Phys.* **2016**, *110*, 282–295. [CrossRef]
31. Balan, R.; Casazza, P.G.; Edidin, D.; Kutyniok, G. A new identity for Parseval frames. *Proc. Am. Math. Soc.* **2007**, *135*, 1007–1015. [CrossRef]
32. Găvruţa, P. On some identities and inequalities for frames in Hilbert spaces. *J. Math. Anal. Appl.* **2006**, *321*, 469–478. [CrossRef]
33. Li, D.W.; Leng, J.S. On some new inequalities for fusion frames in Hilbert spaces. *Math. Inequal. Appl.* **2017**, *20*, 889–900. [CrossRef]
34. Li, D.W.; Leng, J.S. On some new inequalities for continuous fusion frames in Hilbert spaces. *Mediterr. J. Math.* **2018**, *15*, 173. [CrossRef]
35. Poria, A. Some identities and inequalities for Hilbert-Schmidt frames. *Mediterr. J. Math.* **2017**, *14*, 59. [CrossRef]
36. Li, D.W.; Leng, J.S. New inequalities for weaving frames in Hilbert spaces. *arXiv* **2018**, arXiv:1809.00863.