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Article

New Refinement of the Operator Kantorovich Inequality

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Abstract: We focus on the improvement of operator Kantorovich type inequalities. Among the consequences, we improve the main result of the paper [H.R. Moradi, I.H. Gümüş, Z. Heydarbeygi, A glimpse at the operator Kantorovich inequality, Linear Multilinear Algebra, doi:10.1080/03081087.2018.1441799].

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1. Notation and Preliminaries

At the beginning of this paper, we cite the following inequality which is called the operator Kantorovich inequality [1]:

$$\Phi\left(A^{-1}\right) \leq \frac{(M+m)^2}{4Mm}\Phi(A)^{-1} \tag{1}$$

where Φ is a normalized positive linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$, (we represent $\mathcal{H}$ and $\mathcal{K}$ as complex Hilbert spaces throughout the paper) and A is a positive operator with spectrum contained in $[m, M]$ with $0 < m < M$. This is a non-commutative analogue of the classical inequality [2],

$$\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle \leq \frac{(M+m)^2}{4Mm}$$

where $x \in \mathcal{H}$ is a unit vector.

In recent years, various attempts have been made by many authors to improve and generalize the operator Kantorovich inequality. One may see the basic references [3–5] and the excellent survey [6] on this topic. In [7], it was shown that

$$\Phi\left(A^{-1}\right) \leq \Phi\left(m^{\frac{A-MI}{M-m}} M^{\frac{mI-A}{M-m}}\right) \leq \frac{(M+m)^2}{4Mm}\Phi(A)^{-1}. \tag{2}$$

The main aim of the present short paper is to improve both inequalities in (2). Actually, we prove that

$$\begin{aligned}\Phi\left(A^{-1}\right) &\le \Phi\left(\left(A-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(A)\right)^{-1}\right)\\ &\le \Phi\left(\left(m^{\frac{A-MI}{M-m}} M^{\frac{mI-A}{M-m}}\right)^{-1}\right)\\ &\le \frac{(M+m)^{2}}{4Mm}\Phi(A)^{-1}-\left(\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{Mm}\right) r(A)\end{aligned}$$

where $r(A)=\min\left\{\frac{MI-A}{M-m}, \frac{A-mI}{M-m}\right\}=\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{M+m}{2}I\right|$.

In what follows, an operator means a bounded linear one acting on a complex Hilbert space $\mathcal{H}$. As customary, we reserve m, M for scalars and I for the identity operator. A self-adjoint operator A is said to be positive if $\langle Ax, x\rangle \ge 0$ holds for all $x \in \mathcal{H}$. A linear map Φ is positive if $\Phi(A) \ge 0$ whenever $A \ge 0$. It is said to be normalized if $\Phi(I) = I$. We denote by $\sigma(A)$ the spectrum of the operator A.

2. Main Results

Before we present the proof of our theorems, we begin with a general observation. We say that a non-negative function f on $[0,\infty)$ is geometrically convex [8] when

$$f\left(a^{1-v}b^{v}\right) \le f(a)^{1-v} f(b)^{v} \tag{3}$$

for all $a, b > 0$ and $v \in [0,1]$. Equivalently, a function f is geometrically convex if and only if the associated function $F(y) = \log(f(e^{y}))$ is convex.

Example 1 ([9] Example 2.12)**.** *Given real numbers $c_i \ge 0$ and $p_i \in (-\infty, 0] \cup [1,\infty)$ for $i = 1, \cdots, n$, the function $f(t) = \sum_{i=1}^{n} c_i t^{p_i}$ is geometrically convex on $(0,\infty)$.*

Kittaneh and Manasrah [10] Theorem 2.1 obtained a refinement of the weighted arithmetic-geometric mean inequality as follows:

$$a^{1-v}b^{v} \le (1-v)\,a + vb - r\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{4}$$

where $r = \min\{v, 1-v\}$.

Now, if f is a decreasing geometrically convex function, then

$$\begin{aligned}f\left((1-v)\,a+vb\right) &\le f\left(\left((1-v)\,a+vb\right)-r\left(\sqrt{a}-\sqrt{b}\right)^{2}\right)\\ &\le f\left(a^{1-v}b^{v}\right)\\ &\le f(a)^{1-v} f(b)^{v}\\ &\le (1-v)\,f(a)+vf(b)-r\left(\sqrt{f(a)}-\sqrt{f(b)}\right)^{2}\\ &\le (1-v)\,f(a)+vf(b)\end{aligned} \tag{5}$$

where the first inequality follows from the inequality $(1-v)\,a+vb-r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-v)\,a+vb$ and the fact that f is decreasing function, in the second inequality we used (4), the third inequality is obvious by (3), and the fourth inequality again follows from (4) by interchanging a by $f(a)$ and b by $f(b)$.

Of course, each decreasing geometrically convex function is also convex. However, the converse does not hold in general.

The inequality (5) applied to $a=m$, $b=M$, $1-v=\frac{M-t}{M-m}$, and $v=\frac{t-m}{M-m}$ gives

$$\begin{aligned} f(t) &\le f\left(t-\left(\sqrt{m}-\sqrt{M}\right)^2 r(t)\right) \\ &\le f\left(m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}\right) \\ &\le f(m)^{\frac{M-t}{M-m}} f(M)^{\frac{t-m}{M-m}} \\ &\le \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) - \left(\sqrt{f(m)}-\sqrt{f(M)}\right)^2 r(t) \\ &\le \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M) \end{aligned} \tag{6}$$

with $r(t)=\min\left\{\frac{t-m}{M-m}, \frac{M-t}{M-m}\right\}=\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{M+m}{2}\right|$ whenever $t\in[m,M]$.

In order to establish our promised refinement of the operator Kantorovich inequality, we also use the well-known monotonicity principle for bounded self-adjoint operators on Hilbert space (see, e.g., [6] (p. 3)): If $A\in\mathcal{B}(\mathcal{H})$ is a self-adjoint operator, then

$$f(t)\le g(t),\ t\in\sigma(A) \ \Rightarrow\ f(A)\le g(A) \tag{7}$$

provided that f and g are real-valued continuous functions. Under the same assumptions, $h(t)=|t|$ implies $h(A)=|A|$.

Now, we are in a position to state and prove our main results. We remark that the following theorem can be regarded as an extension of [5] Remark 4.14 to the context of geometrical convex functions.

Theorem 1. *Let $A\in\mathcal{B}(\mathcal{H})$ be a self-adjoint operator with $\sigma(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$ and Φ be a normalized positive linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$. If f is strictly positive decreasing geometrically convex function, then*

$$\begin{aligned} \Phi\left(f\left(A-\left(\sqrt{m}-\sqrt{M}\right)^2 r(A)\right)\right) &\le \Phi\left(f\left(m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}}\right)\right) \\ &\le \mu(m,M,f)\, f(\Phi(A)) - \left(\sqrt{f(m)}-\sqrt{f(M)}\right)^2 \Phi(r(A)) \end{aligned}$$

where $r(A)=\min\left\{\frac{A-mI}{M-m}, \frac{MI-A}{M-m}\right\}=\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{M+m}{2}I\right|$ and

$$\mu(m,M,f)=\max\left\{\frac{1}{f(t)}\left(\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)\right):\ t\in[m,M]\right\}.$$

Proof. On account of the assumptions, from parts of (6), we have

$$\begin{aligned} f\left(t-\left(\sqrt{m}-\sqrt{M}\right)^{2}r(t)\right) &\leq f\left(m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}\right) \\ &\leq L\left(t\right)-\left(\sqrt{f\left(m\right)}-\sqrt{f\left(M\right)}\right)^{2}r(t) \end{aligned} \tag{8}$$

where

$$L\left(t\right)=\frac{M-t}{M-m}f\left(m\right)+\frac{t-m}{M-m}f\left(M\right).$$

Note that inequality (8) holds for all $t\in[m,M]$. On the other hand, $\sigma\left(A\right)\subseteq[m,M]$, which, by virtue of monotonicity principle (7) for operator functions, yields the series of inequalities

$$\begin{aligned} f\left(A-\left(\sqrt{m}-\sqrt{M}\right)^{2}r(A)\right) &\leq f\left(m^{\frac{MI-A}{M-m}}M^{\frac{A-mI}{M-m}}\right) \\ &\leq L\left(A\right)-\left(\sqrt{f\left(m\right)}-\sqrt{f\left(M\right)}\right)^{2}r(A). \end{aligned}$$

It follows from the linearity and the positivity of the map Φ that

$$\begin{aligned} \Phi\left(f\left(A-\left(\sqrt{m}-\sqrt{M}\right)^{2}r(A)\right)\right) &\leq \Phi\left(f\left(m^{\frac{MI-A}{M-m}}M^{\frac{A-mI}{M-m}}\right)\right) \\ &\leq \Phi\left(L\left(A\right)\right)-\left(\sqrt{f\left(m\right)}-\sqrt{f\left(M\right)}\right)^{2}\Phi(r(A)). \end{aligned}$$

Now, by using [5] Corollary 4.12 we get

$$\begin{aligned} \Phi\left(f\left(A-\left(\sqrt{m}-\sqrt{M}\right)^{2}r(A)\right)\right) &\leq \Phi\left(f\left(m^{\frac{MI-A}{M-m}}M^{\frac{A-mI}{M-m}}\right)\right) \\ &\leq \Phi\left(L\left(A\right)\right)-\left(\sqrt{f\left(m\right)}-\sqrt{f\left(M\right)}\right)^{2}\Phi(r(A)) \\ &\leq \mu\left(m,M,f\right)f\left(\Phi\left(A\right)\right)-\left(\sqrt{f\left(m\right)}-\sqrt{f\left(M\right)}\right)^{2}\Phi(r(A)). \end{aligned}$$

This completes the proof. □

As discussed extensively in [6] Cahpter 2, for $f\left(t\right)=t^{p}$, we have

$$\begin{aligned} \mu\left(m,M,t^{p}\right) &= \max\left\{\frac{1}{t^{p}}\left(\frac{M-t}{M-m}m^{p}+\frac{t-m}{M-m}M^{p}\right):\ t\in[m,M]\right\} \\ &= \frac{\left(mM^{p}-Mm^{p}\right)}{\left(p-1\right)\left(M-m\right)}\left(\frac{p-1}{p}\frac{M^{p}-m^{p}}{mM^{p}-Mm^{p}}\right)^{p}. \end{aligned}$$

Now, the following fact can be easily deduced from Theorem 1 and Example 1.

Corollary 1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator with $\sigma(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$ and Φ be a normalized positive linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$. Then for any $p < 0$,*

$$\begin{aligned}\Phi\left(A^{p}\right) &\leq \Phi\left(\left(A-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(A)\right)^{p}\right) \\ &\leq \Phi\left(\left(m^{\frac{A-MI}{M-m}} M^{\frac{mI-A}{M-m}}\right)^{p}\right) \\ &\leq K(m, M, p)\, \Phi(A)^{p}-\left(m^{p/2}-M^{p/2}\right)^{2} \Phi(r(A))\end{aligned}$$

where

$$K(m, M, p)=\frac{\left(m M^{p}-M m^{p}\right)}{(p-1)(M-m)}\left(\frac{p-1}{p} \frac{M^{p}-m^{p}}{m M^{p}-M m^{p}}\right)^{p}.$$

In particular,

$$\begin{aligned}\Phi\left(A^{-1}\right) &\leq \Phi\left(\left(A-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(A)\right)^{-1}\right) \\ &\leq \Phi\left(\left(m^{\frac{A-MI}{M-m}} M^{\frac{mI-A}{M-m}}\right)^{-1}\right) \\ &\leq \frac{(M+m)^{2}}{4 M m} \Phi(A)^{-1}-\left(\frac{\left(\sqrt{M}-\sqrt{m}\right)^{2}}{M m}\right) \Phi(r(A)).\end{aligned}$$

We note that $K(m, M, -1)=\frac{(M+m)^{2}}{4 M m}$ is the original Kantorovich constant.

Theorem 2. *Let all the assumptions of Theorem 1 hold. Then*

$$\begin{aligned}f\left(\Phi(A)-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(\Phi(A))\right) &\leq f\left(m^{\frac{MI-\Phi(A)}{M-m}} M^{\frac{\Phi(A)-mI}{M-m}}\right) \\ &\leq \mu(m, M, f)\, \Phi(f(A))-\left(\sqrt{f(m)}-\sqrt{f(M)}\right)^{2} r(\Phi(A)).\end{aligned}$$

Proof. By applying a standard functional calculus for the operator $\Phi(A)$ such that $mI \leq \Phi(A) \leq MI$, we get from (8)

$$\begin{aligned}f\left(\Phi(A)-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(\Phi(A))\right) &\leq f\left(m^{\frac{MI-\Phi(A)}{M-m}} M^{\frac{\Phi(A)-mI}{M-m}}\right) \\ &\leq \Phi(L(A))-\left(\sqrt{f(m)}-\sqrt{f(M)}\right)^{2} r(\Phi(A)).\end{aligned}$$

We thus have

$$\begin{aligned}
f\left(\Phi(A)-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(\Phi(A))\right) &\leq f\left(m^{\frac{MI-\Phi(A)}{M-m}} M^{\frac{\Phi(A)-mI}{M-m}}\right) \\
&\leq L(\Phi(A))-\left(\sqrt{f(m)}-\sqrt{f(M)}\right)^{2} r(\Phi(A)) \\
&= \Phi(L(A))-\left(\sqrt{f(m)}-\sqrt{f(M)}\right)^{2} r(\Phi(A)) \\
&\leq \mu(m, M, f)\, \Phi(f(A))-\left(\sqrt{f(m)}-\sqrt{f(M)}\right)^{2} r(\Phi(A))
\end{aligned}$$

where at the last step we used the basic inequality [5] Corollary 4.12.

Hence, the proof is complete. □

As a corollary of Theorem 2 we have:

Corollary 2. *Let all the assumptions of Corollary 1 hold. Then for any $p<0$*

$$\begin{aligned}
\Phi(A)^{p} &\leq\left(\Phi(A)-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(\Phi(A))\right)^{p} \\
&\leq\left(m^{\frac{MI-\Phi(A)}{M-m}} M^{\frac{\Phi(A)-mI}{M-m}}\right)^{p} \\
&\leq K(m, M, p)\, \Phi\left(A^{p}\right)-\left(\sqrt{m^{p}}-\sqrt{M^{p}}\right)^{2} r(\Phi(A)).
\end{aligned}$$

Remark 1. *Notice that the inequalities in Corollary 2 are stronger than the inequalities obtained in [11] Corollary 2.1.*

Recall that if f is operator convex, the solidarities [12] or the perspective [13] of f is defined by

$$\mathcal{P}_{f}(A \mid B)=A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}.$$

Using a series of inequalities (6) we have the upper bounds of the perspective for non-negative decreasing geometrically convex function (not necessary operator convex f). We use the same symbol $\mathcal{P}_{f}(A \mid B)$ for a simplicity.

Proposition 1. *Let $A, B>0$ with $mA \leq B \leq MA$ for some scalars $0<m<M$. For a non-negative decreasing geometrically convex function f, we have*

$$\begin{aligned}
\mathcal{P}_{f}(A \mid B) &\leq A^{1/2} f\left(A^{-1/2} B A^{-1/2}-\left(\sqrt{m}-\sqrt{M}\right)^{2} r(A, B)\right) A^{1/2} \\
&\leq A^{1/2} f\left(m^{\frac{MI-A^{-1/2} B A^{-1/2}}{M-m}} M^{\frac{A^{-1/2} B A^{-1/2}-mI}{M-m}}\right) A^{1/2} \\
&\leq A^{1/2} f(m)^{\frac{MI-A^{-1/2} B A^{-1/2}}{M-m}} f(M)^{\frac{A^{-1/2} B A^{-1/2}-mI}{M-m}} A^{1/2} \\
&\leq \frac{Mf(m)-mf(M)}{M-m} A+\frac{f(M)-f(m)}{M-m} B-\left(\sqrt{f(m)}-\sqrt{f(M)}\right)^{2} A^{1/2} r(A, B) A^{1/2} \\
&\leq \frac{Mf(m)-mf(M)}{M-m} A+\frac{f(M)-f(m)}{M-m} B,
\end{aligned}$$

where

$$r(A,B) = \min\left\{\frac{A^{-1/2}BA^{-1/2} - mI}{M-m}, \frac{MI - A^{-1/2}BA^{-1/2}}{M-m}\right\}$$
$$= \frac{1}{2}I - \frac{1}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{M+m}{2}I\right|.$$

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