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# Iterative Algorithms for Pseudomonotone Variational Inequalities and Fixed Point Problems of Pseudocontractive Operators

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**Abstract:** In this paper, we are interested in the pseudomonotone variational inequalities and fixed point problem of pseudocontractive operators in Hilbert spaces. An iterative algorithm has been constructed for finding a common solution of the pseudomonotone variational inequalities and fixed point of pseudocontractive operators. Strong convergence analysis of the proposed procedure is given. Several related corollaries are included.

**Keywords:** pseudomonotone variational inequality; fixed point; pseudocontractive operators; strong convergence

**MSC:** 47H10; 47J25; 47J40.

## 1. Introduction

Let $H$ be a real Hilbert space endowed with inner product and induced norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $\emptyset \neq C \subset H$ be a closed and convex set.

In this article, our study is related to a classical variational inequality (VI) of seeking an element $\tilde{u} \in C$ verifying

$$\langle f(\tilde{u}), u - \tilde{u} \rangle \geq 0, \ \forall u \in C, \tag{1}$$

where $f : H \to H$ is a given operator, under the following assumptions:

(i) $VI(C, f)$, the solution set of (1), is nonempty;
(ii) $f$ is pseudomonotone on $H$, i.e.,

$$\langle f(\tilde{u}), u - \tilde{u} \rangle \geq 0 \Rightarrow \langle f(u), u - \tilde{u} \rangle \geq 0, \ \forall u, \tilde{u} \in H; \tag{2}$$

(iii) $f$ is $\kappa$-Lipschitz continuous on $H$ (for some $\kappa > 0$), i.e.,

$$\| f(x) - f(y) \| \leq \kappa \| x - y \|, \ \forall x, y \in H.$$

Numerical iterative methods have been presented, developed and adopted widely as algorithmic solutions to the concept of variational inequalities. This notion, that mainly involves some important operators, plays a key role in applied mathematics, such as obstacle problems, optimization problems, complementarity problems as a unified framework for the study of a large number of significant real-word problems arising in physics, engineering, economics and so on. For more information, the reader can refer to [1–12].

For solving VI (1) in which the involved operator $f$ may be monotone, several iterative algorithms have been introduced and studied, see, e.g., [13–18]. Among them, the more popular iterative technique is the projected gradient rule ([19–23]): for the fixed previous iteration $x_{n-1}$, calculate the current iteration $x_n$ via the following manner

$$x_n = P_C[x_{n-1} - \tau f(x_{n-1})],\ n \geq 1, \tag{3}$$

where $P_C$ means the projection operator from $H$ onto $C$ and the positive constant $\tau$ is the step-size.

The projected gradient rule (3) is an effective technique for solving VI (1). However, the involved operator $f$ should be strongly monotone or inverse strongly monotone. In order to overcome this flaw, in [21], Korpelevich put forward an extragradient technique: for the fixed previous iteration $x_{n-1}$, calculate the current iteration $x_n$ via the following manner

$$\begin{cases} y_{n-1} = P_C[x_{n-1} - \tau f(x_{n-1})], \\ x_n = P_C[x_{n-1} - \tau f(y_{n-1})],\ n \geq 1, \end{cases} \tag{4}$$

where the step-size $\tau \in (0, 1/\kappa)$.

Korpelevich's algorithm (4) provides an important idea for solving monotone variational inequality. Please refer to the references [24–27] for several important extended version of Korpelevich's algorithm.

The another motivation of this paper is to study the following fixed point equation:

$$\text{find } x \in C \ \text{ such that } \ x = Tx, \tag{5}$$

where $T : C \to C$ is a pseudocontractive operator.

Now, it is well-known that fixed point algorithm of successive approximation is one of the most important techniques in numerical mathematics ([28–40]). Focusing on the research with pseudocontractive operators originated in their relations with the important class of monotone operators. Algorithmic approximation theories and experiments of pseudocontractive operators have been studied extensively in the literature, see, for example, [41–47].

Motivated and inspired by the work in this field, the purpose of this paper is to investigate the problem of pseudomonotone variational inequality (1) and fixed point of pseudocontractive operators. We construct an iterative algorithm for seeking a common solution of the pseudomonotone variational inequalities and fixed point of pseudocontractive operators. Strong convergence analysis of the proposed procedure is given. Several related corollaries are included.

## 2. Preliminaries

Let $H$ be a real Hilbert space. Let $C \subset H$ be a nonempty, closed and convex set. Recall that an operator $f : C \to C$ is said to be monotone if

$$\langle f(x) - f(y), x - y \rangle \geq 0,\ \forall x, y \in C.$$

An operator $T : C \to C$ is said to be pseudocontractive if

$$\|Tu - Tu^{\ddagger}\|^2 \leq \|u - u^{\ddagger}\|^2 + \|(I-T)u - (I-T)u^{\ddagger}\|^2$$

for all $u, u^{\ddagger} \in C$.

Recall that an operator $f : C \to C$ is called weakly sequentially continuous, if for any given sequence $\{x_n\} \subset C$ satisfying $x_n \rightharpoonup \tilde{x}$, we conclude that $f(x_n) \rightharpoonup f(\tilde{x})$.

Recall that the metric projection $P_C : H \to C$ is an orthographic projection from $H$ onto $C$, which possesses the following characteristic: for given $x \in H$,

$$\langle x - P_C[x], y - P_C[x] \rangle \leq 0, \forall y \in C. \tag{6}$$

The following symbols will be used in the sequel.

- $u_n \rightharpoonup z^{\dagger}$ denotes the weak convergence of $u_n$ to $z^{\dagger}$.
- $u_n \to z^{\dagger}$ stands for the strong convergence of $u_n$ to $z^{\dagger}$.
- $Fix(T)$ means the set of fixed points of $T$.
- $\omega_w(u_n) = \{u^{\dagger} : \exists \{u_{n_i}\} \subset \{u_n\}$ such that $u_{n_i} \rightharpoonup u^{\dagger} (i \to \infty)\}$.

**Lemma 1** ([1]). *Let $H$ be a real Hilbert space. Then, we have*

$$\|\delta u + (1-\delta)u^{\dagger}\|^2 = \delta\|u\|^2 + (1-\delta)\|u^{\dagger}\|^2 - \delta(1-\delta)\|u - u^{\dagger}\|^2,$$

*$\forall u, u^{\dagger} \in H$ and $\forall t \in [0,1]$.*

**Lemma 2** ([45]). *Let $C$ a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be an $L$-Lipschitz pseudocontractive operator. Let $0 < \eta < \frac{1}{\sqrt{1+L^2}+1}$. Then,*

$$\|u^{\dagger} - T((1-\eta)\tilde{u} + \eta T\tilde{u})\|^2 \leq \|\tilde{u} - u^{\dagger}\|^2 + (1-\eta)\|\tilde{u} - T((1-\eta)\tilde{u} + \eta T\tilde{u})\|^2,$$

*for all $\tilde{u} \in C$ and $u^{\dagger} \in Fix(T)$.*

**Lemma 3** ([18]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $f : H \to H$ be a continuous and pseudomonotone operator. Then $x^{\dagger} \in VI(C, f)$ iff $x^{\dagger}$ solves the following dual variational inequality*

$$\langle f(u^{\dagger}), u^{\dagger} - x^{\dagger} \rangle \geq 0, \ \forall u^{\dagger} \in H.$$

**Lemma 4** ([47]). *Let $H$ be a real Hilbert space, $C$ a nonempty closed convex subset of $H$. Let $T : C \to C$ be a continuous pseudocontractive operator. Then*

*(i) $Fix(T)$ is a closed convex subset of $C$;*
*(ii) $T$ is demi-closed, i.e., $u_n \rightharpoonup \tilde{u}$ and $T(u_n) \to u^{\dagger}$ imply that $T(\tilde{u}) = u^{\dagger}$.*

**Lemma 5** ([15]). *Let $\{\mu_n\} \subset (0, \infty)$, $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\}$ be three real number sequences. If $\mu_{n+1} \leq (1-\gamma_n)\mu_n + \delta_n$ for all $n \geq 0$ with $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$, then $\lim_{n\to\infty} \mu_n = 0$.*

## 3. Main Results

Let $\emptyset \neq C$ be a convex and closed subset of a real Hilbert space $H$. Let the operator $f$ be pseudomonotone on $H$, weakly sequentially continuous and Lipschitz continuous on $C$ with Lipschitz constant $\kappa > 0$. Let $T : C \to C$ be an $L$-Lipschitz pseudocontractive operator with $L \geq 1$.

Next, we first present the following iterative algorithm for solving pseudomonotone variational inequality and fixed point problem of pseudocontractive operator $T$. In what follows, assume that $\Lambda := VI(C, f) \cap Fix(T) \neq \emptyset$.

**Remark 1.** *By virtue of (6), we know that $u^\dagger \in VI(C,f) \Leftrightarrow u^\dagger = P_C[u^\dagger - \tau f(u^\dagger)]$ for all $\tau > 0$. Thus, if at some iterative step $x_n = P_C[x_n - f(x_n)]$, then $x_n$ is a solution of variational inequality (1) and hence $\omega_w(x_n) \subset VI(C,f)$.*

**Remark 2.** *For given $x_n$, we can find $m(x_n)$ such that (14) holds. In fact, we can choose $m(x_n)$ such that $\gamma^{m(x_n)} \leq \frac{\theta}{\mu\kappa}$ due to the Lipschitz continuity of $f$. So, (14) is well-defined. At the same time, there exists a positive $\varsigma > 0$ such that $\gamma^{m(x_n)} \geq \varsigma > 0$ for all $x_n$. As a matter of fact, if $m(x_n) = 0$, then $\gamma^{m(x_n)} = \varsigma = 1$. If $m(x_n) > 0$, then we have $\frac{\kappa\mu\gamma^{m(x_n)}}{\gamma} > \theta$, which implies that $0 < \frac{\gamma\theta}{\mu\kappa} < \gamma^{m(x_n)} < 1$ for all $n$.*

**Proposition 1.** *If $x_n \neq P_C[x_n - f(x_n)]$, then $x_n - y_n + \mu\gamma^{m(x_n)} f(y_n) \neq 0$.*

**Proof.** Let $x^* = P_\Lambda(u)$. Owing to $x_n \in C$ and $y_n \in C$, we have

$$\langle f(x^*), x_n - x^*\rangle \geq 0, \tag{7}$$

and

$$\langle f(x^*), y_n - x^*\rangle \geq 0. \tag{8}$$

Applying the pseudomonotonicity (2) of $f$ to (7) and (8), we obtain

$$\langle f(x_n), x_n - x^*\rangle \geq 0, \tag{9}$$

and

$$\langle f(y_n), y_n - x^*\rangle \geq 0. \tag{10}$$

Since $y_n = P_C[x_n - \mu\gamma^{m(x_n)} f(x_n)]$, using the characteristic (6) of projection $P_C$, we have

$$\langle x_n - \mu\gamma^{m(x_n)} f(x_n) - y_n, y_n - x^*\rangle \geq 0. \tag{11}$$

Hence,

$$\begin{aligned}
\langle x_n - y_n + \mu\gamma^{m(x_n)} f(y_n), x_n - x^*\rangle &= \langle x_n - y_n - \mu\gamma^{m(x_n)} f(x_n), x_n - x^*\rangle + \mu\gamma^{m(x_n)}\langle f(x_n), x_n - x^*\rangle \\
&\quad + \mu\gamma^{m(x_n)}\langle f(y_n), x_n - y_n\rangle + \mu\gamma^{m(x_n)}\langle f(y_n), y_n - x^*\rangle \\
(\text{by } (9) \text{ and } (10)) \quad &\geq \langle x_n - y_n - \mu\gamma^{m(x_n)} f(x_n), x_n - x^*\rangle + \mu\gamma^{m(x_n)}\langle f(y_n), x_n - y_n\rangle \\
&= \langle x_n - y_n - \mu\gamma^{m(x_n)}(f(x_n) - f(y_n)), x_n - y_n\rangle \\
&\quad + \langle x_n - y_n - \mu\gamma^{m(x_n)} f(x_n), y_n - x^*\rangle \\
(\text{by } (11)) \quad &\geq \langle x_n - y_n - \mu\gamma^{m(x_n)}(f(x_n) - f(y_n)), x_n - y_n\rangle \\
&= \|x_n - y_n\|^2 - \mu\gamma^{m(x_n)}\langle f(x_n) - f(y_n), x_n - y_n\rangle \\
&\geq \|x_n - y_n\|^2 - \mu\gamma^{m(x_n)}\|f(x_n) - f(y_n)\|\|x_n - y_n\| \\
&\geq (1-\theta)\|x_n - y_n\|^2 \ (\text{by } (14)).
\end{aligned} \tag{12}$$

Since $P_C[x_n - f(x_n)] \neq x_n$, it follows that $P_C[x_n - \gamma f(x_n)] \neq x_n$ for all $\gamma > 0$. Thus, $y_n \neq x_n$. According to (12), we deduce $x_n - y_n + \mu\gamma^{m(x_n)} f(y_n) \neq 0$. □

**Remark 3.** *In case 1, we have $f(x_n) \neq 0$ (by Remark 1) and $x_n - y_n + \mu\gamma^{m(x_n)} f(y_n) \neq 0$ (by Remark 2) for all $n \geq 0$. According to Proposition 1, the sequence $\{u_n\}$ is well-defined and hence the sequence $\{x_n\}$ is well-defined.*

Now, in this position, we give the convergence analysis of the iterative sequence $\{x_n\}$ generated by Algorithm 1.

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**Algorithm 1:** Iterative procedures for VI and FP.

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Let $u \in C$ be a fixed point. Let $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ be three real number sequences in $(0,1)$. Let $\gamma \in (0,1)$, $\mu \in (0,1)$, $\theta \in (0,1)$ and $\tau \in (0,2)$ be four constants.

Step 1. Let $x_0 \in C$ be an initial value. Set $n = 0$.

Step 2. Assume that the sequence $\{x_n\}$ has been constructed and then calculate $P_C[x_n - f(x_n)]$.

Step 3. Case 1. If $P_C[x_n - f(x_n)] \neq x_n$, then calculate the sequence $\{y_n\}$ by the following manner

$$y_n = P_C[x_n - \mu\gamma^{m(x_n)} f(x_n)], \tag{13}$$

where $m(x_n) = \min\{0, 1, 2, 3, \cdots\}$ and satisfies

$$\mu\gamma^{m(x_n)}\|f(x_n) - f(y_n)\| \leq \theta\|x_n - y_n\|, \tag{14}$$

and consequently, calculate the sequences $\{u_n\}$, $\{z_n\}$ and $\{x_{n+1}\}$ by the following rule

$$\begin{cases} u_n = P_C\Big[x_n - \tau(1-\theta)\|x_n - y_n\|^2 \dfrac{x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2}\Big], \\ z_n = (1-\sigma_n)u_n + \sigma_n T[(1-\delta_n)u_n + \delta_n T u_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n. \end{cases} \tag{15}$$

Case 2. If $P_C[x_n - f(x_n)] = x_n$, then calculate the sequence $\{x_{n+1}\}$ via the following form

$$\begin{cases} z_n = (1-\sigma_n)x_n + \sigma_n T[(1-\delta_n)x_n + \delta_n T x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n. \end{cases}$$

Step 4. Set $n := n + 1$ and return to Step 2.

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**Theorem 1.** *Suppose that the iterative parameters $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ satisfy the following assumptions:*

*(C1):* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
*(C2):* $0 < \underline{\sigma} < \sigma_n < \overline{\sigma} < \delta_n < \overline{\delta} < \frac{1}{\sqrt{1+L^2}+1}, \forall n \geq 0.$

*Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $P_\Lambda(u)$.*

**Proof.** Step 1. the sequence $\{x_n\}$ is bounded. First, we consider Case 1. In this case, from (15) and (12), we have

$$\begin{aligned} \|u_n - x^*\|^2 &\leq \Big\|x_n - x^* - \tau(1-\theta)\|x_n - y_n\|^2 \frac{x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2}\Big\|^2 \\ &= \|x_n - x^*\|^2 + \frac{\tau^2(1-\theta)^2\|x_n - y_n\|^4}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2} \\ &\quad - \frac{2\tau(1-\theta)\|x_n - y_n\|^2}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2}\langle x_n - y_n + \mu\gamma^{m(x_n)} f(y_n), x_n - x^*\rangle \\ &\leq \|x_n - x^*\|^2 - \frac{(2-\tau)\tau(1-\theta)^2\|x_n - y_n\|^4}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2} \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{16}$$

In the light of (15) and Lemmas 1 and 2, we obtain

$$\begin{aligned}\|z_n - x^*\|^2 &= \|(1-\sigma_n)(u_n - x^*) + \sigma_n(T[(1-\delta_n)u_n + \delta_n Tu_n] - x^*)\|^2 \\ &= (1-\sigma_n)\|u_n - x^*\|^2 - \sigma_n(1-\sigma_n)\|T[(1-\delta_n)u_n + \delta_n Tu_n] - u_n\|^2 \\ &\quad + \sigma_n\|T[(1-\delta_n)u_n + \delta_n Tu_n] - x^*\|^2 \\ &\le (1-\sigma_n)\|u_n - x^*\|^2 - \sigma_n(1-\sigma_n)\|T[(1-\delta_n)u_n + \delta_n Tu_n] - u_n\|^2 \\ &\quad + \sigma_n(\|u_n - x^*\|^2 + (1-\delta_n)\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2) \\ &= \|u_n - x^*\|^2 - \sigma_n(\delta_n - \sigma_n)\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2 \\ &\le \|u_n - x^*\|^2.\end{aligned} \tag{17}$$

By (15), (16) and (17), we get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|\alpha_n(u - x^*) + (1-\alpha_n)(z_n - x^*)\| \\ &\le (1-\alpha_n)\|z_n - x^*\| + \alpha_n\|u - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|u - x^*\|.\end{aligned}$$

By induction, we can deduce that $\|x_{n+1} - x^*\| \le \max\{\|u - x^*\|, \|x_0 - x^*\|\}$. Hence, the sequence $\{x_n\}$ is bounded. It is easy to check that the sequence $\{x_n\}$ is also bounded in Case 2.

Step 2. $\omega_w(x_n) \subset \Lambda$. We firstly discuss Case 1. On account of (15), we achieve

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + (1-\alpha_n)(z_n - x^*)\|^2 \\ &\le (1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle.\end{aligned} \tag{18}$$

By virtue of (16), (17) and (18), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le (1-\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n)\sigma_n(\delta_n - \sigma_n)\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2 \\ &\quad - (1-\alpha_n)\frac{(2-\tau)\tau(1-\theta)^2\|x_n - y_n\|^4}{\|x_n - y_n + \mu\gamma^{m(x_n)}f(y_n)\|^2} + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\Big[-(1-\alpha_n)\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2}{\alpha_n} \\ &\quad - (1-\alpha_n)\frac{(2-\tau)\tau(1-\theta)^2\|x_n - y_n\|^4}{\alpha_n\|x_n - y_n + \mu\gamma^{m(x_n)}f(y_n)\|^2} + 2\langle u - x^*, x_{n+1} - x^*\rangle\Big].\end{aligned} \tag{19}$$

Write $s_n = \|x_n - x^*\|^2$ and

$$\begin{aligned}t_n = &-(1-\alpha_n)\frac{(2-\tau)\tau(1-\theta)^2\|x_n - y_n\|^4}{\alpha_n\|x_n - y_n + \mu\gamma^{m(x_n)}f(y_n)\|^2} + 2\langle u - x^*, x_{n+1} - x^*\rangle \\ &- (1-\alpha_n)\sigma_n(\delta_n - \sigma_n)\frac{\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|^2}{\alpha_n},\end{aligned} \tag{20}$$

for all $n \ge 0$.

We can adapt (19) as

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n t_n \tag{21}$$

for all $n \ge 0$.

Now, we show that $\limsup_{n\to\infty} t_n$ is finite. First, thanks to (20), we deduce that $t_n \le 2\langle u - x^*, x_{n+1} - x^*\rangle \le 2\|u - x^*\|\|x_{n+1} - x^*\|$. This together with the boundedness of $\{x_n\}$ implies that $\limsup_{n\to\infty} t_n$ has a upper bound.

Next, we show that $\limsup_{n\to\infty} t_n$ has a lower bound. As a matter of fact, we can prove that $\limsup_{n\to\infty} t_n \geq -1$. Assume the contrary that $\limsup_{n\to\infty} t_n < -1$. If so, there exists $N$ such that $t_n < -1$ when $n \geq N$. Hence, for all $n \geq N$, from (21), we deduce

$$\begin{aligned} s_{n+1} &\leq (1-\alpha_n)s_n + \alpha_n t_n \\ &\leq (1-\alpha_n)s_n - \alpha_n \\ &= s_n - \alpha_n(1+s_n) \\ &\leq s_n - \alpha_n. \end{aligned}$$

It follows that $s_{n+1} \leq s_N - \sum_{k=N}^{n} \alpha_k$, which implies that $\limsup_{n\to\infty} s_n \leq s_N - \limsup_{n\to\infty} \sum_{k=N}^{n} \alpha_k = -\infty$. It is a contradiction. So, $-1 \leq \limsup_{n\to\infty} t_n \leq +\infty$. Thus, we can select a subsequence $\{x_{n_i}\} \subset \{x_n\}$ (because of the boundedness of $\{x_n\}$) verifying $x_{n_i} \rightharpoonup x^\dagger \in C$ and

$$\begin{aligned} \limsup_{n\to\infty} t_n = \lim_{i\to\infty} t_{n_i} = \lim_{i\to\infty} \Big[ &- \frac{(2-\tau)\tau(1-\theta)^2\|x_{n_i} - y_{n_i}\|^4}{\alpha_{n_i}\|x_{n_i} - y_{n_i} + \mu\gamma^{m(x_{n_i})} f(y_{n_i})\|^2} + 2\langle u - x^*, x_{n_i+1} - x^*\rangle \\ &- \sigma_{n_i}(\delta_{n_i} - \sigma_{n_i}) \frac{\|u_{n_i} - T[(1-\delta_{n_i})u_{n_i} + \delta_{n_i} T u_{n_i}]\|^2}{\alpha_{n_i}} \Big]. \end{aligned} \tag{22}$$

Based on the boundedness of $\{x_{n_i+1}\}$, without loss of generality, assume that $\lim_{i\to\infty} 2\langle u - x^*, x_{n_i+1} - x^*\rangle$ exists. Hence, according to (22), we deduce that the following limit

$$\lim_{i\to\infty} \Big[ \frac{(2-\tau)\tau(1-\theta)^2\|x_{n_i} - y_{n_i}\|^4}{\alpha_{n_i}\|x_{n_i} - y_{n_i} + \mu\gamma^{m(x_{n_i})} f(y_{n_i})\|^2} + \sigma_{n_i}(\delta_{n_i} - \sigma_{n_i}) \frac{\|u_{n_i} - T[(1-\delta_{n_i})u_{n_i} + \delta_{n_i} T u_{n_i}]\|^2}{\alpha_{n_i}} \Big] \tag{23}$$

exists.

Since $\lim_{i\to\infty} \alpha_{n_i} = 0$ and $\liminf_{i\to\infty} \sigma_{n_i}(\delta_{n_i} - \sigma_{n_i}) > 0$, it follows from (23) that

$$\lim_{i\to\infty} \frac{\|x_{n_i} - y_{n_i}\|^4}{\|x_{n_i} - y_{n_i} + \mu\gamma^{m(x_{n_i})} f(y_{n_i})\|^2} = 0 \tag{24}$$

and

$$\lim_{i\to\infty} \|u_{n_i} - T[(1-\delta_{n_i})u_{n_i} + \delta_{n_i} T u_{n_i}]\| = 0. \tag{25}$$

Note that $\|x_{n_i} - y_{n_i} + \mu\gamma^{m(x_{n_i})} f(y_{n_i})\|$ is bounded. In virtue of this fact and (24), we derive

$$\lim_{i\to\infty} \|x_{n_i} - y_{n_i}\| = 0 \tag{26}$$

Combining (14) and (26), we obtain

$$\lim_{i\to\infty} \|f(x_{n_i}) - f(y_{n_i})\| = 0. \tag{27}$$

As a result of (15), we have the following estimate

$$\begin{aligned} \|u_n - x_n\| &= \Big\| P_C\Big[x_n - \tau(1-\theta)\|x_n - y_n\|^2 \frac{x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2}\Big] - P_C[x_n] \Big\| \\ &\leq \frac{\tau(1-\theta)\|x_n - y_n\|^2}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|}. \end{aligned}$$

This together with (24) implies that

$$\lim_{i\to\infty}\|u_{n_i}-x_{n_i}\|=0. \tag{28}$$

Applying the characterization (6) of projection $P_C$, we have

$$\langle x_{n_i}-\mu\gamma^{m(x_{n_i})}f(x_{n_i})-y_{n_i},y_{n_i}-x^\dagger\rangle\geq 0,\ \forall x^\dagger\in C.$$

It yields

$$\langle f(x_{n_i}),x^\dagger-x_{n_i}\rangle\geq\langle f(x_{n_i}),y_{n_i}-x_{n_i}\rangle+\frac{1}{\mu\gamma^{m(x_{n_i})}}\langle y_{n_i}-x^\dagger,y_{n_i}-x_{n_i}\rangle,\ \forall x^\dagger\in C. \tag{29}$$

Noting that $\{f(x_{n_i})\}$ and $\{y_{n_i}\}$ are bounded, $\frac{\gamma\theta}{\kappa}<\mu\gamma^{m(x_{n_i})}\leq\mu$ due to Remark 2, in view of (26) and (29), we obtain

$$\liminf_{i\to\infty}\langle f(x_{n_i}),x^\dagger-x_{n_i}\rangle\geq 0,\ \forall x^\dagger\in C. \tag{30}$$

Thanks to (30), we can choose a positive real numbers sequence $\{\epsilon_j\}$ satisfying $\lim_{j\to\infty}\epsilon_j=0$. For each $\epsilon_j$, there exists the smallest positive integer $k_i$ such that

$$\langle f(x_{n_{i_j}}),x^\dagger-x_{n_{i_j}}\rangle+\epsilon_j\geq 0,\ \forall j\geq k_i. \tag{31}$$

Moreover, for each $j>0$, $f(x_{n_{i_j}})\neq 0$ (by Remark 3), letting $w(x_{n_{i_j}})=\frac{f(x_{n_{i_j}})}{\|f(x_{n_{i_j}})\|^2}$, then $\langle f(x_{n_{i_j}}),w(x_{n_{i_j}})\rangle=1$. By virtue of (31), we have

$$\langle f(x_{n_{i_j}}),x^\dagger+\epsilon_j w(x_{n_{i_j}})-x_{n_{i_j}}\rangle\geq 0,$$

which implies, together with the pseudomonotonicity of $f$ on $H$, that

$$\langle f(x^\dagger+\epsilon_j w(x_{n_{i_j}})),x^\dagger+\epsilon_j w(x_{n_{i_j}})-x_{n_{i_j}}\rangle\geq 0.$$

It follows that

$$\langle f(x^\dagger),x^\dagger-x_{n_{i_j}}\rangle\geq\langle f(x^\dagger)-f(x^\dagger+\epsilon_j w(x_{n_{i_j}})),x^\dagger+\epsilon_j w(x_{n_{i_j}})-x_{n_{i_j}}\rangle+\langle f(x^\dagger),-\epsilon_j w(x_{n_{i_j}})\rangle. \tag{32}$$

Since the sequence $\{x_{n_{i_j}}\}$ is bounded, without loss of generality, we assume that $x_{n_{i_j}}\rightharpoonup v\in C$ as $j\to\infty$. Furthermore, $f(x_{n_{i_j}})\rightharpoonup f(v)$ due to the weakly sequentially continuity of $f$. Assume that $f(v)\neq 0$ (otherwise, $v\in VI(C,f)$ and $\omega_w(x_n)\subset VI(C,f)$). Thus, we have

$$\liminf_{j\to\infty}\|f(x_{n_{i_j}})\|\geq\|f(v)\|,$$

and consequently,

$$\lim_{j\to\infty}\|\epsilon_j w(x_{n_{i_j}})\|=\lim_{j\to\infty}\frac{\epsilon_j}{\|f(x_{n_{i_j}})\|}=0.$$

This together with (32) and $f$ being Lipschitz continuous, we deduce

$$\langle f(x^\dagger),x^\dagger-v\rangle\geq 0. \tag{33}$$

It follows from Lemma 3 that $v \in VI(C, f)$ and hence $\omega_w(x_n) \subset VI(C, f)$.

Since $T$ is $L$-Lipschitzian, we have

$$\begin{aligned}\|u_n - Tu_n\| &\leq \|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\| + \|T[(1-\delta_n)u_n + \delta_n Tu_n] - Tu_n\| \\ &\leq \|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\| + L\delta_n\|u_n - Tu_n\|,\end{aligned}$$

which yields

$$\|u_n - Tu_n\| \leq \frac{1}{1-\delta_n L}\|u_n - T[(1-\delta_n)u_n + \delta_n Tu_n]\|. \tag{34}$$

On the basis of (25), (28) and (34), we derive

$$\lim_{j\to\infty} \|x_{n_{i_j}} - Tx_{n_{i_j}}\| = 0. \tag{35}$$

Consequently, applying Lemma 4 to (35) to deduce that $v \in Fix(T)$. Thus, $v \in VI(C, f) \cap Fix(T) = \Lambda$.

In case 2, we have $x_n \in VI(C, f)$ and the following estimate (by the similar argument as (19))

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq\ & (1-\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n)\sigma_n(\delta_n - \sigma_n)\|x_n - T[(1-\delta_n)x_n + \delta_n Tx_n]\|^2 \\ & + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\ =\ & (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\Big[-(1-\alpha_n)\sigma_n(\delta_n - \sigma_n)\frac{\|x_n - T[(1-\delta_n)x_n + \delta_n Tx_n]\|^2}{\alpha_n} \\ & + 2\langle u - x^*, x_{n+1} - x^*\rangle\Big].\end{aligned}$$

Consequently, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\lim_{j\to\infty} \sigma_{n_j}(\delta_{n_j} - \sigma_{n_j})\frac{\|x_{n_j} - T[(1-\delta_{n_i})x_{n_j} + \delta_{n_j} Tx_{n_j}]\|^2}{\alpha_{n_j}} = 0.$$

It follows that

$$\lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}]\| = 0.$$

Thus, we also deduce that $\omega_w(x_n) \subset \Lambda$.

Step 3. $x_n \to P_\Lambda(u)$.

In Case 1 or Case 2, we have

$$\limsup_{n\to\infty}\langle u - x^*, x_{n+1} - x^*\rangle = \langle u - x^*, v - x^*\rangle \leq 0. \tag{36}$$

From (16), (17) and (18), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u - x^*) + (1-\alpha_n)(z_n - x^*)\|^2 \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle.\end{aligned} \tag{37}$$

Finally, applying Lemma 5 with (36) to (37), we conclude that $x_n \to x^*$. This completes the proof. □

**Remark 4.** *We assume that $f$ is $\kappa$-Lipschitz continuous. However, the information of $\kappa$ is not necessary priority to be known. That is, we need not to estimate the value of $\kappa$.*

**Remark 5.** *It is obvious that monotonicity implies pseudomonotonicity. Hence, our theorem holds when the involved operator $f$ is monotone.*

Assume that the above Algorithm 2 does not terminate in a finite iterations.

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**Algorithm 2:** Iterative procedures for VI.

---

Step 1. Fixed four constants $\gamma \in (0,1)$, $\mu \in (0,1)$, $\theta \in (0,1)$ and $\tau \in (0,2)$. Let $x_0 \in C$ be an initial value. Set $n = 0$.

Step 2. Assume that the sequence $\{x_n\}$ has been constructed and then calculate $P_C[x_n - f(x_n)]$. If $P_C[x_n - f(x_n)] = x_n$, then stop. Otherwise, continuously proceed the following steps.

Step 3. Calculate

$$y_n = P_C[x_n - \mu\gamma^{m(x_n)} f(x_n)],$$

where $m(x_n) = \min\{0, 1, 2, 3, \cdots\}$ and satisfies

$$\mu\gamma^{m(x_n)}\|f(x_n) - f(y_n)\| \leq \theta\|x_n - y_n\|.$$

Step 4. Let $u \in C$ be a fixed point. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Compute the sequence $\{x_{n+1}\}$ via the following form

$$x_{n+1} = \alpha_n u + (1-\alpha_n)P_C\Big[x_n - \tau(1-\theta)\|x_n - y_n\|^2 \frac{x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)}{\|x_n - y_n + \mu\gamma^{m(x_n)} f(y_n)\|^2}\Big].$$

Step 5. Set $n := n+1$ and return to Step 2.

---

**Corollary 1.** *Suppose that $VI(C, f) \neq \emptyset$. Assume that the iterative parameter $\{\alpha_n\}$ satisfies condition (C1) in Theorem 1. Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $P_{VI(C,f)}(u)$.*

**Corollary 2.** *Suppose that $Fix(T) \neq \emptyset$. Assume that the iterative parameters $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ satisfy the conditions (C1) and (C2) in Theorem 1. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to $P_{Fix(T)}(u)$.*

---

**Algorithm 3:** Iterative procedures for FP.

---

Step 1. Let $x_0 \in C$ be an initial value. Set $n = 0$.

Step 2. Assume that the sequence $\{x_n\}$ has been constructed. Let $u \in C$ be a fixed point. Let $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ be three real number sequences in $(0,1)$. Compute the sequences $\{z_n\}$ and $\{x_{n+1}\}$ via the following iterations

$$\begin{cases} z_n = (1-\sigma_n)x_n + \sigma_n T[(1-\delta_n)x_n + \delta_n T x_n], \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n. \end{cases}$$

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## 4. Applications

Let $\emptyset \neq C$ be a convex and closed subset of a real Hilbert space $H$. Recall that an operator $T : C \to C$ is said to be $\alpha$-strictly pseudocontractive if there exists a constant $\alpha \in (0,1)$ satisfying

$$\|Tz - Tz^\dagger\|^2 \leq \|z - z^\dagger\|^2 + \alpha\|(I-T)z - (I-T)z^\dagger\|^2$$

for all $z, z^\dagger \in C$.

**Remark 6.** *It is easy to check that the class of pseudocontractive operators strictly includes the class of strictly pseudocontractive operators.*

**Proposition 2** ([48])**.** *Let $\emptyset \neq C$ be a convex and closed subset of a real Hilbert space $H$. Let $T : C \to C$ is said to be an $\alpha$-strictly pseudocontractive operator. Then,*

*(i)* $T$ *is* $\frac{1+\alpha}{1-\alpha}$*-Lipschitz;*
*(ii)* $I - T$ *is demi-closed at* $0$.

*Now, by using Remark 6 and Proposition 2, we can apply Theorem 1 for solving pseudomonotone variational inequalities and fixed point problem of strictly pseudocontractive operators.*

**Theorem 2.** *Let $\emptyset \neq C$ be a convex and closed subset of a real Hilbert space $H$. Let the operator $f$ be pseudomonotone on $H$, weakly sequentially continuous and Lipschitz continuous on $C$ with Lipschitz constant $\kappa > 0$. Let $T : C \to C$ be an $\alpha$-strictly pseudocontractive operator. Suppose that the iterative parameters $\{\alpha_n\}$, $\{\sigma_n\}$ and $\{\delta_n\}$ satisfy the following assumptions:*

*(C1):*$\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
*(C2):*$0 < \underline{\sigma} < \sigma_n < \overline{\sigma} < \delta_n < \overline{\delta} < \frac{1}{\sqrt{1+L^2}+1} (\forall n \geq 0)$ *where* $L = \frac{1+\alpha}{1-\alpha}$.

*Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $P_\Lambda(u)$.*

**Remark 7.** *In [49], Anh and Phuong introduced an iteration algorithm for solving pseudomonotone variational inequalities and fixed point problem of strictly pseudocontractive operators. Theorem 2 extends the main result of ([49] Theorem 3.3) from weak convergence to strong convergence.*

**Remark 8.** *In [50], Strodiot, Nguyen and Vuong presented a shrinking projection algorithm for solving variational inequalities and fixed point problem of strictly pseudocontractive operators. Note that the the computation of projection $P_{C_{n+1}}$ (([50] Algorithm 1-VI) is expensive. Our Algorithm 1 is more applicable.*

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