mathematics

MDPI

Article

Direct and Inverse Fractional Abstract Cauchy Problems

Mohammed AL Horani [1], Angelo Favini [2,*] and Hiroki Tanabe [3]

[1] Department of Mathematics, The University of Jordan, Amman 11942, Jordan; horani@ju.edu.jo
[2] Dipartimento di Matematica, Universita di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy
[3] Takarazuka, Hirai Sanso 12-13, Osaka 665-0817, Japan; bacbx403@jttk.zaq.ne.jp
* Correspondence: angelo.favini@unibo.it

Received: 7 September 2019; Accepted: 21 October 2019; Published: 25 October 2019

check for updates

Abstract: We are concerned with a fractional abstract Cauchy problem for possibly degenerate equations in Banach spaces. This form of degeneration may be strong and some convenient assumptions about the involved operators are required to handle the direct problem. Moreover, we succeeded in handling related inverse problems, extending the treatment given by Alfredo Lorenzi. Some basic assumptions on the involved operators are also introduced allowing application of the real interpolation theory of Lions and Peetre. Our abstract approach improves previous results given by Favini–Yagi by using more general real interpolation spaces with indices θ, p, $p \in (0, \infty]$ instead of the indices θ, ∞. As a possible application of the abstract theorems, some examples of partial differential equations are given.

Keywords: fractional derivative; abstract Cauchy problem; C_0-semigroup; inverse problem

---

1. Introduction

Consider the abstract equation

$$BMu - Lu = f \tag{1}$$

where B, M, L are closed linear operators on the complex Banach space E, the domain of L is contained in domain of M, i.e., $D(L) \subseteq D(M)$, $0 \in \rho(L)$, the resolvent set of L, $f \in E$ and u is the unknown. The first approach to handle existence and uniqueness of the solution u to (1) was given by Favini–Yagi [1], see in particular the monograph [2]. By using real interpolation spaces, see [3,4], suitable assumptions on the operators B, M, L guarantee that (1) has a unique solution. Such a result was improved by Favini, Lorenzi and Tanabe in [5], see also [6–8]. In order to describe the results, we list the basic assumptions:

(H_1) Operator B has a resolvent $(z - B)^{-1}$ for any $z \in \mathbb{C}$, $\mathrm{Re}\, z < a$, $a > 0$ satisfying

$$\|(z - B)^{-1}\|_{\mathcal{L}(E)} \leq \frac{c}{|Re\, z| + 1}, \quad Re\, z < a\,, \tag{2}$$

where $\mathcal{L}(E)$ denotes the space of all continuous linear operators from E into E.

(H_2) Operators L, M satisfy

$$\|M(\lambda M - L)^{-1}\|_{\mathcal{L}(E)} \leq \frac{c}{(|\lambda| + 1)^{\beta}} \tag{3}$$

for any $\lambda \in \Sigma_\alpha := \{z \in \mathbb{C} : \ Re\, z \geq -c(1 + |Im\, z|)^{\alpha}, \ c > 0, \ 0 < \beta \leq \alpha \leq 1\}$.

(H_3) Let A be the possibly multivalued linear operator $A = LM^{-1}$, $D(A) = M(D(L))$. Then A and B commute in the resolvent sense:

$$B^{-1}A^{-1} = A^{-1}B^{-1}.$$

Let $(E, D(B))_{\theta,\infty}$, $0 < \theta < 1$, denote the real interpolation space between E and $D(B)$. The main result holds

Theorem 1. *Let $\alpha + \beta > 1$, $2 - \alpha - \beta < \theta < 1$. Then under hypotheses (H_1)–(H_3), Equation (1) admits a unique strict solution u such that Lu, $BMu \in (E, D(B))_{\omega,\infty}$, $\omega = \theta - 2 + \alpha + \beta$, provided that $f \in (E, D(B))_{\theta,\infty}$.*

It is straightforward to verify that if B generates a bounded c_0-group in E, then assumption (H_1) holds for B. Analogously, if $-B$ generates a bounded c_0-semigroup in E, then assumption (H_1) holds for B. It was also shown, in a previous paper, that Theorem 1 works well for solving degenerate equations on the real axis, too, see [9].

The first aim of this paper is to extend Theorem 1 to the interpolation spaces $(E, D(B))_{\theta,p}$, $1 < p < \infty$. This affirmation is not immediate. Section 2 is devoted to this proof. In Section 3, we apply the abstract results to solve concrete differential equations. In Section 4, we handle related inverse problems. In Section 5, we study abstract equations generalizing second-order equations in time. In Section 6, we present our conclusions and remarks. For some related results, we refer to Guidetti [10] and Bazhlekova [11].

2. Fundamental Results

To begin with, we recall, from Favini–Yagi [2], p. 16, that if E_0, E_1 are two Banach spaces such that (E_0, E_1) is an interpolation couple, i.e., there exists a locally convex topological space X such that $E_i \subset X$, $i = 0, 1$, continuously, then the following injections

$$E_0 \cap E_1 \subset_d (E_0, E_1)_{\zeta,q} \subset_d (E_0, E_1)_{\eta,1} \subset_d (E_0, E_1)_{\eta,\infty} \subset_d (E_0, E_1)_{\xi,q} \subset E_0 + E_1$$

are true for $1 \le q < \infty$, $0 < \xi < \eta < \zeta < 1$, where $\subset_d$ denotes continuous and dense embedding. Moreover,

$$(E_0, E_1)_{\theta,q} \subset_d (E_0, E_1)_{\theta,r} \text{ for } 1 \le q < r < \infty, \quad 0 < \theta < 1.$$

Taking into account the previous embedding and Theorem 1, we easily deduce that if ϵ, ϵ_1 are suitable small positive numbers, since $(E, D(B))_{\theta+\epsilon,q} \subset (E, D(B))_{\theta,\infty}$, then Equation (1) admits a unique solution u with Lu, $BMu \in (E, D(B))_{\theta-2+\alpha+\beta,\infty}$ and Lu, $BMu \in (E, D(B))_{\theta-2+\alpha+\beta-\epsilon_1,q}$, that is a weaker result than case $q = \infty$.

Our aim is to extend Theorem 1 to $1 < p < \infty$. In order to establish the corresponding result, we need the following lemma concerning multiplicative convolution. We recall that $L^p_*(\mathbb{R}^+) = L^p\left(R^+; t^{-1}dt\right)$ and that the multiplicative convolution of two (measurable) functions $f, g : \mathbb{R}^+ \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_0^\infty f(xt^{-1})g(t)t^{-1}\,dt$$

where the integral exists a.e. for $x \in \mathbb{R}_+$.

Lemma 1. *For any $f_1 \in L^p_*(\mathbb{R}^+)$ and $g \in L^1_*(\mathbb{R}^+)$, the multiplicative convolution $f_1 * g \in L^p_*(\mathbb{R}^+)$ and satisfies*

$$\|f_1 * g\|_{L^p_*} \le \|f_1\|_{L^p_*} \|g\|_{L^1_*}$$

Consider now the chain of estimates

$$\begin{aligned} t^{\theta+\alpha+\beta-2}\|B(B+t)^{-1}v\| &\leq t^{\theta+\alpha+\beta-2}\int_0^\infty \frac{(1+y)^{3-\alpha-\beta-\theta}}{(1+y+t)}\,(1+y)^\theta\|B(B+1+y)^{-1}f\|\,\frac{dy}{1+y} \\ &\leq t^{\theta+\alpha+\beta-2}\int_0^\infty \frac{y^{3-\alpha-\beta-\theta}}{y+t}\,y^\theta\,\|B(B+y)^{-1}f\|\,\frac{dy}{y} \\ &= t^{\theta+\alpha+\beta-2}\int_0^\infty \frac{y^{3-\alpha-\beta-\theta}}{y(1+ty^{-1})}\,y^\theta\,\|B(B+y)^{-1}f\|\,\frac{dy}{y} \\ &= \int_0^\infty \frac{(ty^{-1})^{\theta+\alpha+\beta-2}}{1+ty^{-1}}\,y^\theta\,\|B(B+y)^{-1}f\|\,\frac{dy}{y} \end{aligned}$$

where

$$v = (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}B(B-z)^{-1}f\,dz, \quad T = ML^{-1},$$

$\Gamma = \Gamma_\alpha$ being the oriented contour

$$\Gamma = \{z = a - c(1+|y|)^\alpha + iy, \quad -\infty < y < \infty\},$$

with $a \in (c, c+a_0)$. Such a function v is the unique solution to $BTv - v = f$, that is, u with $v = Lu$ satisfies (1).

Let $f_1(y) = y^\theta\,\|B(B+y)^{-1}f\|$, $\quad g(y) = \dfrac{y^{\theta+\alpha+\beta-2}}{1+y}$, $\quad y \in \mathbb{R}^+$, and note that $f \in (E, D(B))_{\theta,p}$ if and only if $f_1 \in L^p_*(\mathbb{R}^+)$. Moreover, $g \in L^p_*(\mathbb{R}^+)$ since $\theta > 2-\alpha-\beta$ and obviously $\theta < 3-\alpha-\beta$. Therefore, from Lemma 1, we deduce that $f \in (E, D(B))_{\omega,p}$, where $\omega = \theta+\alpha+\beta-2$. Thus, we can establish the fundamental result concerning Equation (1) .

Theorem 2. *Let B, M, L be three closed linear operators on the Banach space E satisfying (H_1)–(H_3), $0 < \beta \leq \alpha \leq 1$. Then for all $f \in (E, D(B))_{\theta,p}$, $2-\alpha-\beta < \theta < 1$, $1 < p < \infty$, Equation (1) admits a unique solution u. Moreover, Lu, $BMu \in (E, D(B))_{\omega,p}$, $\omega = \theta+\alpha+\beta-2$.*

3. Fractional Derivative

Let $\tilde{\alpha} > 0$, $m = \lceil\tilde{\alpha}\rceil$ is the smallest integer greater or equal to $\tilde{\alpha}$, $I = (0, T)$ for some $T > 0$. Define

$$g_\beta(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\,t^{\beta-1} & t > 0, \\ 0 & t \leq 0, \end{cases} \qquad \beta \geq 0,$$

where $\Gamma(\beta)$ is the Gamma function. Note that $g_0(t) = 0$ because $\Gamma(0)^{-1} = 0$. The Riemann–Liouville fractional derivative of order $\tilde{\alpha}$, or, more precisely, the so-called left handed Riemann–Liouville fractional derivative of order $\tilde{\alpha}$, is defined for all $f \in L^1(I)$, $g_{m-\tilde{\alpha}} * f \in W^{m,1}(I)$ by

$$D_t^{\tilde{\alpha}} f(t) = D_t^m (g_{m-\tilde{\alpha}} * f)(t) = D_t^m\, J_t^{m-\tilde{\alpha}} f(t)$$

where $D_t^m := \dfrac{d^m}{dt^m}$, $m \in \mathbb{N}$. $D_t^{\tilde{\alpha}}$ is a left inverse of $J_t^{\tilde{\alpha}}$, but in general it is not a right inverse. The Riemann–Liouville fractional integral of order $\tilde{\alpha} > 0$ is defined as:

$$J_t^{\tilde{\alpha}} f(t) := (g_{\tilde{\alpha}} * f)(t),\ \ f \in L^1(I),\ t > 0,\ \ J_t^0 f(t) := f(t).$$

If X is a complex Banach space, $\tilde{\alpha} > 0$, then we define the operator $\mathcal{J}_{\tilde{\alpha}}$ as:

$$D\left(\mathcal{J}_{\tilde{\alpha}}\right) := L^p(I; X),\ \ \mathcal{J}_{\tilde{\alpha}} u = g_{\tilde{\alpha}} * u,\ \ p \in [1, \infty).$$

Define the spaces $R^{\tilde{\alpha},p}(I;X)$ and $R_0^{\tilde{\alpha},p}(I;X)$ as follows. If $\tilde{\alpha} \notin \mathbb{N}$, set

$$R^{\tilde{\alpha},p}(I;X) := \{u \in L^p(I;X) \,:\, g_{m-\tilde{\alpha}} * u \in W^{m,p}(I;X)\}\,,$$
$$R_0^{\tilde{\alpha},p}(I;X) := \left\{u \in L^p(I;X) \,:\, g_{m-\tilde{\alpha}} * u \in W_0^{m,p}(I;X)\right\}\,,$$

where

$$W_0^{m,p}(I;X) = \left\{y \in W^{m,p}(I;X),\ y^{(k)}(0) = 0,\ \ k = 0, 1, ..., m-1\right\}\,.$$

For the Sobolev space $W^{\beta,p}(I;X)$ of fractional order $\beta > 0$, we define

$$W_0^{\beta,p}(I;X) = \left\{y \in W^{\beta,p}(I;X),\ y^{(k)}(0) = 0,\ \ k = 0, 1, ..., \lfloor \beta - 1/p \rfloor\right\}\,,$$

$\beta - 1/p \notin \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\lfloor \beta - 1/p \rfloor$ is the greatest integer less or equal to $\beta - 1/p$.
If $\tilde{\alpha} \in \mathbb{N}$, we take

$$R^{\tilde{\alpha},p}(I;X) := W^{\tilde{\alpha},p}(I;X),\ \ R_0^{\tilde{\alpha},p}(I;X) := W_0^{\tilde{\alpha},p}(I;X)\,.$$

Denote the extensions of the operators of fractional differentiation in $L^p(I;X)$ by $\mathcal{L}_{\tilde{\alpha}}$, i.e.,

$$D(\mathcal{L}_{\tilde{\alpha}}) := R_0^{\tilde{\alpha},p}(I;X)\,,\ \ \mathcal{L}_{\tilde{\alpha}} u := D_t^{\tilde{\alpha}} u\,,$$

where $D_t^{\tilde{\alpha}}$ is the Riemann–Liouville fractional derivative. Notice that if $\tilde{\alpha} \in (0,1)$, $u \in D(\mathcal{L}_{\tilde{\alpha}})$, then $(g_{1-\tilde{\alpha}} * u)(0) = 0$.

We illustrate the previous abstract concepts in the following example

Example 1. *For $u \in L^p(I,X) = L^p(0,T;X)$ set $u(t) = 0$ for $t < 0$. Then, if $u \in W_0^{1,p}(I,X)$, we have $u \in W^{1,p}(-\infty,T;X)$. Let $U(\tau)$, $\tau \geq 0$, be the semigroup in $L^p(0,T;X)$ defined by*

$$(U(\tau)u)(t) = u(t-\tau),\ \ t \in I.$$

Clearly $U(\tau) = 0$ if $\tau > T$. For $\mathrm{Re}\lambda > 0$, $t > 0$

$$\begin{aligned}\left(\int_0^\infty e^{-\lambda\tau} U(\tau) u d\tau\right)(t) &= \int_0^\infty e^{-\lambda\tau}(U(\tau)u)(t)d\tau = \int_0^\infty e^{-\lambda\tau} u(t-\tau)d\tau \\ &= \int_0^t e^{-\lambda\tau} u(t-\tau)d\tau = \int_0^t e^{-\lambda(t-s)} u(s)ds. \end{aligned} \tag{4}$$

Since $D(D_t) = W_0^{1,p}(I,X)$, $D_t = d/dt$, equation $(\lambda + D_t)u = f$ is

$$\begin{cases} \lambda u(t) + u'(t) = f(t),\ 0 < t < T, \\ u(0) = 0. \end{cases}$$

The solution is

$$u(t) = \int_0^t e^{-\lambda(t-s)} f(s)ds,$$

i.e.,

$$((\lambda + D_t)^{-1} f)(t) = \int_0^t e^{-\lambda(t-s)} f(s)ds, \quad \lambda > 0. \tag{5}$$

From (4) and (5) it follows that

$$(\lambda + D_t)^{-1} = \int_0^\infty e^{-\lambda\tau} U(\tau)d\tau, \quad \lambda > 0.$$

Therefore $-D_t$ *is the infinitesimal generator of the semigroup* $U(\tau)$, $\tau \geq 0$. *Let* $\tilde{\alpha} > 0$. *Then for* $f \in L^p(I, X)$

$$\begin{aligned}(D_t^{-\tilde{\alpha}} f)(t) &= \frac{1}{\Gamma(\tilde{\alpha})}\int_0^\infty \tau^{\tilde{\alpha}-1}(U(\tau)f)(t)d\tau = \frac{1}{\Gamma(\tilde{\alpha})}\int_0^t \tau^{\tilde{\alpha}-1} f(t-\tau)d\tau \\ &= \frac{1}{\Gamma(\tilde{\alpha})}\int_0^t (t-s)^{\tilde{\alpha}-1} f(s)ds,\ 0 \leq t \leq T.\end{aligned}$$

If $m \in \mathbb{N}$, $m-1 < \tilde{\alpha} < m$,

$$\begin{aligned}&(D_t^{-\tilde{\alpha}} f)(t) = 0\ \forall t \in [0,T] \Longleftrightarrow \int_0^t (t-s)^{\tilde{\alpha}-1} f(s)ds = 0\ \forall t \in [0,T] \\ &\Longrightarrow \int_0^\tau (\tau-t)^{m-\tilde{\alpha}-1}\int_0^t (t-s)^{\tilde{\alpha}-1} f(s)dsdt = 0\ \forall \tau \in [0,T] \\ &\Longrightarrow \int_0^\tau \int_s^\tau (\tau-t)^{m-\tilde{\alpha}-1}(t-s)^{\tilde{\alpha}-1}dt f(s)ds = 0\ \forall \tau \in [0,T] \\ &\Longrightarrow \frac{\Gamma(m-\tilde{\alpha})\Gamma(\tilde{\alpha})}{\Gamma(m)}\int_0^\tau (\tau-s)^{m-1} f(s)ds = 0\ \forall \tau \in [0,T] \Longrightarrow f(\tau) = 0\ \forall \tau \in [0,T].\end{aligned}$$

If $\tilde{\alpha} = m \in \mathbb{N}$ *and*

$$(D_t^{-\tilde{\alpha}} f)(t) = (D_t^{-m} f)(t) = \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} f(s)ds = 0\ \forall t \in [0,T] \Longrightarrow f(t) = 0\ \forall t \in [0,T].$$

Therefore $D_t^{-\tilde{\alpha}}$ *has an inverse which is denoted by* $D_t^{\tilde{\alpha}}$.*We have*

$$D_t^{\tilde{\alpha}+\beta} = D_t^{\tilde{\alpha}} D_t^{\beta} \quad \forall \tilde{\alpha}, \beta \in \mathbb{R}.$$

Therefore, if $m \in \mathbb{N}$, $m-1 < \tilde{\alpha} < m$,

$$D_t^{\tilde{\alpha}} = D_t^m D_t^{\tilde{\alpha}-m} = D_t^m g_{m-\tilde{\alpha}} *, \quad D(D_t^{\tilde{\alpha}}) = \{u; g_{m-\tilde{\alpha}} * u \in D(D_t^m) = W_0^{m,p}(I;X)\} = R_0^{\tilde{\alpha},p}(I;X).$$

Let us now list the main properties of $\mathcal{L}_{\tilde{\alpha}}$, see [11], Lemma 1.8, p. 15.

Lemma 2. *Let* $\tilde{\alpha} > 0$, $1 < p < \infty$, *X a complex Banach space, and* $\mathcal{L}_{\tilde{\alpha}}$ *be the operator introduced above. Then*
(a) $\mathcal{L}_{\tilde{\alpha}}$ *is closed, linear and densely defined*
(b) $\mathcal{L}_{\tilde{\alpha}} = \mathcal{J}_{\tilde{\alpha}}^{-1}$
(c) $\mathcal{L}_{\tilde{\alpha}} = \mathcal{L}_1^{\tilde{\alpha}}$, *the* $\tilde{\alpha}-$*th power of the operator* $\mathcal{L}_1$
(d) if $\tilde{\alpha} \in (0,2)$, *operator* $\mathcal{L}_{\tilde{\alpha}}$ *is positive with spectral angle* $\omega_{\mathcal{L}_{\tilde{\alpha}}} = \tilde{\alpha}\pi/2$
(e) if $\tilde{\alpha} \in (0,1]$, *then* $\mathcal{L}_{\tilde{\alpha}}$ *is m-accretive*
(f) $R_0^{\tilde{\alpha},p}(I;X) \hookrightarrow C^{\tilde{\alpha}-1/p}(I;X)$, $\tilde{\alpha} > 1/p$, $\tilde{\alpha} - 1/p \notin \mathbb{N}$, *see [11], Theorem 1.10, p. 17*
(g) if $\tilde{\alpha}\gamma - 1/p \notin \mathbb{N}_0$,

$$\left(L^p(I;X), R_0^{\tilde{\alpha},p}(I;X)\right)_{\gamma,p} = W_0^{\tilde{\alpha}\gamma,p}(I;X),$$

see [11], Proposition 11, p. 18 .

Statement (e) implies that if $\tilde{\alpha} \in (0,1]$,

$$\|\lambda(\lambda + \mathcal{L}_{\tilde{\alpha}})^{-1}\|_{L^p(I;X)} \leq C, \ \ |\arg\lambda| < \pi(1 - \frac{\tilde{\alpha}}{2}).$$

However, this reads equivalently $\|(\lambda - \mathcal{L}_{\tilde{\alpha}})^{-1}\| \leq C/|\lambda|$ provided that λ is in a sector of the complex plane containing $Re\,\lambda \leq 0$. Therefore, if $\tilde{\alpha} \leq 1$, operator $\mathcal{L}_{\tilde{\alpha}} = \dfrac{d}{dt^{\tilde{\alpha}}} = D_t^{\tilde{\alpha}}$ satisfies assumption (H_1) in Theorem 1. Therefore, we can handle abstract equations of the type

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + f(t), \quad 0 \leq t \leq T,$$

in a Banach space X with an initial condition $\left(g_{1-\tilde{\alpha}} * u\right)(0) = 0$. Then the results follow easily from the abstract model.

Example 2. *Let M be the multiplication operator in $L^p(\Omega)$, Ω a bounded open set in $\mathbb{R}^n$ with a C^n boundary $\partial\Omega$, $1 < p < \infty$, by $m(x)$, m is continuous and bounded, and take $L = \Delta - c$, $D(L) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $c > 0$. Then it is seen in Favini–Yagi [2], pp. 79–80,*

$$\|M(zM - L)^{-1}f\|_{L^p(\Omega)} \leq \frac{c}{(1+|z|)^{1/p}}\,\|f\|_{L^p(\Omega)}$$

for all z in a sector containing $Re\,z \geq 0$.

In order to solve our problem, $0 < \tilde{\alpha} \leq 1$,

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + f(t), \quad 0 \leq t \leq T,$$

we must recall, see (g) in Lemma 2, that if $\tilde{\alpha}\gamma - 1/p \notin \mathbb{N}$, the interpolation space

$$\left(L^p(I;X), R_0^{\tilde{\alpha},p}(I;X)\right)_{\gamma,p} = W_0^{\tilde{\alpha}\gamma,p}(I;X).$$

Therefore, using Theorem 2, for any $f \in W_0^{\tilde{\alpha}\theta,p}(I;X)$, $1 - \frac{1}{p} < \theta < 1$, $1 < p < \infty$, $\tilde{\alpha}\theta - \frac{1}{p} \notin \mathbb{N}_0$, the problem above admits a unique strict solution y such that

$$\Delta y,\ D_t^{\tilde{\alpha}} m(\cdot)y \in W_0^{\tilde{\alpha}(\theta+\frac{1}{p}-1),p}(I;X).$$

Remark 1. *Since $\frac{-1}{p} < \tilde{\alpha}\theta - \frac{1}{p} < 1$, then the only integer that $\tilde{\alpha}\theta - \frac{1}{p}$ can take is the zero integer.*

We refer to to the monograph [2] for many further examples of concrete degenerate partial differential equations to which Theorem 2 applies.

4. Inverse Problems

Given the problem

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + f(t)z + h(t), \quad 0 \leq t \leq T, \tag{6}$$

then corresponding to an initial condition and following the strategy in various previous papers, see in particular Lorenzi [12], we could study existence and regularity of solutions (y, f) to the above problem such that $\Phi[My(t)] = g(t)$, where g is a complex-valued function on $[0, T]$. This is, of course, an inverse problem. Applying Φ to both sides of Equation (6) we get

$$D_t^{\tilde{\alpha}} g(t) = \Phi[Ly(t)] + \Phi[h(t)] + f(t)\Phi[z].$$

If $\Phi[z] \neq 0$, we obtain necessarily

$$f(t) = \frac{D_t^{\tilde{\alpha}} g(t) - \Phi[Ly(t)] - \Phi[h(t)]}{\Phi[z]}.$$

Therefore,

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + h(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}\,z - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{D_t^{\tilde{\alpha}}g(t)}{\Phi[z]}\,z\,.$$

If L_1 is defined by

$$D(L_1) = D(L), \quad L_1 y = -\frac{\Phi[Ly(t)]}{\Phi[z]}\,z\,,$$

one can introduce assumptions on the given operators ensuring that the direct problem

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + L_1 y + h(t) - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{D_t^{\tilde{\alpha}}g(t)}{\Phi[z]}\,z$$

has a unique strict solution, see [13]. The main step is to verify that assumption (H_2) holds for the operators $L + L_1$ and M.

Introduce the multivalued linear operator $A := LM^{-1}$, $D(A) = M(D(L))$ such that (H_2) holds. This means that $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \dfrac{c}{(|\lambda| + 1)^{\beta}}$, $\lambda \in \Sigma_\alpha$. Theorem 1 in [13], pp. 148–149, affirms that if L, L_1, M are closed linear operators on X, $D(L) \subseteq D(L_1) \subseteq D(M)$, $0 \in \rho(L)$, such that (H_2) holds and $L_1 \in \mathcal{L}\left(D(L), X_A^{\theta_1}\right)$, $1 - \beta < \theta_1 < 1$, where

$$X_A^{\theta_1} = \left\{u \in X,\ \sup_{t>0} t^{\theta_1}\|A^0(t - A)^{-1}u\|_X < \infty\right\},$$

with $A^0(t - A)^{-1} = -I + t(t - A)^{-1}$, then

$$\|M(\lambda M - L - L_1)^{-1}\|_{\mathcal{L}(X)} \leq c(1 + |\lambda|)^{-\beta}, \quad \forall\, \lambda \in \Sigma_\alpha\,,\ |\lambda| \text{ large}\,.$$

In order to apply this theorem in our case, we must suppose that z belongs to $X_A^{\theta_1}$ for some $\theta_1 \in (1 - \beta, 1)$. Then

$$\begin{aligned} &D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + L_1 y + h(t) - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{D_t^{\tilde{\alpha}}g(t)}{\Phi[z]}\,z \\ &\big(g_{1-\tilde{\alpha}} * My\big)(0) = 0 \end{aligned}$$

will admit a unique strict solution y provided that

$$h(t) - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{D_t^{\tilde{\alpha}}g(t)}{\Phi[z]}\,z \in W_0^{\tilde{\alpha}\theta,p}(I;X)$$

with $\tilde{\alpha}\theta - 1/p \notin \mathbb{N}$ and then $D_t^{\tilde{\alpha}}(My(t))$ and $Ly(t) \in W_0^{\tilde{\alpha}(\theta+\alpha+\beta-2),p}(I;X)$, $\tilde{\alpha}(\theta + \alpha + \beta - 2) \notin \mathbb{N}$. Notice that if $\tilde{\alpha}\theta - 1/p \notin \mathbb{N}$, then $\tilde{\alpha}(\theta + \alpha + \beta - 2) - 1/p = \tilde{\alpha}\theta - \frac{1}{p} + \tilde{\alpha}(\alpha + \beta - 2) \notin \mathbb{N}$.

5. Application: Generalized Second-Order Abstract Equation

Let us consider the abstract equation, generalizing second-order equation in time,

$$B_2 C B_1 u + B B_1 u + A u = f$$

where A, B, C are some closed linear operators in the complex Banach space X, B_1, B_2 are suitable operators defined on suitable Banach spaces. The change of variables $B_1 u = v$ transforms the given equation to the system

$$B_1 u = v\,,$$
$$B_2 C v + B v + A u = f\,,$$

which can be written in the matrix form

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & -I \\ A & B \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

The basic idea is to use a convenient space and a domain of operator matrices. Noting

$$\mathbb{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad M = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}, \quad L = \begin{bmatrix} 0 & I \\ -A & -B \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

it assumes the form

$$(\mathbb{B}M - L)U = F\,, \quad U = (u, v)^T\,.$$

In order to simplify the argument, we take $D(B) \subseteq D(A) \cap D(C)$. Moreover, we assume that for all $z \in \Sigma_\alpha$, where

$$\Sigma_\alpha := \Big\{ z \in \mathbb{C} :\; Re\, z \geq -c(1 + |Im\, z|)^\alpha,\; c > 0,\; 0 < \beta \leq \alpha \leq 1\,, \quad \alpha + \beta > 1 \Big\}\,,$$

the involved operators satisfy

$$\|C(zC + B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c_1}{(1 + |z|)^\beta}\,, \tag{7}$$

which guarantees that the problem is of parabolic type. Take $Y = D(B) \times X$ with the usual product norm. Then it is shown in Favini–Yagi [2], page 184, that the resolvent estimate

$$\|M(zM - L)^{-1}\|_{\mathcal{L}(Y)} \leq \frac{c}{(1 + |z|)^\beta}\,, \quad \forall z \in \Sigma_\alpha$$

holds. Therefore assumption (H_2) is satisfied.

Take B_1 the Riemann–Liouville fractional derivative of order $\tilde{\alpha}$, $0 < \tilde{\alpha} \leq 1$, in $L^p(0, T; D(B))$, $1 < p < \infty$; similarly, take B_2 the Riemann–Liouville fractional derivative of order $\tilde{\beta}$, $0 < \tilde{\beta} \leq 1$, in $L^p(0, T; X)$, $1 < p < \infty$. Then assumptions (H_1) and (H_2) hold. Therefore, according to Theorem 2, see also Bazhlekova [11], problem

$$D_t^{\tilde{\alpha}} u = v\,,$$
$$D_t^{\tilde{\beta}} C v + B v + A u = f(t)\,, \quad 0 \leq t \leq T\,,$$
$$\big(g_{1-\tilde{\alpha}} * u\big)(0) = 0\,, \quad \big(g_{1-\tilde{\beta}} * Cv\big)(0) = 0$$

admits a unique strict solution (u, v) in $L^p(0, T; D(B)) \times L^p(0, T; X)$, provided that $D(B) \subseteq D(A) \cap D(C)$, $f \in W_0^{\tilde{\beta}\theta, p}(I; X)$, $2 - \alpha - \beta < \theta < 1$, $0 < \beta \leq \alpha \leq 1$, $\alpha + \beta > 1$, $\tilde{\beta}\theta - 1/p \neq 0$. Moreover, $D_t^{\tilde{\alpha}} u = v \in W_0^{\tilde{\alpha}\omega, p}(I; D(B))$, $D_t^{\tilde{\beta}} C v \in W_0^{\tilde{\beta}\omega, p}(I; X)$, $Au + Bv \in W_0^{\tilde{\beta}\omega, p}(I; X)$, $\omega = \theta + \alpha + \beta - 2$, $\tilde{\alpha}\omega = \tilde{\alpha}\theta + \tilde{\alpha}(\alpha + \beta - 2) - \frac{1}{p} \notin \mathbb{N}_0$, $\tilde{\beta}\omega = \tilde{\beta}\theta + \tilde{\beta}(\alpha + \beta - 2) - \frac{1}{p} \notin \mathbb{N}_0$.

Example 3. *Consider the problem*

$$D_t^{\tilde{\beta}}\left(m(x) D_t^{\tilde{\alpha}} u\right) - \Delta D_t^{\tilde{\alpha}} u + A(x; D) u = f(x, t)\,, \quad (x, t) \in \Omega \times [0, T]\,,$$
$$u(x, 0) = 0\,, \quad x \in \Omega\,,$$
$$m(x) D_t^{\tilde{\alpha}} u(x, 0) = 0\,, \quad x \in \Omega\,,$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $m \in L^\infty$, $m(x) \geq 0$ in Ω, $A(x;D)$ is a second order linear differential operator on Ω with continuous coefficients in $\overline{\Omega}$, $f(x,t)$ is a scalar valued continuous function on $\overline{\Omega} \times [0,T]$, then we take $B = -\Delta_1$, the Laplacian with respect to x, $D(B) = H_0^1(\Omega) \cap H^2(\Omega)$, $X = L^2(\Omega)$. Therefore, (H_2) holds with $\alpha = 1$, $\beta = 1/2$.

6. Conclusions

It was shown that the degenerate problem including Riemann–Liouville fractional derivative can be handled by means of a general abstract equation. Applications to degenerate fractional differential equations with some related inverse problems were studied. Moreover, generalized second-order abstract equations were well-treated.

Author Contributions: All authors have equally contributed to this work. All authors wrote, read, and approved the final manuscript.

Funding: There is no external fund.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Favini, A.; Yagi, A. Multivalued Linear Operators and Degenerate Evolution Equations. *Ann. Mat. Pura Appl.* **1993**, *163*, 353–384. [CrossRef]
2. Favini, A.; Yagi, A. *Degenerate Differential Equations in Banach Spaces*; Marcel Dekker. Inc.: New York, NY, USA, 1999.
3. Lions, J.L.; Peetre, J. Sur Une Classe d'Espaces d'Interpolation,Publications Mathmatiques de l'IHS. *Publ. Math. Inst. Hautes Etudes Sci.* **1964**, *19*, 5–68. [CrossRef]
4. Lions, J.L.; Magenes, E. *Non-Homogeneous Boundary Value Problems and Applications*; Springer: Berlin, Germany, 1972; Volume 1, p. 187.
5. Favini, A.; Lorenzi, A.; Tanabe, H. Degenerate Integrodifferential Equations of Parabolic Type With Robin Boundary Conditions: L^p-Theory. *J. Math. Anal. Appl.* **2017**, *447*, 579–665. [CrossRef]
6. Favini, A.; Lorenzi, A.; Tanabe, H. Direct and Inverse Degenerate Parabolic Differential Equations with Multi-Valued Operators. *Electron. J. Differ. Equ.* **2015**, *2015*, 1–22.
7. Favini, A.; Lorenzi, A.; Tanabe, H. Singular integro-differential equations of parabolic type. *Adv. Differ. Equ.* **2002**, *7*, 769–798.
8. Favini, A.; Tanabe, H. Degenerate Differential Equations and Inverse Problems. In Proceedings of the Partial Differential Equations, Osaka, Japan, 21–24 August 2013; pp. 89–100.
9. Al Horani, M.; Fabrizio, M.; Favini, A.; Tanabe, H. Direct and Inverse Problems for Degenerate Differential Equations. *Ann. Univ. Ferrara* **2018**, *64*, 227–241. [CrossRef]
10. Guidetti, D. On Maximal Regularity For The Cauchy-Dirichlet Mixed Parabolic Problem with Fractional Time Derivative. *arXiv* **2018**, arXiv:1807.05913.
11. Bazhlekova, E.G. *Fractional Evolution Equations in Banach Spaces*; Eindhoven University of Technology: Eindhoven, The Netherlands, 2001.
12. Lorenzi, A. *An Introduction to Identification Problems Via Functional Analysis*; VSP: Utrecht, The Netherland, 2001.
13. Favini, A.; Lorenzi, A.; Marinoschi, G.; Tanabe, H. Perturbation Methods and Identifcation Problems for Degenerate Evolution Systems. In *Advances in Mathematics, Contributions at the Seventh Congress of Romanian Mathematicians, Brasov, 2011*; Beznea, L., Brinzanescu, V., Iosifescu, M., Marinoschi, G., Purice, R., Timotin, D., Eds.; Publishing House of the Romanian Academy: Bucharest, Romania, 2013; pp. 145–156.