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Modified Relaxed CQ Iterative Algorithms for the Split Feasibility Problem †

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Abstract: The split feasibility problem models inverse problems arising from phase retrievals problems and intensity-modulated radiation therapy. For solving the split feasibility problem, Xu proposed a relaxed CQ algorithm that only involves projections onto half-spaces. In this paper, we use the dual variable to propose a new relaxed CQ iterative algorithm that generalizes Xu's relaxed CQ algorithm in real Hilbert spaces. By using projections onto half-spaces instead of those onto closed convex sets, the proposed algorithm is implementable. Moreover, we present modified relaxed CQ algorithm with viscosity approximation method. Under suitable conditions, global weak and strong convergence of the proposed algorithms are proved. Some numerical experiments are also presented to illustrate the effectiveness of the proposed algorithms. Our results improve and extend the corresponding results of Xu and some others.

Keywords: split feasibility problem; relaxed CQ algorithm; convergence; Hilbert space

1. Introduction

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let I denote the identity operator on H. Let C and Q be nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively.

The split feasibility problem can mathematically be formulated as the problem of finding a point $u^* \in C$ with the property

$$Au^* \in Q, \tag{1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP (SFP) in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and medical image reconstruction [2], with particular progress in intensity-modulated radiation therapy [3,4]. It has been found that the SFP can also be used in the air traffic flow management problems. Many researchers studied the SFP and introduced various algorithms to solve it (see [5–15] and references therein).

The original algorithm introduced in [1] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A) and thus does not become popular. A more popular algorithm that solves the SFP (Equation (1)) seems to be the following CQ algorithm of Byrne [2,16]:

$$u_{n+1} = P_C(u_n - \mu A^*(I - P_Q)Au_n), \tag{2}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, and A^* is the adjoint of A and $\mu \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A. The CQ algorithm only involves the computations of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions (e.g., C and Q are the closed balls or half-spaces). It remains however a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions though theoretically we can prove (weak) convergence of the algorithm.

We assume that the SFP (Equation (1)) is consistent, and use Φ to denote the solution set of the SFP (Equation (1)), i.e.,

$$\Phi = \{u \in C : Au \in Q\}.$$

Thus, the set Φ is closed, convex and nonempty.

The CQ algorithm is found to be a gradient-projection method (GPM) in convex minimization (it is also a special case of the proximal forward-backward splitting method). We can reformulate the SFP (Equation (1)) as an optimization problem [17]. We may introduce the (convex) objective function

$$g(u) := \frac{1}{2}\|(I - P_Q)Au\|^2 \tag{3}$$

and consider the convex minimization problem

$$\min_{u \in C} g(u). \tag{4}$$

The objective function g is continuously differentiable with gradient given by

$$\nabla g(u) = A^*(I - P_Q)Au. \tag{5}$$

Because $I - P_Q$ is (firmly) nonexpansive, we obtain that ∇g is Lipschitz continuous with Lipschitz constant $L = \|A\|^2$. It is well known that the gradient-projection algorithm (GPM), for solving the minimization problem in Equation (4), generates the following iterative sequence $\{u_n\}$:

$$u_{n+1} = P_C(u_n - \mu \nabla g(u_n)), \tag{6}$$

where μ is chosen in the interval $(0, 2/L)$ with L being the Lipschitz constant of ∇g. For solving the problem in Equation (4), the GPM with gradient ∇g given as in Equation (5) is the CQ algorithm in Equation (2).

By Equation (4), the SFP (Equation (1)) can be written as the following convex separable minimization problem:

$$\min_{u \in H_1} \iota_C(u) + g(u), \tag{7}$$

where $g(u)$ is defined by Equation (3) and $\iota_C(u)$ is an indicator function of the set C defined by

$$\iota_C(u) = \begin{cases} 0, & u \in C, \\ +\infty, & u \notin C. \end{cases} \tag{8}$$

Chen et al. [18] designed and discussed an efficient algorithm for minimizing the sum of two proper lower semi-continuous convex functions, i.e.,

$$\min_{u \in R^n} g_1(u) + g_2(u), \tag{9}$$

where $g_1, g_2 \in \Gamma_0(\mathbb{R}^n)$ (all proper lower semi-continuous convex functions from $\mathbb{R}^n$ to $(-\infty, +\infty]$) and g_2 is differentiable on R^n with $1/\beta$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$. For $g \in \Gamma_0(\mathbb{R}^n)$

and $\rho \in (0, +\infty)$, the proximal operator of g with order ρ, denoted by $prox_{\rho g}$, is defined by: for each $x \in \mathbb{R}^n$,

$$prox_{\rho g}(x) = arg \min_{y \in \mathbb{R}^n} \{g(y) + \frac{1}{2\rho}\|x - y\|^2\}. \tag{10}$$

To solve the convex separable problem in Equation (9), they obtained the following fixed point formulation: the point u^* is a solution of Equation (9) if and only if there exists $e^* \in \mathbb{R}^n$ such that

$$\begin{cases} e^* = (I - prox_{\frac{\mu}{\lambda} g_1})(u^* - \mu \nabla g_2(u^*) + (1 - \lambda)e^*), \\ u^* = u^* - \mu \nabla g_2(u^*) - \lambda e^*, \end{cases}$$

where $\lambda > 0$ and $\mu > 0$. They introduced the following Picard iterative sequence:

$$\begin{cases} e_{n+1} = (I - prox_{\frac{\mu}{\lambda} g_1})(u_n - \mu \nabla g_2(u_n) + (1 - \lambda)e_n), \\ u_{n+1} = u_n - \mu \nabla g_2(u_n) - \lambda e_{n+1}. \end{cases} \tag{11}$$

It was shown [18] that, under appropriate conditions, the sequence $\{u_n\}$ converges to a solution of the problem in Equation (9). Moreover, u is the primal variable and e is the dual variable of the primal-dual form (see [18]) related to Equation (9).

For solving the SFP (Equation (1)), we note that the CQ algorithm and many related iterative algorithms (see [19–24]) only involves the computations of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions. However, in some cases it is impossible or needs too much work to exactly compute an orthogonal projection. Therefore, if it is the case, the efficiency of projection type methods will be seriously affected. To overcome this difficulty, Fukushima [25] suggested a so-called relaxed projection method to calculate the projection onto a level set of a convex function by computing a sequence of projections onto half-spaces containing the original level set. Theoretical analysis and numerical experiments show the efficiency of his method.

Let C and Q be level sets of convex functions, i.e.,

$$C = \{u \in H_1 : c(u) \leq 0\}, \;\; Q = \{v \in H_2 : q(v) \leq 0\}, \tag{12}$$

where $c : H_1 \to R$ and $q : H_2 \to R$ are convex and lower semi-continuous functions with the subdifferentials

$$\partial c(u) = \{z \in H_1 : c(x) \geq c(u) + \langle x - u, z \rangle, \;\; x \in H_1\} \neq \emptyset$$

for all $u \in C$ and

$$\partial q(v) = \{w \in H_2 : q(y) \geq q(v) + \langle y - v, w \rangle, \;\; y \in H_2\} \neq \emptyset$$

for all $v \in Q$. Set

$$C_n = \{u \in H_1 : c(u_n) + \langle \xi_n, u - u_n \rangle \leq 0\}, \tag{13}$$

where $\xi_n \in \partial c(u_n)$, and

$$Q_n = \{v \in H_2 : q(Au_n) + \langle \eta_n, v - Au_n \rangle \leq 0\}, \tag{14}$$

where $\eta_n \in \partial q(Au_n)$. Obviously, C_n and Q_n are half-spaces and the projections onto half-spaces C_n and Q_n have closed forms.

In the setting of finite-dimensional spaces, relaxed projection method was followed by Yang [26], who introduced the following relaxed CQ algorithms for solving the SFP (Equation (1)) where the closed convex subsets C and Q are level sets of convex functions:

$$u_{n+1} = P_{C_n}(u_n - \mu A^T(I - P_{Q_n})Au_n), \ \ n \geq 1, \tag{15}$$

where $\mu \in (0, 2/L)$ with L being the largest eigenvalue of matrix A^TA, C_n and Q_n are given in Equations (13) and (14), respectively. Due to the special form of C_n and Q_n, the proposed algorithm can be easily implemented.

Recently, for the purpose of generality, the SFP (Equation (1)) is studied in a more general setting. For instance, Xu [27] considered the SFP (Equation (1)) where H_1 and H_2 are infinite-dimensional Hilbert spaces. Xu [27] proposed the following relaxed CQ algorithm where C and Q are given in Equation (12):

$$u_{n+1} = P_{C_n}(u_n - \mu A^*(I - P_{Q_n})Au_n), \ \ n \geq 1, \tag{16}$$

where $\mu \in (0, 2/\|A\|^2)$, C_n and Q_n are given in Equations (13) and (14), respectively. Since the projections P_{C_n} and P_{Q_n} have closed-form expressions, the above relaxed CQ algorithm is implementable. In [27], the relaxed CQ algorithm has the weak convergence result. He and Zhao [28] introduced a Halpern-type relaxed CQ algorithm such that the strong convergence is guaranteed. Some relaxed algorithms have been proposed to solve the SFP (Equation (1)) (see [29–31]).

Inspired and motivated by the works mentioned above, for solving the SFP (Equation (1)) in real Hilbert spaces, we use the dual variable to propose a new relaxed CQ iterative algorithm:

$$\begin{cases} t_n = u_n - \mu_n A^*(I - P_{Q_n})Au_n, \\ e_{n+1} = (I - P_{C_n})(t_n + (1 - \lambda)e_n), \\ u_{n+1} = t_n - \lambda e_{n+1}, \end{cases} \tag{17}$$

where e_0 and $u_0 \in H_1$ are arbitrarily chosen, $0 < \lambda \leq 1$ and $0 < \mu_n \leq \frac{2}{\|A\|^2}$. Taking $\lambda = 1$, the proposed algorithm in Equation (17) becomes the relaxed CQ algorithm in Equation (16) (Xu [27]). Moreover, we present modified relaxed CQ algorithm with viscosity approximation method. Proposed two relaxed CQ iterative algorithms which only involve orthogonal projections onto half-spaces, so that the algorithms are implementable. Under suitable conditions, global weak and strong convergence of the proposed algorithms are proved. Some numerical experiments are also presented to illustrate the effectiveness of the proposed algorithms. Our results improve and extend the corresponding results of Xu and some others.

The rest of this paper is organized as follows. In the next section, some necessary concepts and important facts are collected. The weak convergence theorem of the proposed algorithm is established in Section 3. In Section 4, we modify the proposed algorithm by viscosity method so that it has strong convergence result. Finally, we give some numerical experiments to illustrate the efficiency of the proposed iterative methods.

2. Preliminaries

In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(u_n) = \{u : \exists u_{n_j} \rightharpoonup u\}$ to stand for the weak ω-limit set of $\{u_n\}$.

Definition 1. *A mapping $S : H \to H$ is said to be nonexpansive if*

$$\|Su - Sv\| \leq \|u - v\|$$

for all $u, v \in H$.

Definition 2. *A mapping $S : H \to H$ is said to be firmly nonexpansive if $2S - I$ is nonexpansive or, equivalently,*

$$\langle u - v, Su - Sv \rangle \geq \|Su - Sv\|^2$$

for all $u, v \in H$.

Alternatively, a mapping $S : H \to H$ is firmly nonexpansive if and only if S can be expressed as

$$S = \frac{1}{2}(I + U),$$

where I denotes the identity mapping on H and $U : H \to H$ is a nonexpansive mapping.

Definition 3. *A mapping $h : H \to H$ is said to be ρ-contraction if there exists a constant $\rho \in [0, 1)$ such that*

$$\|h(u) - h(v)\| \leq \rho\|u - v\|$$

for all $u, v \in H$.

Definition 4. *A mapping $h : C \to H$ is said to be η-strongly monotone if there exists a positive constant η such that*

$$\langle h(u) - h(v), u - v \rangle \geq \eta\|u - v\|^2$$

for all $u, v \in C$.

It is obvious that, if h is a ρ-contraction, then $I - h$ is a (1-ρ)-strongly monotone mapping. Recall the variational inequality problem [32] is to find a point $u^* \in C$ such that

$$\langle Fu^*, u - u^* \rangle \geq 0$$

for all $u \in C$, where C is a nonempty closed convex subset of H and $F : C \to H$ is a nonlinear operator. It is well known that [33] if $F : C \to H$ is a Lipschitzian and strongly monotone operator, then the above variational inequality problem has a unique solution.

Definition 5. *A mapping $S : H \to H$ is said to be α-inverse strongly monotone (α-ism) if there exists a positive constant α such that*

$$\langle u - v, Su - Sv \rangle \geq \alpha\|Su - Sv\|^2$$

for all $u, v \in H$.

Recall that the metric (nearest point) projection from H onto a nonempty closed convex subset C of H, denoted by P_C, is defined as follows: for each $u \in H$,

$$P_C(u) = arg \min_{v \in C}\{\|u - v\|\}.$$

Then, P_C is characterized by the inequality (for $u \in H$)

$$\langle u - P_C u, z - P_C u \rangle \leq 0, \ \ \forall z \in C.$$

It is well known that P_C and $I - P_C$ are firmly nonexpansive and 1-ism.

Definition 6. *A function $h : H \to R$ is said to be weakly lower semi-continuous (w-lsc) at u if $u_n \rightharpoonup u$ implies*

$$h(u) \leq \liminf_{n \to \infty} h(u_n).$$

Lemma 1. *[34] Let K be a nonempty closed convex subset of real Hilbert space H. Let $\{u_n\}$ be a sequence which satisfies the following properties:*

(a) *every weak limit point of $\{u_n\}$ lies in K; and*
(b) $\lim_{n\to\infty} \|u_n - u\|$ *exists for every $u \in K$.*

Then, $\{u_n\}$ converges weakly to a point in K.

Lemma 2. *[35] Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n \delta_n, \\ s_{n+1} \leq s_n - \eta_n + \mu_n, \end{cases}$$

for each $n \geq 0$, where $\{\lambda_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\mu_n\}$ are two sequences in $\mathbb{R}$ such that:

(a) $\Sigma_{n=1}^{\infty} \lambda_n = \infty$;
(b) $\lim_{n\to\infty} \mu_n = 0$; *and*
(c) $\lim_{l\to\infty} \eta_{n_l} = 0$ *implies* $\limsup_{l\to\infty} \delta_{n_l} \leq 0$ *for any subsequence* $\{n_l\} \subset \{n\}$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 3. *[36] Let H be a real Hilbert space. Then, for all $t \in [0,1]$ and $u, v \in H$,*

$$\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u-v\|^2.$$

3. Weak Convergence Theorems

The CQ algorithm in Equation (2) involves two projections P_C and P_Q and hence might be hard to be implemented in the case where one of them fails to have a closed-form expression. Now, we use the dual variable to propose a new relaxed CQ algorithm for solving the SFP (Equation (1)) where the closed convex subsets C and Q are level sets of convex functions. We just need projections onto half-spaces, thus the algorithm is implementable in this case.

Let

$$C = \{u \in H_1 : c(u) \leq 0\}, \ \ Q = \{v \in H_2 : q(v) \leq 0\}, \tag{18}$$

where $c : H_1 \to R$ and $q : H_2 \to R$ are convex and lower semi-continuous functions. We assume that c and q are subdifferentiable on H_1 and H_2, respectively. Namely, the subdifferentials,

$$\partial c(u) = \{z \in H_1 : c(x) \geq c(u) + \langle x-u, z\rangle, \ \ x \in H_1\} \neq \emptyset$$

for all $u \in C$ and

$$\partial q(v) = \{w \in H_2 : q(y) \geq q(v) + \langle y-v, w\rangle, \ \ y \in H_2\} \neq \emptyset$$

for all $v \in Q$. We also assume that ∂c and ∂q are bounded operators (i.e., bounded on bounded sets). In this paper, we solve the SFP (Equation (1)) under the above assumptions. We note that every convex function defined on a finite-dimensional Hilbert space is subdifferentible and its subdifferential operator is a bounded operator.

Set

$$C_n = \{u \in H_1 : c(u_n) + \langle \xi_n, u-u_n\rangle \leq 0\}, \tag{19}$$

where $\xi_n \in \partial c(u_n)$, and

$$Q_n = \{v \in H_2 : q(Au_n) + \langle \eta_n, v - Au_n\rangle \leq 0\}, \tag{20}$$

where $\eta_n \in \partial q(Au_n)$. Obviously, C_n and Q_n are half-spaces and it is easy to verify the $C \subseteq C_n$ and $Q \subseteq Q_n$ for every $n \geq 0$ from the subdifferentiable inequality.

Algorithm 1. *Let u_0, $e_0 \in H_1$ be arbitrary. For $n \geq 1$, let*

$$\begin{cases} t_n = u_n - \mu_n A^*(I - P_{Q_n})Au_n, \\ e_{n+1} = (I - P_{C_n})(t_n + (1-\lambda)e_n), \\ u_{n+1} = t_n - \lambda e_{n+1}, \end{cases} \tag{21}$$

where $0 < \lambda \leq 1$, $0 < \mu_n \leq \frac{2}{\|A\|^2}$.

Theorem 1. *Suppose $0 < \lambda \leq 1$ and $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\|A\|^2}$. Let $\{(e_n, u_n)\}$ be the sequence generated by Algorithm 1, then the sequence $\{u_n\}$ converges weakly to a point $u^* \in \Phi$ and the sequence $\{(e_n, u_n)\}$ weakly converges to the point $(0, u^*)$.*

Proof. First, we show that $\lim_{n\to\infty} \|u_n - u\|$ exists for any $u \in \Phi$. Taking $u \in \Phi$, we have $u \in C \subseteq C_n$ and $Au \in Q \subseteq Q_n$ for all $n \in N$. We know that $I - P_{C_n}$ and $I - P_{Q_n}$ are 1-ism for all $n \in N$. Thus, from Algorithm 1, we have

$$\begin{aligned} \|e_{n+1}\|^2 &= \|(I - P_{C_n})(t_n + (1-\lambda)e_n)\|^2 \\ &= \|(I - P_{C_n})(t_n + (1-\lambda)e_n) - (I - P_{C_n})u\|^2 \\ &\leq \langle e_{n+1}, t_n - u + (1-\lambda)e_n\rangle \\ &= \langle e_{n+1}, t_n - u\rangle + (1-\lambda)\langle e_n, e_{n+1}\rangle \end{aligned} \tag{22}$$

and

$$\begin{aligned} \|u_{n+1} - u\|^2 &= \|t_n - \lambda e_{n+1} - u\|^2 \\ &= \|t_n - u\|^2 - 2\lambda\langle t_n - u, e_{n+1}\rangle + \lambda^2\|e_{n+1}\|^2. \end{aligned} \tag{23}$$

Thus, from Equations (22) and (23), we have

$$\begin{aligned} &\|u_{n+1} - u\|^2 + \lambda\|e_{n+1}\|^2 \\ =&\|t_n - u\|^2 - 2\lambda\langle t_n - u, e_{n+1}\rangle + \lambda^2\|e_{n+1}\|^2 + \lambda\|e_{n+1}\|^2 \\ \leq&\|t_n - u\|^2 + 2\lambda(1-\lambda)\langle e_n, e_{n+1}\rangle - \lambda(1-\lambda)\|e_{n+1}\|^2 \\ =&\|t_n - u\|^2 + \lambda(1-\lambda)(2\langle e_n, e_{n+1}\rangle - \|e_{n+1}\|^2). \end{aligned} \tag{24}$$

Since

$$2\langle e_n, e_{n+1}\rangle - \|e_{n+1}\|^2 = \|e_n\|^2 - \|e_{n+1} - e_n\|^2,$$

we obtain

$$\begin{aligned} &\|u_{n+1} - u\|^2 + \lambda\|e_{n+1}\|^2 \\ \leq& \|t_n - u\|^2 + \lambda(1-\lambda)\|e_n\|^2 - \lambda(1-\lambda)\|e_{n+1} - e_n\|^2. \end{aligned} \tag{25}$$

It follows from

$$\begin{aligned} &\langle u_n - u, A^*(I - P_{Q_n})Au_n\rangle \\ =& \langle Au_n - Au, (I - P_{Q_n})Au_n\rangle \\ =& \langle Au_n - Au, (I - P_{Q_n})Au_n - (I - P_{Q_n})Au\rangle \\ \geq& \|(I - P_{Q_n})Au_n - (I - P_{Q_n})Au\|^2 \\ =& \|(I - P_{Q_n})Au_n\|^2 \end{aligned} \tag{26}$$

that

$$\begin{aligned} &\|t_n - u\|^2 \\ &= \|u_n - \mu_n A^*(I - P_{Q_n})Au_n - u\|^2 \\ &= \|u_n - u\|^2 - 2\mu_n\langle u_n - u, A^*(I - P_{Q_n})Au_n\rangle + \mu_n^2\|A\|^2\|(I - P_{Q_n})Au_n\|^2 \\ &\le \|u_n - u\|^2 - 2\mu_n\|(I - P_{Q_n})Au_n\|^2 + \mu_n^2\|A\|^2\|(I - P_{Q_n})Au_n\|^2 \\ &= \|u_n - u\|^2 - \mu_n(2 - \mu_n\|A\|^2)\|(I - P_{Q_n})Au_n\|^2. \end{aligned} \tag{27}$$

By Equations (25) and (27), we obtain

$$\begin{aligned} &\|u_{n+1} - u\|^2 + \lambda\|e_{n+1}\|^2 \\ &\le \|u_n - u\|^2 - \mu_n(2 - \mu_n\|A\|^2)\|(I - P_{Q_n})Au_n\|^2 \\ &\quad + \lambda(1-\lambda)\|e_n\|^2 - \lambda(1-\lambda)\|e_{n+1} - e_n\|^2 \\ &= \|u_n - u\|^2 + \lambda\|e_n\|^2 - \lambda^2\|e_n\|^2 - \lambda(1-\lambda)\|e_{n+1} - e_n\|^2 \\ &\quad - \mu_n(2 - \mu_n\|A\|^2)\|(I - P_{Q_n})Au_n\|^2. \end{aligned} \tag{28}$$

Let $s_n = \|u_n - u\|^2 + \lambda\|e_n\|^2$, then the sequence $\{s_n\}$ is lower bounded. By the assumptions on $\{\mu_n\}$ and λ, from Equation (28) we can get $s_{n+1} \le s_n$, which implies that the sequence $\{s_n\}$ is non-increasing and thus $\lim_{n\to\infty} s_n$ exists. Thus, it follows that $\{s_n\}$ is bounded and hence $\{u_n\}$ is bounded.

Moreover, from Equation (28), we also have

$$\lambda^2\|e_n\|^2 + \lambda(1-\lambda)\|e_{n+1} - e_n\|^2 + \mu_n(2 - \mu_n\|A\|^2)\|(I - P_{Q_n})Au_n\|^2 \le s_n - s_{n+1},$$

which implies that

$$\lim_{n\to\infty} \|(I - P_{Q_n})Au_n\| = 0 \tag{29}$$

and

$$\lim_{n\to\infty} \|e_n\| = 0. \tag{30}$$

Thus, $\lim_{n\to\infty} \|u_n - u\|^2 = \lim_{n\to\infty}(s_n - \lambda\|e_n\|^2) = \lim_{n\to\infty} s_n$ exists.

Next, we prove $\omega_\omega(u_n) \subseteq \Phi$. From Algorithm 1, we have

$$\|u_n - t_n\| = \mu_n\|A^*(I - P_{Q_n})Au_n\| \tag{31}$$

and

$$\|u_{n+1} - t_n\| = \|-\lambda e_{n+1}\|. \tag{32}$$

Combining Equations (29) and (30), we get

$$\lim_{n\to\infty} \|u_n - t_n\| = \lim_{n\to\infty} \|u_{n+1} - t_n\| = 0. \tag{33}$$

It follows from Algorithm 1 that

$$\begin{aligned} u_{n+1} = t_n - \lambda e_{n+1} &= t_n - \lambda(I - P_{C_n})(t_n + (1-\lambda)e_n) \\ &= t_n - \lambda\big(t_n + (1-\lambda)e_n - P_{C_n}(t_n + (1-\lambda)e_n)\big) \\ &= (1-\lambda)t_n - \lambda(1-\lambda)e_n + \lambda P_{C_n}(t_n + (1-\lambda)e_n), \end{aligned} \tag{34}$$

which implies that

$$P_{C_n}(t_n + (1-\lambda)e_n) = \frac{1}{\lambda}u_{n+1} - \frac{1-\lambda}{\lambda}t_n + (1-\lambda)e_n. \tag{35}$$

Let $z_{n+1} = \frac{1}{\lambda} u_{n+1} - \frac{1-\lambda}{\lambda} t_n + (1-\lambda)e_n$, then $z_{n+1} \in C_n$. Since ∂c is bounded on bounded sets, there exists a constant $\xi > 0$ such that $\|\xi_n\| \leq \xi$ for all $n \in N$. It follows that

$$\begin{aligned} c(u_n) \leq -\langle \xi_n, u_{n+1} - u_n \rangle &\leq -\langle \xi_n, \frac{1}{\lambda} u_{n+1} - \frac{1-\lambda}{\lambda} t_n + (1-\lambda)e_n - u_n \rangle \\ &= -\langle \xi_n, \frac{1}{\lambda} u_{n+1} - \frac{1}{\lambda} t_n + t_n + (1-\lambda)e_n - u_n \rangle \\ &\leq \xi(\|\frac{1}{\lambda}(u_{n+1} - t_n)\| + \|t_n - u_n\| + \|(1-\lambda)e_n\|). \end{aligned} \tag{36}$$

By Equations (30), (33) and (36), we have

$$\limsup_{n\to\infty} c(u_n) \leq 0. \tag{37}$$

Assume that $\widehat{u} \in \omega_w(u_n)$, i.e., there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup \widehat{u}$ as $j \to \infty$. By the weak lower semicontinuity of c and Equation (37), we have

$$c(\widehat{u}) \leq \liminf_{j\to\infty} c(u_{n_j}) \leq 0. \tag{38}$$

Therefore, $\widehat{u} \in C$.

Now, we show that $A\widehat{u} \in Q$. Since $\eta_n \in \partial q(Au_n)$, so we have $\|\eta_n\| \leq \eta$ for all $n \in N$. It follows from $P_{Q_n} Au_n \in Q_n$ that

$$q(Au_n) + \langle \eta_n, P_{Q_n} Au_n - Au_n \rangle \leq 0, \tag{39}$$

which implies that

$$q(Au_n) \leq \langle \eta_n, Au_n - P_{Q_n} Au_n \rangle \leq \eta \|Au_n - P_{Q_n} Au_n\|. \tag{40}$$

It follows from Equation (28), the weak lower semicontinuity of q and the fact that $Au_{n_j} \rightharpoonup A\widehat{u}$ that

$$q(A\widehat{u}) \leq \liminf_{j\to\infty} q(Au_{n_j}) \leq 0. \tag{41}$$

Namely, $A\widehat{u} \in Q$.

Thus, $\widehat{u} \in \Phi$, hence $\omega_w(u_n) \subseteq \Phi$. By Lemma 1, we have $u_n \rightharpoonup u^*$ and the sequence $\{(e_n, u_n)\}$ weakly converges to the point $(0, u^*)$, where $u^* \in \Phi$. This completes the proof. □

Remark 1. *When $\lambda = 1$, Algorithm 1 becomes the relaxed CQ algorithm in Equation (16) proposed by Xu [27] for solving the SFP where the closed convex subsets C and Q are level sets of convex functions. Thus, Theorem 1 extends the related results of Xu [27] for solving the SFP (Equation (1)).*

4. Strong Convergence Theorems

In this section, we modify the proposed Algorithm 1 to show that the algorithm has strong convergence. It is known that the viscosity approximation method is often used to approximate a fixed point of a nonexpansive mapping U in Hilbert spaces with the strong convergence, which it is defined as follows [37]:

$$u_{n+1} = \beta_n h(u_n) + (1-\beta_n)U(u_n)$$

for each $n \geq 1$, where $\{\beta_n\} \subseteq [0,1]$ and h is a contractive mapping. Now, we adapt the viscosity approximation method to get the strong convergence result for solving the SFP (Equation (1)) where the closed convex subsets C and Q are given in Equation (18).

Algorithm 2. *Let $h : H_1 \to H_1$ be a ρ-contraction mapping and C_n and Q_n given in Equations (19) and (20), respectively. Let u_0, $e_0 \in H_1$ be arbitrary. For $n \geq 0$, let*

$$\begin{cases} t_n = u_n - \mu_n A^*(I - P_{Q_n})Au_n, \\ \overline{e}_n = (I - P_{C_n})(t_n + (1-\lambda)e_n), \\ \overline{u}_n = t_n - \lambda \overline{e}_n, \\ e_{n+1} = \beta_n h(e_n) + (1-\beta_n)\overline{e}_n, \\ u_{n+1} = \beta_n h(u_n) + (1-\beta_n)\overline{u}_n, \end{cases}$$

where $0 < \lambda \leq 1$, $0 < \mu_n \leq \frac{2}{\|A\|^2}$ *and* $\{\beta_n\} \subset [0,1]$.

Theorem 2. *Assume that* $\{\beta_n\}$, $\{\mu_n\}$, λ *and* ρ *satisfy the following assumptions:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < \frac{2}{\|A\|^2}$; *and*
(iii) $0 \leq \rho < \frac{1}{\sqrt{2}}$ *and* $0 < \lambda < 1$.

Then, the sequence (e_n, u_n) *generated by Algorithm 2 strongly converges to* $(0, u^*)$, *where* $u^* \in \Phi$ *and* u^* *solves the following variational inequality problem:*

$$\langle (I - h)u^*, u - u^* \rangle \geq 0 \tag{42}$$

for any $u \in \Phi$.

Proof. Let $u^* \in \Phi$ be unique solution of the variational inequality problem (42). Then, $u^* \in C \subset C_n$ and $Au^* \in Q \subset Q_n$ for all $n \geq 0$. It follows from Equation (28) that

$$\begin{aligned} &\|\overline{u}_n - u^*\|^2 + \lambda\|\overline{e}_n\|^2 \\ \leq &\|u_n - u^*\|^2 + \lambda\|e_n\|^2 - \mu_n(2 - \mu_n\|A\|^2)\|(I - P_{Q_n})Au_n\|^2 - \lambda^2\|e_n\|^2 \\ &- \lambda(1-\lambda)\|\overline{e}_n - e_n\|^2. \end{aligned} \tag{43}$$

In particular, we have

$$\|\overline{u}_n - u^*\|^2 + \lambda\|\overline{e}_n\|^2 \leq \|u_n - u^*\|^2 + \lambda\|e_n\|^2. \tag{44}$$

From Algorithm 2, we get

$$\begin{aligned} &\|u_{n+1} - u^*\|^2 + \lambda\|e_{n+1}\|^2 \\ =&\|\beta_n h(u_n) + (1-\beta_n)\overline{u}_n - u^*\|^2 + \lambda\|\beta_n h(e_n) + (1-\beta_n)\overline{e}_n\|^2 \\ \leq&\beta_n\|h(u_n) - u^*\|^2 + (1-\beta_n)\|\overline{u}_n - u^*\|^2 + \lambda\big(\beta_n\|h(e_n)\|^2 + (1-\beta_n)\|\overline{e}_n\|^2\big) \\ \leq&2\beta_n\big(\|h(u_n) - h(u^*)\|^2 + \|h(u^*) - u^*\|^2\big) + (1-\beta_n)\|\overline{u}_n - u^*\|^2 \\ &+ \lambda[2\beta_n\big(\|h(e_n) - h(0)\|^2 + \|h(0)\|^2\big) + (1-\beta_n)\|\overline{e}_n\|^2] \\ \leq&2\beta_n\rho^2\|u_n - u^*\|^2 + 2\beta_n\|h(u^*) - u^*\|^2 + (1-\beta_n)\|\overline{u}_n - u^*\|^2 \\ &+ 2\lambda\beta_n\rho^2\|e_n\|^2 + 2\lambda\beta_n\|h(0)\|^2 + \lambda(1-\beta_n)\|\overline{e}_n\|^2 \\ =&(1-\beta_n)(\|\overline{u}_n - u^*\|^2 + \lambda\|\overline{e}_n\|^2) + 2\beta_n\rho^2(\|u_n - u^*\|^2 + \lambda\|e_n\|^2) \\ &+ 2\beta_n(\|h(u^*) - u^*\|^2 + \lambda\|h(0)\|^2) \\ \leq&(1-\beta_n)(\|u_n - u^*\|^2 + \lambda\|e_n\|^2) + 2\beta_n\rho^2(\|u_n - u^*\|^2 + \lambda\|e_n\|^2) \\ &+ 2\beta_n(\|h(u^*) - u^*\|^2 + \lambda\|h(0)\|^2) \\ =&[1 - \beta_n(1-2\rho^2)](\|u_n - u^*\|^2 + \lambda\|e_n\|^2) \\ &+ \beta_n(1-2\rho^2)\frac{2}{1-2\rho^2}(\|h(u^*) - u^*\|^2 + \lambda\|h(0)\|^2). \end{aligned} \tag{45}$$

Setting $s_n = \|u_n - u^*\|^2 + \lambda\|e_n\|^2$, we have

$$s_{n+1} \le [1-\beta_n(1-2\rho^2)]s_n + \beta_n(1-2\rho^2)\frac{2}{1-2\rho^2}(\|h(u^*)-u^*\|^2 + \lambda\|h(0)\|^2). \tag{46}$$

It follows from induction that

$$s_n \le \max\left\{s_0, \frac{2}{1-2\rho^2}(\|h(u^*)-u^*\|^2 + \lambda\|h(0)\|^2)\right\}$$

for each $n \ge 0$, which implies that $\{u_n\}$ and $\{e_n\}$ are bounded. In addition, $\{\overline{u}_n\}$, $\{\overline{e}_n\}$, $\{h(u_n)\}$ and $\{h(e_n)\}$ are bounded. It follows from Algorithm 2 that

$$\begin{aligned}
&\|u_{n+1}-u^*\|^2 + \lambda\|e_{n+1}\|^2\\
=&\|\beta_n h(u_n) + (1-\beta_n)\overline{u}_n - u^*\|^2 + \lambda\|\beta_n h(e_n) + (1-\beta_n)\overline{e}_n\|^2\\
=&\beta_n^2\|h(u_n)-u^*\|^2 + 2\beta_n(1-\beta_n)\langle h(u_n)-u^*, \overline{u}_n - u^*\rangle + (1-\beta_n)^2\|\overline{u}_n - u^*\|^2\\
&+\lambda\big(\beta_n^2\|h(e_n)\|^2 + 2\beta_n(1-\beta_n)\langle h(e_n), \overline{e}_n\rangle + (1-\beta_n)^2\|\overline{e}_n\|^2\big)\\
=&\beta_n^2\|h(u_n)-u^*\|^2 + (1-\beta_n)^2\|\overline{u}_n-u^*\|^2 + \lambda\beta_n^2\|h(e_n)\|^2 + \lambda(1-\beta_n)^2\|\overline{e}_n\|^2\\
&+2\beta_n(1-\beta_n)\big(\langle h(u_n)-h(u^*), \overline{u}_n-u^*\rangle + \langle h(u^*)-u^*, \overline{u}_n-u^*\rangle\big)\\
&+2\lambda\beta_n(1-\beta_n)(\langle h(e_n)-h(0), \overline{e}_n\rangle + \langle h(0), \overline{e}_n\rangle)\\
\le&\beta_n^2\|h(u_n)-u^*\|^2 + (1-\beta_n)^2\|\overline{u}_n-u^*\|^2 + \lambda\beta_n^2\|h(e_n)\|^2 + \lambda(1-\beta_n)^2\|\overline{e}_n\|^2\\
&+\beta_n(1-\beta_n)(\|h(u_n)-h(u^*)\|^2 + \|\overline{u}_n-u^*\|^2)\\
&+2\beta_n(1-\beta_n)\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle\\
&+\lambda\beta_n(1-\beta_n)(\|h(e_n)-h(0)\|^2 + \|\overline{e}_n\|^2) + 2\lambda\beta_n(1-\beta_n)\langle h(0), \overline{e}_n\rangle\\
\le&\beta_n^2\|h(u_n)-u^*\|^2 + (1-\beta_n)\|\overline{u}_n-u^*\|^2 + \lambda\beta_n^2\|h(e_n)\|^2 + \lambda(1-\beta_n)\|\overline{e}_n\|^2\\
&+\beta_n(1-\beta_n)\rho^2\|u_n-u^*\|^2 + 2\beta_n(1-\beta_n)\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle\\
&+\lambda\beta_n(1-\beta_n)\rho^2\|e_n\|^2 + 2\lambda\beta_n(1-\beta_n)\langle h(0), \overline{e}_n\rangle.
\end{aligned} \tag{47}$$

Thus, by Equations (44) and (47), we have

$$\begin{aligned}
s_{n+1} \le&[1-\beta_n(1-(1-\beta_n)\rho^2)]s_n + \beta_n[\beta_n(\|h(u_n)-u^*\|^2 + \lambda\|h(e_n)\|^2)\\
&+2(1-\beta_n)\big(\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle + \lambda\langle h(0), \overline{e}_n\rangle\big)]\\
=&(1-\lambda_n)s_n + \lambda_n\delta_n,
\end{aligned} \tag{48}$$

where

$$\lambda_n = \beta_n(1-(1-\beta_n)\rho^2)$$

and

$$\begin{aligned}
\delta_n =&\frac{2(1-\beta_n)(\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle + \lambda\langle h(0), \overline{e}_n\rangle)}{1-(1-\beta_n)\rho^2}\\
&+\frac{\beta_n(\|h(u_n)-u^*\|^2 + \lambda\|h(e_n)\|^2)}{1-(1-\beta_n)\rho^2}.
\end{aligned} \tag{49}$$

On the other hand, from Equations (43) and (47), we have

$$
\begin{aligned}
& s_{n+1} \\
\leq & [1-\beta_n(1-(1-\beta_n)\rho^2)]s_n + \beta_n^2(\|h(u_n)-u^*\|^2+\lambda\|h(e_n)\|^2) \\
& + 2\beta_n(1-\beta_n)(\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle + \lambda\langle h(0), \overline{e}_n\rangle) \\
& - (1-\beta_n)\big[\mu_n(2-\mu_n\|A\|^2)\|(I-P_{Q_n})Au_n\|^2 + \lambda^2\|e_n\|^2 + \lambda(1-\lambda)\|\overline{e}_n-e_n\|^2\big] \\
\leq & s_n + \beta_n^2(\|h(u_n)-u^*\|^2+\lambda\|h(e_n)\|^2) \\
& + 2\beta_n(1-\beta_n)(\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle + \lambda\langle h(0), \overline{e}_n\rangle) \\
& - (1-\beta_n)\big[\mu_n(2-\mu_n\|A\|^2)\|(I-P_{Q_n})Au_n\|^2 + \lambda^2\|e_n\|^2 + \lambda(1-\lambda)\|\overline{e}_n-e_n\|^2\big].
\end{aligned} \tag{50}
$$

Now, by setting

$$a_n = \beta_n^2(\|h(u_n)-u^*\|^2+\lambda\|h(e_n)\|^2) + 2\beta_n(1-\beta_n)(\langle h(u^*)-u^*, \overline{u}_n-u^*\rangle + \lambda\langle h(0), \overline{e}_n\rangle)$$

and

$$\eta_n = (1-\beta_n)\big[\mu_n(2-\mu_n\|A\|^2)\|(I-P_{Q_n})Au_n\|^2 + \lambda^2\|e_n\|^2 + \lambda(1-\lambda)\|\overline{e}_n-e_n\|^2\big],$$

Equation (50) can be rewritten in the following form:

$$s_{n+1} \leq s_n - \eta_n + a_n \tag{51}$$

for each $n \geq 0$. By the assumptions on $\{\beta_n\}$ and ρ, we have

$$\sum_{k=0}^{\infty} \lambda_n = \infty, \quad \lim_{n\to\infty} a_n = 0.$$

To use Lemma 2, it suffices to verify that, for any subsequence$\{n_l\} \subset \{n\}$, $\lim_{l\to\infty} \eta_{n_l} = 0$ implies

$$\limsup_{l\to\infty} \delta_{n_l} \leq 0. \tag{52}$$

Since $\lim_{l\to\infty} \eta_{n_l} = 0$, from the assumptions on λ and $\{\mu_n\}$, we obtain

$$\lim_{l\to\infty} \|(I-P_{Q_{n_l}})Au_{n_l}\| = \lim_{l\to\infty} \|e_{n_l}\| = \lim_{l\to\infty} \|\overline{e}_{n_l}\| = 0. \tag{53}$$

From

$$
\begin{aligned}
\|u_{n_l} - \overline{u}_{n_l}\| &= \|u_{n_l} - u_{n_l} + \mu_{n_l}A^*(I-P_{Q_{n_l}})Au_{n_l} + \lambda\overline{e}_{n_l}\| \\
&= \|\mu_{n_l}A^*(I-P_{Q_{n_l}})Au_{n_l} + \lambda\overline{e}_{n_l}\| \\
&\leq \mu_{n_l}\|A\|\|(I-P_{Q_{n_l}})Au_{n_l}\| + \lambda\|\overline{e}_{n_l}\|,
\end{aligned}
$$

we obtain

$$\lim_{l\to\infty} \|u_{n_l} - \overline{u}_{n_l}\| = 0. \tag{54}$$

In a similar way to the proof of Theorem 1, we can get $\omega_w(u_{n_l}) \subseteq \Phi$. Since

$$\lim_{l\to\infty}(1-(1-\beta_{n_l})\rho^2) = 1-\rho^2$$

and

$$\lim_{l\to\infty} \beta_{n_l}(\|h(u_{n_l})-u^*\|^2 + \lambda\|h(e_{n_l})\|^2) = 0,$$

to get Equation (52), we only need to verify

$$\limsup_{l\to\infty}(\langle h(u^*)-u^*, \overline{u}_{n_l}-u^*\rangle + \lambda\langle h(0), \overline{e}_{n_l}\rangle) \leq 0.$$

From Equation (54), we can take subsequence $\{(e_{n_{l_j}}, u_{n_{l_j}})\}$ of $\{(e_{n_l}, u_{n_l})\}$ such that $u_{n_{l_j}} \rightharpoonup \widetilde{u}$ as $j \to \infty$ and

$$\begin{aligned} &\limsup_{l\to\infty}(\langle h(u^*) - u^*, \overline{u}_{n_l} - u^*\rangle + \lambda\langle h(0), \overline{e}_{n_l}\rangle) \\ &= \lim_{j\to\infty}(\langle h(u^*) - u^*, \overline{u}_{n_{l_j}} - u^*\rangle + \lambda\langle h(0), \overline{e}_{n_{l_j}}\rangle) \\ &= \lim_{j\to\infty}\langle h(u^*) - u^*, u_{n_{l_j}} - u^*\rangle \\ &= \langle h(u^*) - u^*, \widetilde{u} - u^*\rangle. \end{aligned} \tag{55}$$

Since $\omega_w(u_{n_l}) \subset \Phi$ and u^* is a solution of the variational inequality problem in Equation (42), it follows from Equation (55) that

$$\limsup_{l\to\infty}(\langle h(u^*) - u^*, \overline{u}_{n_l} - u^*\rangle + \lambda\langle h(0), \overline{e}_{n_l}\rangle) \leq 0.$$

Thus, it follows from Lemma 2 that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(\|u_n - u^*\|^2 + \lambda\|e_n\|^2) = 0,$$

which implies that $u_n \to u^*$, $e_n \to 0$ and $(e_n, u_n) \to (0, u^*)$, where $u^* \in \Phi$ and u^* solves the variational inequality problem in Equation (42). This completes the proof. □

5. Numerical Results

In this section, we provide some numerical experiments and show the performance of the proposed modified relaxed CQ iterative Algorithm 1 for solving the SFP (Equation (1)) where the closed convex subsets C and Q are level sets of convex functions. All codes were written in MATLAB and were performed on a personal Lenovo computer with Pentium(R) Dual-Core CPU @ 2.4GHz and RAM 2.00 GB.

Example 1. *We consider the SFP (Equation (1)) as follows: $H_1 = H_2 = R^2$, the matrix $A = (a_{i,j})_{N\times N}$ and $a_{i,j} \in (0,1)$ are generated randomly, the nonempty closed convex set $C = \{u \in R^2 | \; c(u) \leq 0\}$ and $Q = \{v \in R^2 | \; q(v) \leq 0\}$, where*

$$c(u) = -u_1 + u_2^2$$

and

$$q(v) = v_1 + v_2^2$$

for all $u = (u_1, u_2)^T \in R^2$ and $v = (v_1, v_2)^T \in R^2$.

Now, we compare the proposed modified relaxed CQ Algorithm 1 with the relaxed CQ algorithm in Equation (16) proposed by Xu [27] to solve Example 1. In the implementation, we took $\mu_n = \frac{1.55}{\|A\|^2}$ and $p(u_n) < \varepsilon = 10^{-4}$ as the stopping criterion, where

$$p(u_n) = \|u_n - P_{C_n}u_n\| + \|Au_n - P_{Q_n}Au_n\|.$$

We took different u_0 and e_0 as initial points. In Case 1, we took $u_0 = (5, -4)^T$ and $e_0 = (0, -1)^T$. In Case 2, we took $u_0 = (-10, 4)^T$ and $e_0 = (-2, 10)^T$.

We tried different values of λ for solving this example. When the parameter $\lambda = 1$, Algorithm 1 becomes the relaxed CQ algorithm in Equation (34). We report the numerical results in Tables 1 and 2. In the tables, "Iter." denotes the terminating iterative numbers, and $C(u)$ and $Q(Au)$ denote the value of $c(u)$ and $q(Au)$ at the terminal point, respectively.

Table 1. Numerical results for solving Example 1 with different λ.

$u_0 = (5, -4)^T, e_0 = (0, -1)^T$			
λ	**Iter.**	$C(u)$	$Q(Au)$
1	7	1.4363×10^{-7}	-2.7797×10^{-4}
0.9	5	-0.1930	-0.0082
0.8	6	-0.1940	-0.1720
0.7	6	-0.3159	-0.2697
0.6	6	-0.3916	-0.3400
0.5	6	-0.4044	-0.3793
0.4	6	-0.4158	-0.3316
0.3	8	-0.1115	-0.7977
0.2	7	-0.4482	-0.1027
0.1	10	-0.3799	-0.3628

Table 2. Numerical results for solving Example 1 with different λ.

$u_0 = (-10, 4)^T, e_0 = (-2, 10)^T$			
λ	**Iter.**	$C(u)$	$Q(Au)$
1	9	9.4358×10^{-7}	-0.7504
0.9	7	-0.1351	-0.7958
0.8	4	-0.2007	-0.8981
0.7	8	-0.1166	-0.8570
0.6	6	-0.3025	-0.8072
0.5	7	-0.1696	-0.4115
0.4	9	-0.0020	-1.0547
0.3	11	-0.0030	-1.0185
0.2	8	-0.4556	-0.2518
0.1	11	-0.5349	-0.4600

Example 2. *Let $H_1 = H_2 = R^3$, $A = (a_{i,j})_{N\times N}$ and $a_{i,j} \in (0,1)$ are generated randomly, the nonempty closed convex set $C = \{u \in R^3 | \, c(u) \leq 0\}$ and $Q = \{v \in R^3 | \, q(v) \leq 0\}$, where*

$$c(u) = -u_1 + u_2^2 + u_3^2$$

and

$$q(v) = v_1 + v_2^2 + v_3^2$$

for all $u = (u_1, u_2, u_3)^T \in R^3$ and $v = (v_1, v_2, v_3)^T \in R^3$.

Similar to Example 1, we compared the proposed modified relaxed CQ Algorithm 1 with the relaxed CQ algorithm in Equation (34) proposed by Xu [27] to solve this example. We took $\mu_n = \frac{1.55}{\|A\|^2}$ and the same stopping criterion as in Example 1. We took different u_0 and e_0 as initial points. The numerical results are given in Tables 3 and 4.

Table 3. Numerical results for solving Example 2 with different λ.

$u_0 = (5, -4, 3)^T, e_0 = (0, 1, -1)^T$			
λ	**Iter.**	$C(u)$	$Q(Au)$
1	13	2.7532×10^{-9}	1.2270×10^{-4}
0.9	9	-0.0094	-6.6177×10^{-5}
0.8	7	-0.0954	-0.0055
0.7	7	-0.0228	-0.0247
0.6	8	-0.4571	-0.0030
0.5	8	-0.6276	-0.0322
0.4	9	-0.8269	-0.0114
0.3	10	-0.9781	-0.0132
0.2	11	-1.1005	-0.0017
0.1	16	-1.1647	-0.0139

Table 4. Numerical results for solving Example 2 with different λ.

$u_0 = (-2, 1, 15)^T, e_0 = (3, -2, 1)^T$			
λ	**Iter.**	$C(u)$	$Q(Au)$
1	9	6.8996×10^{-7}	-1.0408
0.9	8	-0.0250	-1.0249
0.8	7	-0.0369	-0.9827
0.7	7	-0.3059	-0.8919
0.6	9	-0.1699	-0.9188
0.5	10	-0.2647	-0.8608
0.4	9	-0.1480	-0.9394
0.3	10	-0.0890	-0.9616
0.2	10	-0.3267	-0.8054
0.1	12	-0.2652	-0.8550

We can see in Tables 1–4 that Algorithm 1 was efficient and behaved better than the relaxed CQ algorithm in Equation (34) when choosing a suitable parameter λ for solving Examples 1 and 2.

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