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Multipolar Fuzzy p-Ideals of BCI-Algebras

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Abstract: The notion of (normal) m-polar $(\in, \in)$-fuzzy p-ideals of BCI-algebras is introduced, and several properties are investigated. Relations between an m-polar $(\in, \in)$-fuzzy ideal and an m-polar $(\in, \in)$-fuzzy p-ideal are displayed, and conditions for an m-polar $(\in, \in)$-fuzzy ideal to be an m-polar $(\in, \in)$-fuzzy p-ideal are provided. Characterization of m-polar $(\in, \in)$-fuzzy p-ideals are considered. Given an m-polar $(\in, \in)$-fuzzy ideal (resp., m-polar $(\in, \in)$-fuzzy p-ideal), a normal m-polar $(\in, \in)$-fuzzy ideal (resp., normal m-polar $(\in, \in)$-fuzzy p-ideal) is established. Using an m-polar $(\in, \in)$-fuzzy ideal, the quotient structure of BCI-algebras is constructed.

Keywords: (normal) m-polar $(\in, \in)$-fuzzy ideal; (normal) m-polar $(\in, \in)$-fuzzy p-ideal

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1. Introduction

Fuzzy sets, which were introduced by Zadeh [1], deal with possibilistic uncertainty and are connected with imprecision of states, perceptions, and preferences. Since the introduction of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields such as statistics, graph theory, medical and life science, engineering, business and social science, computer networks, decision making, artificial intelligence, pattern recognition, robotics, and automata theory, etc. BCK/BCI-algebras, which are created from two distinct approaches, set theory and proposition calculus, first appeared in the mathematical literature in 1966 (see [2,3]). BCK and BCI algebras describe fragments of propositional calculus involving implications known as BCK and BCI logics. The various attributes of BCK/BCI-algebra are considered in [4–8]. Ideal theory in BCI-algebras, in particular p-ideal, is studied in [9]. As an extension of fuzzy sets, Zhang [10] introduced the notion of bipolar fuzzy sets. Bipolar fuzzy information is applied in many (algebraic) structures, for instance, Γ-semihypergroups (see [11]), finite state machines (see [12–15]), (ordered) semigroups (see [16–19]), and (hyper) BCK/BCI-algebras (see [20–25]). In many real problems, information sometimes comes from multi-factors, and there are many multi-attribute data that cannot be processed using existing anomalies (e.g., fuzzy anomalies and bipolar fuzzy anomalies, etc.). In 2014, Chen et al. [26] introduced an m-polar fuzzy set, which is an extension of bipolar fuzzy sets. m-polar fuzzy sets have been applied to decision-making problems (see [27]), graph theory (see [28–31]), and BCK/BCI-algebra (see [32]).

In this paper, we introduce the notion of (normal) m-polar $(\in, \in)$-fuzzy p-ideals of BCI-algebras and investigate several properties. We discuss relations between an m-polar $(\in, \in)$-fuzzy ideal and an m-polar $(\in, \in)$-fuzzy p-ideal. We provide conditions for an m-polar $(\in, \in)$-fuzzy ideal to be an m-polar $(\in, \in)$-fuzzy p-ideal. We consider the characterization of m-polar $(\in, \in)$-fuzzy p-ideals. Given an m-polar $(\in, \in)$-fuzzy ideal (resp., m-polar $(\in, \in)$-fuzzy p-ideal), we define a normal m-polar

$(\in, \in)$-fuzzy ideal (resp., normal m-polar $(\in, \in)$-fuzzy p-ideal). Using an m-polar $(\in, \in)$-fuzzy ideal, we construct the quotient structure of BCI-algebras.

2. Preliminaries

First, we would like to briefly present the basic concept for use in this paper.

By a BCI-algebra, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $(\forall u, w, v \in X)\ (((u * w) * (u * v)) * (v * w) = 0)$,
(II) $(\forall u, w \in X)\ ((u * (u * w)) * w = 0)$,
(III) $(\forall u \in X)\ (u * u = 0)$,
(IV) $(\forall u, w \in X)\ (u * w = 0,\ w * u = 0\ \Rightarrow\ u = w)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall u \in X)\ (0 * u = 0)$,

Then X is called a BCK-algebra. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall u \in X)\,(u * 0 = u)\,, \tag{1}$$
$$(\forall u, w, v \in X)\,(u \leq w \ \Rightarrow\ u * v \leq w * v,\ v * w \leq v * u)\,, \tag{2}$$
$$(\forall u, w, v \in X)\,((u * w) * v = (u * v) * w)\,, \tag{3}$$

where $u \leq w$ if and only if $u * w = 0$. A subset S of a BCK/BCI-algebra X is called a subalgebra of X if $u * w \in S$ for all $u, w \in S$. A subset I of a BCK/BCI-algebra X is called an ideal of X if it satisfies

$$0 \in I, \tag{4}$$
$$(\forall u \in X)\,(\forall w \in I)\,(u * w \in I \ \Rightarrow\ u \in I)\,. \tag{5}$$

A subset I of a BCI-algebra X is called a p-ideal of X if it satisfies (4) and

$$(\forall u, w, v \in X)\,((u * v) * (w * v) \in I,\ w \in I \ \Rightarrow\ u \in I)\,. \tag{6}$$

Every BCI-algebra X satisfies the following assertions (see [9]).

$$(\forall u, w, v \in X)(0 * (0 * ((u * v) * (w * v))) = (0 * w) * (0 * u)). \tag{7}$$
$$(\forall u, w \in X)(0 * (0 * (u * w)) = (0 * w) * (0 * u)). \tag{8}$$
$$(\forall u, w \in X)(\forall k \in \mathbb{N})(0 * (u * w)^k = (0 * u^k) * (0 * w^k)). \tag{9}$$

See the books [7] and [8] for more information on BCK/BCI-algebras.

By an m-polar fuzzy set (briefly, mp-fuzzy set) on a set X (see [26]), we mean a function $\hat{\ell} : X \to [0,1]^m$. The membership value of every element $y \in X$ is denoted by

$$\hat{\ell}(y) = \Big((\pi_1 \circ \hat{\ell})(y), (\pi_2 \circ \hat{\ell})(y), \cdots, (\pi_m \circ \hat{\ell})(y)\Big)\,,$$

where $\pi_i : [0,1]^m \to [0,1]$ is the i-th projection for all $i = 1, 2, \cdots, m$.

Given an mp-fuzzy set on a set X, we consider the set

$$U(\hat{\ell}, \hat{t}) := \{y \in X \mid \hat{\ell}(y) \geq \hat{t}\}, \tag{10}$$

that is,

$$U(\hat{\ell}, \hat{t}) := \{y \in X \mid (\pi_1 \circ \hat{\ell})(y) \geq t_1, (\pi_2 \circ \hat{\ell})(y) \geq t_2, \cdots, (\pi_m \circ \hat{\ell})(y) \geq t_m\}, \tag{11}$$

which is called an m-polar level set of $\hat{\ell}$.

By an mp-fuzzy point on a set X, we mean an mp-fuzzy set $\hat{\ell}$ on X of the form

$$\hat{\ell}(z) = \begin{cases} \hat{r} = (r_1, r_2, \cdots, r_m) \in (0,1]^m & \text{if } z = y, \\ \hat{0} = (0, 0, \cdots, 0) & \text{if } z \neq y, \end{cases} \tag{12}$$

and it is denoted by $y_{\hat{r}}$. We say that y is the support of $y_{\hat{r}}$ and $\hat{r}$ is the value of $y_{\hat{r}}$.

We say that an mp-fuzzy point $y_{\hat{r}}$ is contained in an mp-fuzzy set $\hat{\ell}$, denoted by $y_{\hat{r}} \in \hat{\ell}$, if $\hat{\ell}(y) \geq \hat{r}$, that is, $(\pi_i \circ \hat{\ell})(y) \geq r_i$ for all $i = 1, 2, \cdots, m$.

An mp-fuzzy set $\hat{\ell}$ on a BCK/BCI-algebra X is called an m-polar fuzzy ideal (briefly, mp-fuzzy ideal) of X (see [32] Definition 3.7) if the following conditions are valid.

$$(\forall y \in X) \left(\hat{\ell}(0) \geq \hat{\ell}(y) \right), \tag{13}$$

$$(\forall y, z \in X) \left(\hat{\ell}(y) \geq \inf\{\hat{\ell}(y * z), \hat{\ell}(z)\} \right), \tag{14}$$

that is,

$$(\forall y \in X) \left((\pi_i \circ \hat{\ell})(0) \geq (\pi_i \circ \hat{\ell})(y) \right), \tag{15}$$

$$(\forall y, z \in X) \left((\pi_i \circ \hat{\ell})(y) \geq \inf\{(\pi_i \circ \hat{\ell})(y * z), (\pi_i \circ \hat{\ell})(z)\} \right), \tag{16}$$

for all $i = 1, 2, \cdots, m$.

We display an example of an mp-fuzzy ideal that is given by Al-Masarwah and Ahmad.

Example 1 ([32]). *Let $X = \{0, a, 1, 2, 3\}$ be a BCI-algebra with a binary operation "$*$", which is given in the following Cayley table.*

$*$	**0**	a	**1**	**2**	**3**
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

Define a 4-polar fuzzy set $\hat{\ell}$ on X as follows:

$$\hat{\ell} : X \to [0,1]^4,\ y \mapsto \begin{cases} (0.5, 0.6, 0.6, 0.7) & \text{if } y = 0, \\ (0.4, 0.5, 0.5, 0.7) & \text{if } y = a, \\ (0.2, 0.3, 0.3, 0.2) & \text{if } y = 1, 3, \\ (0.3, 0.4, 0.4, 0.5) & \text{if } y = 2. \end{cases}$$

Then $\hat{\ell}$ is a 4-polar $(\in, \in)$-fuzzy ideal of X.

3. *m*-Polar Fuzzy *p*-Ideals

In a BCK/BCI-algebra, the mp-fuzzy ideal is characterized as follows.

Lemma 1. *An mp-fuzzy set $\hat{\ell}$ on a BCK/BCI-algebra X is an mp-fuzzy ideal of X if and only if the following conditions are valid.*

$$(\forall y \in X)(\forall \hat{r} \in [0,1]^m) \left(y_{\hat{r}} \in \hat{\ell} \Rightarrow 0_{\hat{r}} \in \hat{\ell} \right), \tag{17}$$

$$(\forall y, z \in X)(\forall \hat{r}, \hat{t} \in [0,1]^m) \left((y * z)_{\hat{r}} \in \hat{\ell},\, z_{\hat{t}} \in \hat{\ell} \Rightarrow y_{\inf\{\hat{r},\hat{t}\}} \in \hat{\ell} \right). \tag{18}$$

Proof. Straightforward. □

If an mp-fuzzy set $\hat{\ell}$ on a BCK/BCI-algebra X satisfies two conditions (17) and (18), we say that $\hat{\ell}$ is an *m-polar $(\in, \in)$-fuzzy ideal* of X.

Lemma 1 shows that an mp-fuzzy ideal and an m-polar $(\in, \in)$-fuzzy ideal are in agreement.

Definition 1. *An mp-fuzzy set $\hat{\ell}$ on a BCI-algebra X is called an m-polar $(\in, \in)$-fuzzy p-ideal of X if it satisfies* (17) *and*

$$(\forall y, z, x \in X)(\forall \hat{r}, \hat{t} \in [0,1]^m) \left(((y * x) * (z * x))_{\hat{r}} \in \hat{\ell},\, z_{\hat{t}} \in \hat{\ell} \Rightarrow y_{\inf\{\hat{r},\hat{t}\}} \in \hat{\ell} \right). \tag{19}$$

It is easy to show that the condition (19) is equivalent to the following condition:

$$(\forall y, z, x \in X) \left(\hat{\ell}(y) \geq \inf\{\hat{\ell}((y * x) * (z * x)), \hat{\ell}(z)\} \right), \tag{20}$$

that is,

$$(\pi_i \circ \hat{\ell})(y) \geq \inf\{(\pi_i \circ \hat{\ell})((y * x) * (z * x)), (\pi_i \circ \hat{\ell})(z)\} \tag{21}$$

for all $y, z, x \in X$ and $i = 1, 2, \cdots, m$.

Example 2. *Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation "$*$", which is given in Table 1.*

Then X is a BCI-algebra (see [7]). Define a 5-polar fuzzy set $\hat{\ell}$ on X as follows:

$$\hat{\ell} : X \to [0,1]^5,\ y \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.5, 0.9) & \text{if } y = 0, \\ (0.5, 0.6, 0.7, 0.4, 0.7) & \text{if } y = 1, \\ (0.3, 0.4, 0.6, 0.2, 0.5) & \text{if } y = 2, \\ (0.3, 0.4, 0.6, 0.2, 0.5) & \text{if } y = 3. \end{cases}$$

It is routine to check that $\hat{\ell}$ is a 5-polar $(\in, \in)$-fuzzy p-ideal of X.

Table 1. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Theorem 1. *Let I be a subset of a BCI-algebra X, and let $\hat{\ell}_I$ be an mp-fuzzy set on X defined by*

$$\hat{\ell}_I : X \to [0,1]^m,\ y \mapsto \begin{cases} \hat{1} & \text{if } y \in I, \\ \hat{0} & \text{otherwise.} \end{cases}$$

Then $\hat{\ell}_I$ is an m-polar $(\in, \in)$-fuzzy ideal (resp., m-polar $(\in, \in)$-fuzzy p-ideal) of X if and only if I is an ideal (resp., p-ideal) of X.

Proof. Straightforward. □

Proposition 1. *Every m-polar $(\in, \in)$-fuzzy p-ideal $\hat{\ell}$ of a BCI-algebra X satisfies the following inequalities:*

$$(\forall y \in X)(\hat{\ell}(y) \geq \hat{\ell}(0 * (0 * y))), \tag{22}$$

that is, $(\pi_i \circ \hat{\ell})(y) \geq (\pi_i \circ \hat{\ell})(0 * (0 * y))$ for all $y \in X$ and $i = 1, 2, \cdots, m$.

Proof. In (20), if we change x to y and z to 0, then

$$\hat{\ell}(y) \geq \inf\{\hat{\ell}((y * y) * (0 * y)), \hat{\ell}(0)\} = \inf\{\hat{\ell}(0 * (0 * y)), \hat{\ell}(0)\} = \hat{\ell}(0 * (0 * y))$$

for all $y \in X$. □

Theorem 2. *In a BCI-algebra, every m-polar $(\in, \in)$-fuzzy p-ideal is an m-polar $(\in, \in)$-fuzzy ideal.*

Proof. Let $\hat{\ell}$ be an m-polar $(\in, \in)$-fuzzy p-ideal of a BCI-algebra X. Since $y * 0 = y$ for all $y \in X$, it follows from (20) that

$$\hat{\ell}(y) \geq \inf\{\hat{\ell}((y * 0) * (z * 0)), \hat{\ell}(z)\} = \inf\{\hat{\ell}(y * z), \hat{\ell}(z)\}$$

for all $y, z \in X$. Therefore, $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy ideal of a BCI-algebra X. □

Proposition 2. *Every m-polar $(\in, \in)$-fuzzy p-ideal $\hat{\ell}$ of a BCI-algebra X satisfies the following inequalities.*

$$(\forall y, z, x \in X)(\hat{\ell}(y * z) \leq \hat{\ell}((y * x) * (z * x))), \tag{23}$$

that is, $(\pi_i \circ \hat{\ell})(y * z) \leq (\pi_i \circ \hat{\ell})((y * x) * (z * x))$, for all $y, z, x \in X$ and $i = 1, 2, \cdots, m$.

Proof. Let $\hat{\ell}$ be an m-polar $(\in, \in)$-fuzzy p-ideal of X. Then it is an m-polar $(\in, \in)$-fuzzy ideal of X by Theorem 2. For any $y, z, x \in X$, we have $((y * x) * (z * x)) * (y * z) = 0$. Hence,

$$\begin{aligned} \hat{\ell}((y * x) * (z * x)) &\geq \inf\{\hat{\ell}(((y * x) * (z * x)) * (y * z)), \hat{\ell}(y * z)\} \\ &= \inf\{\hat{\ell}(0), \hat{\ell}(y * z)\} = \hat{\ell}(y * z) \end{aligned}$$

for all $y, z, x \in X$. □

The following example shows that an m-polar $(\in, \in)$-fuzzy ideal may not be an m-polar $(\in, \in)$-fuzzy p-ideal.

Example 3. *Let $X = \{0, 1, a, b, c\}$ be a set with a binary operation "$*$", which is given in Table 2. Then X is a BCI-algebra (see [7]). Define a 3-polar fuzzy set $\hat{\ell}$ on X as follows:*

$$\hat{\ell} : X \to [0, 1]^3,\ y \mapsto \begin{cases} (0.7, 0.9, 0.6) & \text{if } y = 0, \\ (0.5, 0.8, 0.6) & \text{if } y = 1, \\ (0.3, 0.4, 0.2) & \text{if } y = a, \\ (0.4, 0.6, 0.5) & \text{if } y = b, \\ (0.3, 0.4, 0.2) & \text{if } y = c. \end{cases}$$

It is easy to verify that $\hat{\ell}$ is a 3-polar $(\in, \in)$-fuzzy ideal of X. But it is not a 3-polar $(\in, \in)$-fuzzy p-ideal of X since $(\pi_2 \circ \hat{\ell})(1) = 0.8 < 0.9 = \inf\{(\pi_2 \circ \hat{\ell})((1 * a) * (0 * a)), (\pi_2 \circ \hat{\ell})(0)\}$.

We provide conditions for an m-polar $(\in, \in)$-fuzzy ideal to be an m-polar $(\in, \in)$-fuzzy p-ideal.

Table 2. Cayley table for the binary operation "$*$".

$*$	**0**	**1**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	0	0	c	b	a
1	1	0	c	b	a
a	a	a	0	c	b
b	b	b	a	0	c
c	c	c	b	a	0

Theorem 3. *Let $\hat{\ell}$ be an m-polar $(\in, \in)$-fuzzy ideal of a BCI-algebra X. If $\hat{\ell}$ satisfies the inequality*

$$(\forall y, z, x \in X)(\hat{\ell}(y * z) \geq \hat{\ell}((y * x) * (z * x))), \tag{24}$$

then $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X.

Proof. Combining (14) and (24), we get

$$\hat{\ell}(y) \geq \inf\{\hat{\ell}(y * z), \hat{\ell}(z)\} \geq \inf\{\hat{\ell}((y * x) * (z * x)), \hat{\ell}(z)\}$$

for all $y, z, x \in X$. □

Lemma 2. *Let X be a BCI-algebra. Then every m-polar $(\in, \in)$-fuzzy ideal of X satisfies the following inequality*

$$(\forall y \in X)(\hat{\ell}(y) \leq \hat{\ell}(0 * (0 * y))). \tag{25}$$

That is, $(\pi_i \circ \hat{\ell})(y) \leq (\pi_i \circ \hat{\ell})(0 * (0 * y))$ for all $y \in X$ and $i = 1, 2, \cdots, m$.

Proof. Using (13) and (14), we have

$$\hat{\ell}(0 * (0 * y)) \geq \inf\{\hat{\ell}((0 * (0 * y)) * y), \hat{\ell}(y)\} = \inf\{\hat{\ell}(0), \hat{\ell}(y)\} = \hat{\ell}(y)$$

for all $y \in X$. □

Theorem 4. *Let $\hat{\ell}$ be an m-polar $(\in, \in)$-fuzzy ideal of a BCI-algebra X. If $\hat{\ell}$ satisfies the inequality*

$$(\forall y \in X)(\hat{\ell}(0 * (0 * y)) \leq \hat{\ell}(y)), \tag{26}$$

that is, $(\pi_i \circ \hat{\ell})(0 * (0 * y)) \leq (\pi_i \circ \hat{\ell})(y)$ for all $y \in X$ and $i = 1, 2, \cdots, m$, then $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X.

Proof. For any $y, z, x \in X$ and $i = 1, 2, \cdots, m$, we have

$$\begin{aligned}
(\pi_i \circ \hat{\ell})((y * x) * (z * x)) &\leq (\pi_i \circ \hat{\ell})(0 * (0 * ((y * x) * (z * x)))) \\
&= (\pi_i \circ \hat{\ell})((0 * z) * (0 * y)) \\
&= (\pi_i \circ \hat{\ell})(0 * (0 * (y * z))) \\
&\leq (\pi_i \circ \hat{\ell})(y * z),
\end{aligned}$$

and hence, $\hat{\ell}((y * x) * (z * x)) \leq \hat{\ell}(y * z)$ for all $y, z, x \in X$. Therefore, $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X by Theorem 3. □

Theorem 5. *Let $\hat{\ell}$ be an mp-fuzzy set on a BCI-algebra X. Then the following assertions are equivalent.*

(1) *$\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X.*
(2) *The m-polar level set $U(\hat{\ell}, \hat{r})$ of $\hat{\ell}$ is a p-ideal of X for all $\hat{r} \in [0, 1]^m$.*

Proof. Assume that $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X and let $\hat{r} = (r_1, r_2, \cdots, r_m) \in (0,1]^m$. It is clear that $0 \in U(\hat{\ell}, \hat{r})$. Let $y, z, x \in X$ be such that $(y * x) * (z * x) \in U(\hat{\ell}, \hat{r})$ and $z \in U(\hat{\ell}, \hat{r})$. Then $(\pi_i \circ \hat{\ell})((y * x) * (z * x)) \geq r_i$ and $(\pi_i \circ \hat{\ell})(z) \geq r_i$. It follows from (21) that

$$(\pi_i \circ \hat{\ell})(y) \geq \inf\{(\pi_i \circ \hat{\ell})((y * x) * (z * x)), (\pi_i \circ \hat{\ell})(z)\} \geq r_i$$

for $i = 1, 2, \cdots, m$. Hence, $y \in U(\hat{\ell}, \hat{r})$, and therefore $U(\hat{\ell}, \hat{r})$ is a p-ideal of X.

Conversely, suppose that the m-polar level set $U(\hat{\ell}, \hat{r})$ of $\hat{\ell}$ is a p-ideal of X for all $\hat{r} \in [0,1]^m$. If $\hat{\ell}(0) < \hat{\ell}(a)$ for some $a \in X$ and taking $\hat{r} := \hat{\ell}(a)$, then $a \in U(\hat{\ell}, \hat{r})$ and $0 \notin U(\hat{\ell}, \hat{r})$. This is a contradiction, and so $\hat{\ell}(0) \geq \hat{\ell}(y)$ for all $y \in X$. Now, suppose that there exist $a, b, c \in X$ such that $\hat{\ell}(a) < \inf\{\hat{\ell}((a * c) * (b * c)), \hat{\ell}(b)\}$. If we take

$$\hat{r} := \inf\{\hat{\ell}((a * c) * (b * c)), \hat{\ell}(b)\},$$

then $(a * c) * (b * c) \in U(\hat{\ell}, \hat{r})$ and $b \in U(\hat{\ell}, \hat{r})$. Since $U(\hat{\ell}, \hat{r})$ of $\hat{\ell}$ is a p-ideal of X, it follows that $a \in U(\hat{\ell}, \hat{r})$. Hence, $\hat{\ell}(a) \geq \hat{r}$, which is a contradiction. Thus $\hat{\ell}(y) \geq \inf\{\hat{\ell}((y * x) * (z * x)), \hat{\ell}(z)\}$ for all $y, z, x \in X$. Therefore, $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X. □

Corollary 1. *Let $\hat{\ell}$ be an mp-fuzzy set on a BCI-algebra X. If $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X, then the set*

$$I := \{y \in X \mid \hat{\ell}(y) = \hat{\ell}(0)\}$$

is a p-ideal and hence, an ideal of X.

In Example 2, we used definition to check that $\hat{\ell}$ is a 5-polar fuzzy p-ideal. Theorem 5 is a nice tool to verify that an mp-fuzzy set in a BCI-algebra X is an m-polar $(\in, \in)$-fuzzy p-ideal of X. Note that the m-polar level set of $\hat{\ell}$ can be represented as follows:

$$U(\hat{\ell}, \hat{t}) = \bigcap_{i \in \{1,2,\cdots,m\}} U_i(\hat{\ell}, \hat{t}),$$

where $U_i(\hat{\ell}, \hat{t}) = \{y \in X \mid (\pi_i \circ \hat{\ell})(y) \geq t_i\}$ for $i = 1, 2, \cdots, m$. In Example 2, we have

$$U_1(\hat{\ell}, \hat{t}) = \begin{cases} \varnothing & \text{if } t_1 \in (0.7, 1], \\ \{0\} & \text{if } t_1 \in (0.5, 0.7], \\ \{0,1\} & \text{if } t_1 \in (0.3, 0.5], \\ X & \text{if } t_1 \in [0, 0.3], \end{cases} \qquad U_2(\hat{\ell}, \hat{t}) = \begin{cases} \varnothing & \text{if } t_1 \in (0.6, 1], \\ \{0,1\} & \text{if } t_1 \in (0.4, 0.6], \\ X & \text{if } t_1 \in [0, 0.4], \end{cases}$$

$$U_3(\hat{\ell}, \hat{t}) = \begin{cases} \varnothing & \text{if } t_1 \in (0.8, 1], \\ \{0\} & \text{if } t_1 \in (0.7, 0.8], \\ \{0,1\} & \text{if } t_1 \in (0.6, 0.7], \\ X & \text{if } t_1 \in [0, 0.6], \end{cases} \qquad U_4(\hat{\ell}, \hat{t}) = \begin{cases} \varnothing & \text{if } t_1 \in (0.5, 1], \\ \{0\} & \text{if } t_1 \in (0.4, 0.5], \\ \{0,1\} & \text{if } t_1 \in (0.2, 0.4], \\ X & \text{if } t_1 \in [0, 0.2], \end{cases}$$

$$U_5(\hat{\ell}, \hat{t}) = \begin{cases} \varnothing & \text{if } t_1 \in (0.9, 1], \\ \{0\} & \text{if } t_1 \in (0.7, 0.9], \\ \{0,1\} & \text{if } t_1 \in (0.5, 0.7], \\ X & \text{if } t_1 \in [0, 0.5]. \end{cases}$$

Recall that $U_i(\hat{\ell}, \hat{t})$ is a p-ideal of X for all $i = 1, 2, 3, 4, 5$. Thus $U(\hat{\ell}, \hat{t})$ is a p-ideal of X, which implies from Theorem 5 that $\hat{\ell}$ is a 5-polar $(\in, \in)$-fuzzy p-ideal of X.

Theorem 6. *Let $\hat{\ell}$ be an mp-fuzzy set on a BCI-algebra X, and let $f : X \to X$ be an epimorphism. If $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X, then so is the composition $\hat{\ell} \circ f$.*

Proof. Let $y, z \in X$. Then $(\hat{\ell} \circ f)(y) = \hat{\ell}(f(y)) \leq \hat{\ell}(0) = \hat{\ell}(f(0)) = (\hat{\ell} \circ f)(0)$ and

$$\begin{aligned}(\hat{\ell} \circ f)(y) &= \hat{\ell}(f(y)) \geq \inf\{\hat{\ell}(f(y) * x), \hat{\ell}(x)\} \\ &= \inf\{\hat{\ell}(f(y) * f(z)), \hat{\ell}(f(z))\} \\ &= \inf\{\hat{\ell}(f(y * z)), \hat{\ell}(f(z))\} \\ &= \inf\{(\hat{\ell} \circ f)(y * z), (\hat{\ell} \circ f)(z)\}.\end{aligned}$$

Hence, $\hat{\ell} \circ f$ is an m-polar $(\in, \in)$-fuzzy ideal of X. Now, we have

$$\begin{aligned}(\pi_i \circ (\hat{\ell} \circ f))(y) &= \pi_i((\hat{\ell} \circ f)(y)) = \pi_i(\hat{\ell}(f(y))) \\ &= (\pi_i \circ \hat{\ell})(f(y)) \geq (\pi_i \circ \hat{\ell})(0 * (0 * f(x))) \\ &= (\pi_i \circ \hat{\ell})(f(0) * (f(0) * f(y))) \\ &= (\pi_i \circ \hat{\ell})(f(0 * (0 * y))) \\ &= (\pi_i \circ (\hat{\ell} \circ f))(0 * (0 * y))\end{aligned}$$

for all $y \in X$. Therefore, $\hat{\ell} \circ f$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X by Theorem 4. □

Definition 2. *An m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal) $\hat{\ell}$ of a BCI-algebra X is said to be normal if there exists $y \in X$ such that $\hat{\ell}(y) = \hat{1}$, that is, $(\pi_i \circ \hat{\ell})(y) = 1$ for all $i = 1, 2, \cdots, m$.*

Example 4. *The m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal) $\hat{\ell}_I$ in Theorem 1 is normal.*

We note that if an m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal) $\hat{\ell}$ of a BCI-algebra X is normal, then $\hat{\ell}(0) = \hat{1}$. Thus, we have the following characterization.

Theorem 7. *An m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal) $\hat{\ell}$ of a BCI-algebra X is normal if and only if $\hat{\ell}(0) = \hat{1}$, that is, $(\pi_i \circ \hat{\ell})(0) = 1$ for all $i = 1, 2, \cdots, m$.*

Using a non-normal m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal), we provide a way to make a normal m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal).

Theorem 8. *If $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy ideal (resp. m-polar $(\in, \in)$-fuzzy p-ideal) of a BCI-algebra X, then the mp-fuzzy set $\hat{\ell}^+$ on X defined by*

$$\hat{\ell}^+ : X \to [0,1]^m,\ y \mapsto \hat{1} + \hat{\ell}(y) - \hat{\ell}(0), \tag{27}$$

that is, $(\pi_i \circ \hat{\ell})^+(y) = 1 + (\pi_i \circ \hat{\ell})(y) - (\pi_i \circ \hat{\ell})(0)$ for $i = 1, 2, \cdots, m$, is a normal m-polar $(\in, \in)$-fuzzy ideal (resp. normal m-polar $(\in, \in)$-fuzzy p-ideal) of X containing $\hat{\ell}$.

Proof. Assume that $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy ideal of X. For any $y, z \in X$, we have

$$(\pi_i \circ \hat{\ell})(0) = 1 + (\pi_i \circ \hat{\ell})(0) - (\pi_i \circ \hat{\ell})(0) = 1 \geq (\pi_i \circ \hat{\ell})(y)$$

and

$$
\begin{aligned}
(\pi_i \circ \hat{\ell})^+(y) &= 1 + (\pi_i \circ \hat{\ell})(y) - (\pi_i \circ \hat{\ell})(0) \\
&\geq 1 + \inf\{(\pi_i \circ \hat{\ell})(y * y), (\pi_i \circ \hat{\ell})(z)\} - (\pi_i \circ \hat{\ell})(0) \\
&= \inf\{1 + (\pi_i \circ \hat{\ell})(y * z) - (\pi_i \circ \hat{\ell})(0), 1 + (\pi_i \circ \hat{\ell})(z) - (\pi_i \circ \hat{\ell})(0)\} \\
&= \inf\{(\pi_i \circ \hat{\ell})^+(y * z), (\pi_i \circ \hat{\ell})^+(z)\}
\end{aligned}
$$

for all for $i = 1, 2, \cdots, m$. This shows that $\hat{\ell}^+$ is an m-polar $(\in, \in)$-fuzzy ideal of X, and it is normal by Theorem 7. Suppose that $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X. Then $\hat{\ell}^+$ is an m-polar $(\in, \in)$-fuzzy ideal of X by Theorem 2. For any $y \in X$, we get

$$
\begin{aligned}
(\pi_i \circ \hat{\ell})^+(y) &= 1 + (\pi_i \circ \hat{\ell})(y) - (\pi_i \circ \hat{\ell})(0) \\
&\geq 1 + (\pi_i \circ \hat{\ell})(0 * (0 * y)) - (\pi_i \circ \hat{\ell})(0) \\
&= (\pi_i \circ \hat{\ell})^+(0 * (0 * y))
\end{aligned}
$$

for all for $i = 1, 2, \cdots, m$. It follows from Theorem 4 that $\hat{\ell}^+$ is an m-polar $(\in, \in)$-fuzzy p-ideal of X. It is clear that $\hat{\ell}$ is contained in $\hat{\ell}^+$. $\square$

The following example illustrates Theorem 8.

Example 5. *(1) Consider the* 3*-polar* $(\in, \in)$*-fuzzy ideal* $\hat{\ell}$*, which is not normal, of* X *in Example 3. Then* $\hat{\ell}^+$ *is given as follows:*

$$
\hat{\ell}^+ : X \to [0,1]^3,\ y \mapsto \begin{cases} (1,1,1) & \text{if } y = 0, \\ (0.8, 0.9, 1) & \text{if } y = 1, \\ (0.6, 0.5, 0.6) & \text{if } y = a, \\ (0.7, 0.7, 0.9) & \text{if } y = b, \\ (0.6, 0.5, 0.6) & \text{if } y = c, \end{cases}
$$

and it is a normal 3*-polar* $(\in, \in)$*-fuzzy ideal of* X *that contains* $\hat{\ell}$*.*

(2) If we take the 5*-polar* $(\in, \in)$*-fuzzy* p*-ideal* $\hat{\ell}$ *of* X *in Example 2, then it is not normal and* $\hat{\ell}^+$ *is given as follows:*

$$
\hat{\ell}^+ : X \to [0,1]^5,\ y \mapsto \begin{cases} (1,1,1,1,1) & \text{if } y = 0, \\ (0.8, 1, 0.9, 0.9, 0.8) & \text{if } y = 1, \\ (0.8, 0.8, 0.9, 0.8, 0.8) & \text{if } y = 2, \\ (0.8, 0.8, 0.9, 0.8, 0.8) & \text{if } y = 3, \end{cases}
$$

which is a normal 5*-polar* $(\in, \in)$*-fuzzy* p*-ideal of* X *containing* $\hat{\ell}$*.*

Theorem 9. *Let* $\hat{\ell}$ *be an* m*-polar* $(\in, \in)$*-fuzzy ideal (resp.* m*-polar* $(\in, \in)$*-fuzzy* p*-ideal) of a BCI-algebra* X*. Then* $\hat{\ell}$ *is normal if and only if* $\hat{\ell}^+ = \hat{\ell}$*.*

Proof. The sufficiency is clear. Assume that $\hat{\ell}$ is normal. Then

$$
(\pi_i \circ \hat{\ell})^+(y) = 1 + (\pi_i \circ \hat{\ell})(y) - (\pi_i \circ \hat{\ell})(0) = (\pi_i \circ \hat{\ell})(y)
$$

for all $y \in X$ by Theorem 7, completing the proof. $\square$

Corollary 2. *Let* $\hat{\ell}$ *be an* m*-polar* $(\in, \in)$*-fuzzy ideal (resp.* m*-polar* $(\in, \in)$*-fuzzy* p*-ideal) of a BCI-algebra* X*. If* $\hat{\ell}$ *is normal, then* $(\hat{\ell}^+)^+ = \hat{\ell}$*.*

Theorem 10. *Let $\hat{\ell}$ be an m-polar $(\in,\in)$-fuzzy ideal (resp. m-polar $(\in,\in)$-fuzzy p-ideal) of a BCI-algebra X. If there exists an m-polar $(\in,\in)$-fuzzy ideal (resp. m-polar $(\in,\in)$-fuzzy p-ideal) $\hat{\vartheta}$ of X satisfying $\hat{\vartheta}^+ \subseteq \hat{\ell}$, then $\hat{\ell}$ is normal and $\hat{\ell}^+ = \hat{\ell}$.*

Proof. Straightforward. □

Theorem 11. *Let X be a BCI-algebra, and let $\hat{\ell}$ be a non-constant normal m-polar $(\in,\in)$-fuzzy ideal (resp. m-polar $(\in,\in)$-fuzzy p-ideal) of X which is maximal in the poset of normal m-polar $(\in,\in)$-fuzzy ideals (resp. m-polar $(\in,\in)$-fuzzy p-ideals) under set inclusion. Then $\hat{\ell}$ has the values $\hat{0}$ and $\hat{1}$ only.*

Proof. Since $\hat{\ell}$ is normal, we have $\hat{\ell}(0) = \hat{1}$. Let $y \in X$ be such that $\hat{\ell}(y) \neq \hat{1}$. It is sufficient to show that $\hat{\ell}(y) = \hat{0}$. If $\hat{\ell}(y) \neq \hat{0}$, then there exists $a \in X$ such that $\hat{0} < \hat{\ell}(a) < \hat{1}$. Let $\hat{\vartheta}$ be an mp-fuzzy set on X given by

$$\hat{\vartheta} : X \to [0,1]^m,\ y \mapsto \tfrac{1}{2}\left(\hat{\ell}(y) + \hat{\ell}(a)\right).$$

It is clear that $\hat{\vartheta}$ is well-defined. For any $y, z \in X$, we have

$$\hat{\vartheta}(0) = \tfrac{1}{2}\left(\hat{\ell}(0) + \hat{\ell}(a)\right) = \tfrac{1}{2}\left(\hat{1} + \hat{\ell}(a)\right) \geq \tfrac{1}{2}\left(\hat{\ell}(y) + \hat{\ell}(a)\right) = \hat{\vartheta}(y)$$

and

$$\begin{aligned}
\hat{\vartheta}(y) &= \tfrac{1}{2}\left(\hat{\ell}(y) + \hat{\ell}(a)\right) \\
&\geq \tfrac{1}{2}\left(\inf\left\{\hat{\ell}(y*z), \hat{\ell}(z)\right\} + \hat{\ell}(a)\right) \\
&= \tfrac{1}{2}\inf\{\hat{\ell}(y*z) + \hat{\ell}(a), \hat{\ell}(z) + \hat{\ell}(a)\} \\
&= \inf\left\{\tfrac{1}{2}(\hat{\ell}(y*z) + \hat{\ell}(a)), \tfrac{1}{2}(\hat{\ell}(z) + \hat{\ell}(a))\right\} \\
&= \inf\left\{\hat{\vartheta}(y*z), \hat{\vartheta}(z)\right\}.
\end{aligned}$$

Hence, $\hat{\vartheta}$ is an m-polar $(\in,\in)$-fuzzy ideal of X. In addition, we have

$$\hat{\vartheta}(y) = \tfrac{1}{2}\left(\hat{\ell}(y) + \hat{\ell}(a)\right) \geq \tfrac{1}{2}\left(\hat{\ell}(0*(0*y)) + \hat{\ell}(a)\right) = \hat{\vartheta}(0*(0*y))$$

for all $y \in X$. Hence, $\hat{\vartheta}$ is an m-polar $(\in,\in)$-fuzzy p-ideal of X by Theorem 4. Now, we get

$$\hat{\vartheta}^+(y) = \hat{1} + \hat{\vartheta}(y) - \hat{\vartheta}(0) = \hat{1} + \tfrac{1}{2}\left(\hat{\ell}(y) + \hat{\ell}(a)\right) - \tfrac{1}{2}\left(\hat{\ell}(0) + \hat{\ell}(a)\right) = \tfrac{1}{2}\left(\hat{1} + \hat{\ell}(y)\right),$$

and so $\hat{\vartheta}^+(0) = \tfrac{1}{2}\left(\hat{1} + \hat{\ell}(0)\right) = \hat{1}$, which shows that $\hat{\vartheta}$ is normal. Note that

$$\hat{\vartheta}^+(0) = \hat{1} > \hat{\vartheta}^+(a) = \tfrac{1}{2}\left(\hat{1} + \hat{\ell}(a)\right) > \hat{\ell}(a).$$

Hence, $\hat{\vartheta}^+$ is non-constant and $\hat{\ell}$ is not maximal, which is a contradiction. Therefore, $\hat{\ell}$ has the values $\hat{0}$ and $\hat{1}$ only. □

4. Quotient BCI-Algebras via *m*-Polar $(\in,\in)$-Fuzzy Ideals

Lemma 3 ([32] Proposition 3.9)**.** *Every m-polar $(\in,\in)$-fuzzy ideal of a BCI-algebra is order reversing.*

Let $\hat{\ell}$ be an m-polar $(\in,\in)$-fuzzy ideal of a BCI-algebra X. Define a binary relation $\approx$ on X related to $\hat{\ell}$ by

$$y \approx z \;\Leftrightarrow\; \hat{\ell}(0 * (y * z)^k) > 0,\, \hat{\ell}(0 * (z * y)^k) > 0, \tag{28}$$

that is,

$$y \approx z \;\Leftrightarrow\; (\pi_i \circ \hat{\ell})(0 * (y * z)^k) > 0,\, (\pi_i \circ \hat{\ell})(0 * (z * y)^k) > 0 \tag{29}$$

for all $y, z \in X$, $i = 1, 2, \cdots, m$ and a natural number k.

Lemma 4. *The relation $\approx$ is a congruence on X.*

Proof. It is clear that $\approx$ is reflexive and symmetric. Assume that $y \approx z$ and $z \approx x$. Then $(\pi_i \circ \hat{\ell})(0 * (y * z)^k) > 0$, $(\pi_i \circ \hat{\ell})(0 * (z * y)^k) > 0$, $(\pi_i \circ \hat{\ell})(0 * (z * x)^k) > 0$, and $(\pi_i \circ \hat{\ell})(0 * (x * z)^k) > 0$ for a natural number k. Since

$$\begin{aligned}(0 * (y * x)^k) * (0 * (y * z)^k) &= ((0 * y^k) * (0 * x^k)) * ((0 * y^k) * (0 * z^k)) \\ &\leq (0 * z^k) * (0 * x^k) \\ &= 0 * (z * x)^k\end{aligned}$$

by (9) and (I), it follows from Lemma 3 that

$$(\pi_i \circ \hat{\ell})((0 * (y * x)^k) * (0 * (y * z)^k)) \geq (\pi_i \circ \hat{\ell})(0 * (z * x)^k) > 0.$$

Since $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy ideal of X, we have

$$(\pi_i \circ \hat{\ell})(0 * (y * x)^k) \geq \inf\{(\pi_i \circ \hat{\ell})((0 * (y * x)^k) * (0 * (y * z)^k)), (\pi_i \circ \hat{\ell})(0 * (y * z)^k)\} > 0.$$

By a similar way, we have $(\pi_i \circ \hat{\ell})(0 * (x * y)^k) > 0$, and thus $y \approx x$. Hence, $\approx$ is an equivalence relation on X. Let $a, y, z \in X$ be such that $y \approx z$. Then

$$\begin{aligned}&(0 * ((y * a) * (z * a))^k) * (0 * (y * z)^k) \\ &= (((0 * y^k) * (0 * a^k)) * ((0 * z^k) * (0 * a^k))) * (0 * (y * z)^k) \\ &\leq ((0 * y^k) * (0 * z^k)) * (0 * (y * z)^k) \\ &= 0,\end{aligned}$$

and so

$$\begin{aligned}&(\pi_i \circ \hat{\ell})(0 * ((y * a) * (z * a))^k) \\ &\geq \inf\{(\pi_i \circ \hat{\ell})((0 * ((y * a) * (z * a))^k) * (0 * (y * z)^k)), (\pi_i \circ \hat{\ell})(0 * (y * z)^k)\} \\ &> 0.\end{aligned}$$

By a similar way, we get $(\pi_i \circ \hat{\ell})(0 * ((z * a) * (y * a))^k) > 0$. Hence, $y * a \approx z * a$. Now, we have

$$\begin{aligned}&(0 * ((a * y) * (a * z))^k) * (0 * (z * y)^k) \\ &= (((0 * a^k) * (0 * y^k)) * ((0 * a^k) * (0 * z^k))) * (0 * (z * y)^k) \\ &\leq ((0 * z^k) * (0 * y^k)) * (0 * (z * y)^k) \\ &= 0,\end{aligned}$$

which implies from Lemma 3 that

$$\begin{aligned}&(\pi_i \circ \hat{\ell})(0 * ((a * y) * (a * z))^k)\\ &\geq \inf\{(\pi_i \circ \hat{\ell})((0 * ((a * y) * (a * z))^k) * (0 * (z * y)^k)), (\pi_i \circ \hat{\ell})(0 * (z * y)^k)\}\\ &> 0.\end{aligned}$$

Similarly, we obtain $(\pi_i \circ \hat{\ell})(0 * ((a * z) * (a * y))^k) > 0$. Thus, $a * y \approx a * z$. It follows that if $y \approx z$ and $a \approx b$ for all $a, b, y, z \in X$, then $y * a \approx z * a \approx z * b$. Therefore, $\approx$ is a congruence on X. □

Denote by $[y]$ the equivalence class of an element y in X, and the quotient set of X by $\hat{\ell}$ is denoted by $X/\hat{\ell}$, and so

$$X/\hat{\ell} = \{[y] \mid y \in X\}.$$

Theorem 12. *Let $\hat{\ell}$ be an m-polar $(\in, \in)$-fuzzy ideal of a BCI-algebra X. Define a binary operation $\circledast$ on $X/\hat{\ell}$ by*

$$(\forall [y], [z] \in X/\hat{\ell})\left([y] \circledast [z] = [y * z]\right).$$

Then $(X/\hat{\ell}, \circledast, [0])$ is a BCI-algebra.

Proof. We first show that the binary operation $\circledast$ on $X/\hat{\ell}$ is well-defined. Assume that $[y] = [a]$ and $[z] = [b]$ for $a, b, y, z \in X$. Then $y \approx a$ and $z \approx b$. If $x \in [y] \circledast [z]$, then $x \approx y * z \approx a * b$, and so $x \in [a] \circledast [b]$. Thus, $[y] \circledast [z] \subseteq [a] \circledast [b]$. Similarly, we get $[a] \circledast [b] \subseteq [y] \circledast [z]$. Therefore, $\circledast$ is well-defined. Let $[y], [z], [x] \in X/\hat{\ell}$. Then

$$\begin{aligned}&(([y] \circledast [z]) \circledast ([y] \circledast [x])) \circledast ([x] \circledast [z])\\ &= ([y * z] \circledast [y * x]) \circledast [x * z]\\ &= [(y * z) * (y * x)] \circledast [x * z]\\ &= [((y * z) * (y * x)) * (x * z)]\\ &= [0].\end{aligned}$$

By a similar way, we have $([y] \circledast ([y] \circledast [z])) \circledast [z] = [0]$ and $[y] \circledast [y] = [0]$. Suppose that $[y] \circledast [z] = [0]$ and $[z] \circledast [y] = [0]$. Then $[y * z] = [0] = [z * y]$, and so $y * z \approx 0 \approx z * y$. It follows that $(\pi_i \circ \hat{\ell})(0 * (y * z)^k) = (\pi_i \circ \hat{\ell})(0 * ((y * z) * 0)^k) > 0$ and $(\pi_i \circ \hat{\ell})(0 * (z * y)^k) = (\pi_i \circ \hat{\ell})(0 * ((z * y) * 0)^k) > 0$. Thus, $y \approx z$, and so $[y] = [z]$. Consequently, $(X/\hat{\ell}, \circledast, [0])$ is a BCI-algebra. □

Theorem 13. *If $\hat{\ell}$ is an m-polar $(\in, \in)$-fuzzy ideal of a BCI-algebra X, then the mapping*

$$f : X/U(\hat{\ell}, \hat{r}) \to X/\hat{\ell},\ U(\hat{\ell}, \hat{r})_y \mapsto [y] \tag{30}$$

is an epimorphism. Moreover, if $\hat{\ell}(y) = \hat{0}$ for all $y \in X \setminus U(\hat{\ell}, \hat{r})$, then f is an isomorphism.

Proof. Let $y, z \in X$ be such that $U(\hat{\ell}, \hat{r})_y = U(\hat{\ell}, \hat{r})_z$. Then $y \sim z$ and so $0 * (y * z)^k \in U(\hat{\ell}, \hat{r})$ and $0 * (z * y)^k \in U(\hat{\ell}, \hat{r})$ for $k \in \mathbb{N}$. It follows that $(\pi_i \circ \hat{\ell})(0 * (y * z)^k) \geq r_i > 0$ and $(\pi_i \circ \hat{\ell})(0 * (z * y)^k) \geq r_i > 0$. Thus $y \approx z$, and so $[y] = [z]$. This shows that the map f is well-defined. For every $U(\hat{\ell}, \hat{r})_y, U(\hat{\ell}, \hat{r})_z \in X/U(\hat{\ell}, \hat{r})$, we get

$$\begin{aligned}f\left(U(\hat{\ell}, \hat{r})_y \,\tilde{*}\, U(\hat{\ell}, \hat{r})_z\right) &= f\left(U(\hat{\ell}, \hat{r})_{y*z}\right) = [y * z]\\ &= [y] \circledast [z] = f\left(U(\hat{\ell}, \hat{r})_y\right) \circledast f\left(U(\hat{\ell}, \hat{r})_z\right).\end{aligned}$$

Therefore, f is a homomorphism. It is clear that f is onto. Suppose that $U(\hat{\ell},\hat{r})_y \neq U(\hat{\ell},\hat{r})_z$ for some $y, z \in X$. Then $0*(z*y)^k \notin U(\hat{\ell},\hat{r})$ or $0*(z*y)^n \notin U(\hat{\ell},\hat{r})$ for some $k, n \in \mathbb{N}$. It follows from the hypothesis that $\hat{\ell}(0*(z*y)^k) = \hat{0}$ or $\hat{\ell}(0*(z*y)^n) = \hat{0}$. Hence, $[y] \neq [z]$. This completes the proof. □

5. Conclusions

Zhang introduced the notion of bipolar fuzzy sets, which is an extension of fuzzy sets. As an extension of bipolar fuzzy sets, Chen et al. have introduced an m-polar fuzzy set. The purpose of this paper is to apply the notion of m-polar fuzzy sets to p-ideals in BCI-algebras. We have introduced the concept of (normal) m-polar $(\in,\in)$-fuzzy p-ideals of BCI-algebras and have investigated several properties. We have discussed relations between an m-polar $(\in,\in)$-fuzzy p-ideal and an m-polar $(\in,\in)$-fuzzy ideal. We have provided an example of an m-polar $(\in,\in)$-fuzzy ideal that is not an m-polar $(\in,\in)$-fuzzy p-ideal. We have given conditions for an m-polar $(\in,\in)$-fuzzy ideal to be an m-polar $(\in,\in)$-fuzzy p-ideal and have considered the characterization of m-polar $(\in,\in)$-fuzzy p-ideals. Given an m-polar $(\in,\in)$-fuzzy ideal (resp., m-polar $(\in,\in)$-fuzzy p-ideal), we have defined a normal m-polar $(\in,\in)$-fuzzy ideal (resp., normal m-polar $(\in,\in)$-fuzzy p-ideal). Using an m-polar $(\in,\in)$-fuzzy ideal, we have constructed the quotient structure of BCI-algebras. There are several kinds of ideals in BCI-algebras, for example, p-ideal, q-ideal, a-ideal, BCI-implicative ideal, BCI-positive implicative ideal, BCI-commutative ideal, sub-implicative ideal, etc. These different kinds of ideals are basically very relevant to the ideal. Thus, the polarity of p-ideals as studied in this paper will be the basic step in the polarity study of other ideals. The purpose of our research in the future is to continue to think about these things and define new concepts in some algebraic structures.

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