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Existence and Iterative Method for Some Riemann Fractional Nonlinear Boundary Value Problems

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Abstract: In this paper, we prove the existence and uniqueness of solution for some Riemann–Liouville fractional nonlinear boundary value problems. The positivity of the solution and the monotony of iterations are also considered. Some examples are presented to illustrate the main results. Our results generalize those obtained by Wei et al (Existence and iterative method for some fourth order nonlinear boundary value problems. Appl. Math. Lett. 2019, 87, 101–107.) to the fractional setting.

Keywords: fractional differential equation; Green's function; existence and uniqueness of solution; positivity of solution; iterative method

1. Introduction

Forth-order boundary value problems, can be used to model the deformation of the elastic beam, which is considered to be one of the most used elements in structures such as bridges, buildings and aircraft (see, for instance, [1,2]).

In the literature problems of the form

$$u^{(4)}(x) = f(x, u(x), u''(x)),\ x \in (0,1), \tag{1}$$

subject to different types of boundary conditions have been extensively studied (see, for example, [1–11] and the references therein).

Under adequate conditions imposed on f and using different approach, the existence, uniqueness and qualitative properties of solutions have been considered.

In [1], Aftabizadeh considered Equation (1) together with the boundary conditions:

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{2}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Under adequate conditions imposed on f he proved that problem (1)–(2) has a unique solution. To do this, he transforms Equation (1) into a second-order integro-differential equation and apply the Schauder's fixed point theorem.

In [4], by using the method of lower and upper solutions for a fourth-order equation and some restrictive conditions on f, Bai established an existence result to problem (1)–(2).

In [7], Dang et al., to prove the existence and uniqueness of a solution of the problem (1)–(2), they reduced the problem to an operator equation for the right-hand side function and proved the contraction of the operator under some convenient conditions on f. The positivity of the solution and the monotony of iterations are also considered. This idea was also used by Dang and Qey for cantilever beam equation [12].

Recently, in [11], Wei et al. considered the following problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u'(x)), & t \in (0,1), \\ u(0) = u'(0) = u'(1) = u''(1) = 0. \end{cases} \tag{3}$$

Observe that problem (3) cannot be reduced to two second-order problems. Nevertheless, following the idea developed in [7], they proved the existence and uniqueness of this problem.

Motivated by the mentioned works, in this paper, we generalize the results obtained in [11] to the fractional setting.

More precisely, we are concerned with the following problem

$$\begin{cases} D^{\beta}(D^{\alpha}u)(x) = f(x, u(x), D^{\alpha}u(x)), & x \in (0,1), \\ u(0) = D^{\alpha}u(0) = D^{\alpha}u(1) = (D^{\alpha}u)'(1) = 0, \end{cases} \tag{4}$$

where $0 < \alpha \leq 1$, $2 < \beta \leq 3$, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ continuous function satisfying some adequate assumptions. Here D^{α} (resp. D^{β}) denotes the Riemann–Liouville fractional derivative of order α (resp. β).

It is worth mentioning that many authors studied fractional differential equations which were applied in many fields such as physics, mechanics, chemistry, and engineering; (see, for instance [13–32] and the references therein).

Following a different approach, they addressed the question of existence and uniqueness of positive continuous solution.

In [31], the authors considered the two-dimensional fractional Schrödinger equation (FSE) without potential

$$i\frac{\partial \psi}{\partial z} - \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)^{\frac{\alpha}{2}} \psi = 0, \tag{5}$$

for the slowly varying envelope ψ of the optical field and $1 < \alpha \leq 2$.

They transformed Equation (5) into a Dirac–Weyl-like equation, which is used to establish a link with light propagation in the honeycomb lattice (HCL). They discovered a very similar behavior—the conical diffraction. This similarity in behavior is broken if an additional potential is brought into system.

Our paper is organized as follows. In Section 2, we establish some estimates on the Green's function and we prove appropriate inequalities on some integral operators involving the Green' function. In Section 3, under adequate conditions imposed on function f, we prove the existence and uniqueness of a solution of problem (4). Our approach is based on the Banach contraction principle. The positivity of the solution and the monotony of iterations are also considered. Some examples are given to illustrate our existence results.

Throughout this paper, we denote by $C([0,1])$ the set of continuous functions in $[0,1]$. We recall that the space $C([0,1])$ equipped with the uniform norm $\|u\| := \max_{x \in [0,1]} |u(x)|$ is a Banach space.

2. Preliminary Results

2.1. Fractional Calculus

We recall in this section some basic definitions on fractional calculus (see [33–36]).

Definition 1. *The Riemann–Liouville fractional integral of order $\gamma > 0$ for a measurable function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$I^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} f(t)\, dt, \quad x > 0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here Γ is the Euler Gamma function.

Definition 2. *The Riemann–Liouville fractional derivative of order $\gamma > 0$ for a measurable function $f : (0,\infty) \to \mathbb{R}$ is defined as*

$$D^{\gamma} f(x) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^{n} \int_0^x (x-t)^{n-\gamma-1} f(t)\, dt = \left(\frac{d}{dx}\right)^{n} I^{n-\gamma} f(x),$$

provided that the right-hand side is pointwise defined on $(0,\infty)$. Here $n = [\gamma]+1$, where $[\gamma]$ denotes the integer part of γ.

Please note that if $\gamma = m \in \mathbb{N}\backslash\{0\}$, then we obtain the classical derivative of order m.

Lemma 1. *Let $\gamma > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then we have*

- (i) *For $0 < \gamma < \delta$, $D^{\gamma} I^{\delta} u = I^{\delta-\gamma} u$ and $D^{\gamma} I^{\gamma} u = u$.*
- (ii) *$D^{\gamma} u(x) = 0$ if and only if $u(x) = c_1 x^{\gamma-1} + c_2 x^{\gamma-2} + ... + c_n x^{\gamma-m}$,*

  where m is the smallest integer greather than or equal to γ and $c_i \in \mathbb{R}$ $(i = 1,...,m)$ are arbitrary constants.
- (iii) *Assume that $D^{\gamma} u \in C(0,1) \cap L^1(0,1)$, then*

  $$I^{\gamma} D^{\gamma} u(x) = u(x) + c_1 x^{\gamma-1} + c_2 x^{\gamma-2} + ... + c_m x^{\gamma-m},$$

  where m is the smallest integer greather than or equal to γ and $c_i \in \mathbb{R}$ $(i = 1,...,m)$ are arbitrary constants.

Proof. For the convenience of the reader, we provide the proof of property (ii) which plays an important role in the rest of the paper.

The property is clear if $\gamma = m \in \mathbb{N}\backslash\{0\}$. Next we assume that $m-1 < \gamma < m$.

We claim that for $i = 1,2,...,m$,

$$D^{\gamma}(t^{\gamma-i})(x) = 0.$$

Indeed, by elementary calculus, we have

$$I^{m-\gamma}(t^{\gamma-i})(x) = \frac{1}{\Gamma(m-\gamma)} \int_0^x (x-t)^{m-\gamma-1} t^{\gamma-i} dt = \frac{\Gamma(\gamma+1-i)}{\Gamma(m-i+1)} x^{m-i}.$$

Hence

$$D^{\gamma}(t^{\gamma-i})(x) = \left(\frac{d}{dx}\right)^{m} \left(I^{m-\gamma}(t^{\gamma-i})\right)(x) = 0.$$

Therefore, if $u(x) = \sum_{i=1}^{m} c_i x^{\gamma-i}$, then $D^{\gamma} u(x) = 0$.

Conversely, assume that $D^{\gamma} u(x) = 0$.

From Definition 2, we obtain

$$I^{m-\gamma} u(x) = a_0 + a_1 x + ... + a_{m-1} x^{m-1},$$

where $a_i \in \mathbb{R}(i = 0,1,...,m-1)$ are arbitrary constants.

Using property (i), we deduce that

$$\begin{aligned} u(x) &= D^{m-\gamma}(I^{m-\gamma}u)(x) \\ &= \sum_{i=0}^{m-1} a_i D^{m-\gamma}(t^i)(x) \\ &= \sum_{i=0}^{m-1} a_i \frac{\Gamma(1+i)}{\Gamma(1+i-m+\gamma)} x^{i-m+\gamma} \\ &= \sum_{i=1}^{m} c_i x^{\gamma-i}, \end{aligned}$$

where $c_i \in \mathbb{R}(i = 1, ..., m)$ are arbitrary constants. □

2.2. Estimates on the Green's Function

Lemma 2. *Let $2 < \beta \leq 3$ and $\varphi \in C([0,1])$, then the boundary-value problem,*

$$\begin{cases} D^{\beta} v(x) = \varphi(x) \text{ in } (0,1), \\ v(0) = v(1) = v'(1) = 0, \end{cases} \tag{6}$$

has a unique solution

$$v(x) = \int_0^1 G_\beta(x,t)\,\varphi(t)\,dt, \tag{7}$$

where for $x, t \in [0,1]$,

$$\begin{aligned} G_\beta(x,t) &= \frac{1}{\Gamma(\beta)} \begin{cases} G(x,t), & \text{for } 0 \leq x \leq t \leq 1, \\ G(x,t) + (x-t)^{\beta-1}, & \text{for } 0 \leq t \leq x \leq 1, \end{cases} \\ &= \frac{1}{\Gamma(\beta)} (G(x,t) + (\max(x-t,0))^{\beta-1}), \end{aligned} \tag{8}$$

with

$$G(x,t) := x^{\beta-2}(1-t)^{\beta-2}\left[(\beta-1)(t-x) + (\beta-2)\,x\,(1-t)\right] \tag{9}$$

$$= (\beta-1)t(1-x)x^{\beta-2}(1-t)^{\beta-2} - x^{\beta-1}(1-t)^{\beta-1}. \tag{10}$$

$G_\beta(x,t)$ is called Green's function of boundary-value problem (6).

Proof. By means of Lemma 1, we can reduce equation $D^{\beta} v(x) = \varphi(x)$ to an equivalent integral equation

$$v(x) = c_1 x^{\beta-1} + c_2 x^{\beta-2} + c_3 x^{\beta-3} + I^{\beta}\varphi(x), \tag{11}$$

where $(c_1, c_2, c_3) \in \mathbb{R}^3$.

The boundary condition $v(0) = 0$ implies that $c_3 = 0$, while the condition $v(1) = 0$, gives

$$c_1 + c_2 + I^{\beta}\varphi(1) = 0. \tag{12}$$

On the other hand, since $v'(1) = 0$, we obtain

$$(\beta-1)c_1 + (\beta-2)c_2 + I^{\beta-1}\varphi(1) = 0.$$

Hence

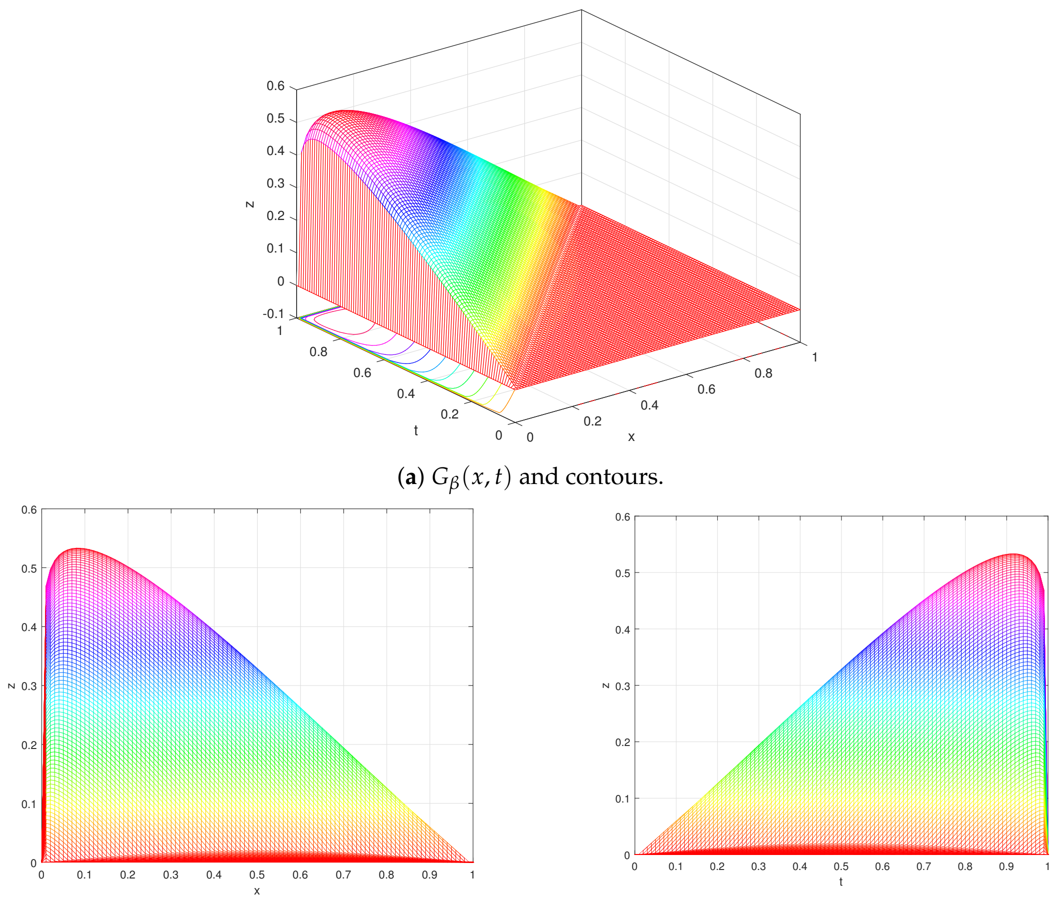
$$c_1 = (\beta-2)\,I^{\beta}\varphi(1) - I^{\beta-1}\varphi(1) \text{ and } c_2 = I^{\beta-1}\varphi(1) - (\beta-1)\,I^{\beta}\varphi(1).$$

Therefore the unique solution of problem (6) is

$$\begin{aligned}
v(x) &= \frac{(\beta-2)}{\Gamma(\beta)}\int_0^1 x^{\beta-1}(1-t)^{\beta-1}\varphi(t)\,dt - \frac{1}{\Gamma(\beta-1)}\int_0^1 x^{\beta-1}(1-t)^{\beta-2}\varphi(t)\,dt \\
&\quad +\Big(\frac{1}{\Gamma(\beta-1)}\int_0^1 x^{\beta-2}(1-t)^{\beta-2}\varphi(t)\,dt - \frac{(\beta-1)}{\Gamma(\beta)}\int_0^1 x^{\beta-2}(1-t)^{\beta-1}\varphi(t)\,dt \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}\varphi(t)\,dt \\
&= \frac{1}{\Gamma(\beta)}\int_0^1 x^{\beta-2}(1-t)^{\beta-2}\left((\beta-1)(t-x)+(\beta-2)x(1-t)\right)\varphi(t)\,dt \\
&\quad +\frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}\varphi(t)\,dt \\
&= \int_0^1 G_\beta(x,t)\,\varphi(t)\,dt.
\end{aligned}$$

The proof is completed. □

In the following, for some values of β we give the representation of the Green function $G_\beta(x,t)$ with the contours and the projections on some coordinate planes (see Figures 1–3). These details give an immediate idea of the behavior of these functions.

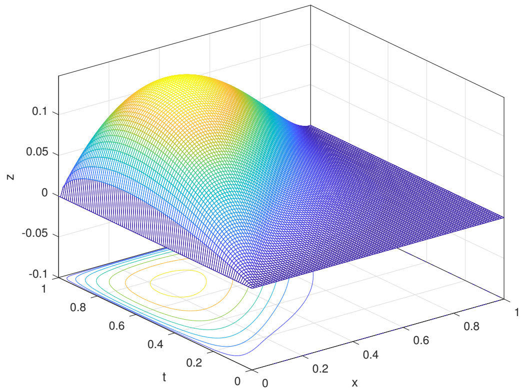


(**a**) $G_\beta(x,t)$ and contours.

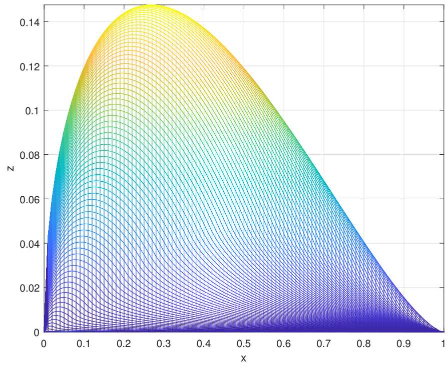
(**b**) Projection of graph of $G_\beta(x,t)$ on the plane xz.

(**c**) Projection of graph of $G_\beta(x,t)$ on the plane tz.

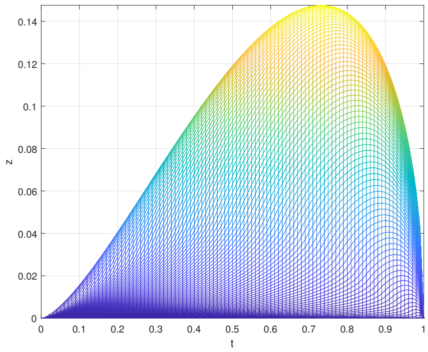
Figure 1. The Green function for $\beta = 2.1$.



(**a**) $G_\beta(x,t)$ and contours.

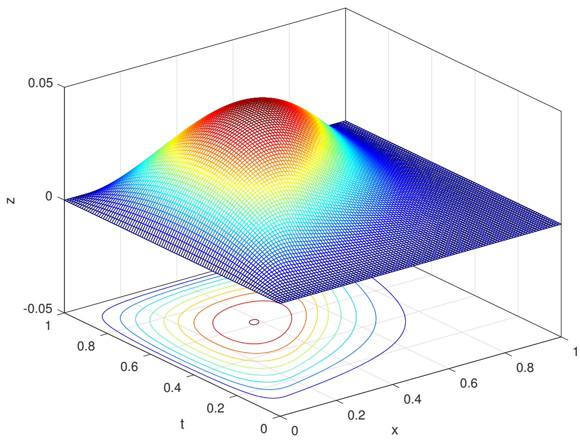


(**b**) Projection of graph of $G_\beta(x,t)$ on the plane xz.

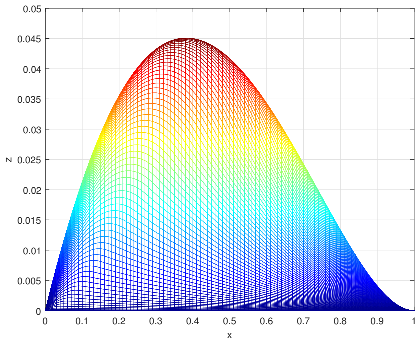


(**c**) Projection of graph of $G_\beta(x,t)$ on the plane tz.

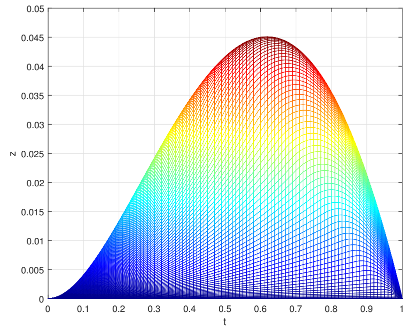
Figure 2. The Green function for $\beta = 5/2$.



(**a**) $G_\beta(x,t)$ and contours.



(**b**) Projection of graph of $G_\beta(x,t)$ on the plane xz.



(**c**) Projection of graph of $G_\beta(x,t)$ on the plane tz.

Figure 3. The Green function for $\beta = 3$.

Proposition 1. *Let $2 < \beta \leq 3$. The Green function $G_\beta(x,t)$ satisfies the following properties.*

(i) $(x,t) \to G_\beta(x,t)$ *is continuous on* $[0,1] \times [0,1]$.

(ii) *For* $0 \leq x \leq t \leq 1$, *we have*

$$(\beta-2)\, H(x,t) \leq \Gamma\,(\beta)\, G_\beta\,(x,t) \leq (\beta-1)\, H(x,t),$$

where $H(x,t) := t\,(1-x)\,x^{\beta-2}\,(1-t)^{\beta-2}$.

(iii) *For* $0 \leq t \leq x \leq 1$, *we have*

$$(\beta-2)\,\overline{H}(x,t) \leq 2\Gamma\,(\beta-1)\, G_\beta\,(x,t) \leq \overline{H}(x,t),$$

where $\overline{H}(x,t) := t^2\,(1-x)^2\,x^{\beta-3}\,(1-t)^{\beta-3}$.

Proof. (i) It is clear.

(ii) Assume that $0 \leq x \leq t \leq 1$. From (8) and (9) we have

$$\begin{aligned}
\Gamma\,(\beta)\, G_\beta\,(x,t) &= x^{\beta-2}\,(1-t)^{\beta-2}\,[(\beta-1)(t-x) + (\beta-2)\,x\,(1-t)] \\
&\leq (\beta-1)x^{\beta-2}\,(1-t)^{\beta-2}\,[(t-x) + x\,(1-t)] \\
&\leq (\beta-1)H(x,t).
\end{aligned}$$

On the other hand, since $t - x \geq 0$, we get

$$\Gamma\,(\beta)\, G_\beta\,(x,t) \geq (\beta-2)H(x,t).$$

(iii) Now, assume that $0 \leq t \leq x \leq 1$.

Since

$$x^{\beta-1}\,(1-t)^{\beta-1} - (x-t)^{\beta-1} = (\beta-1)t(1-x)\int_0^1 (x-t+st(1-x))^{\beta-2}ds,$$

it follows from (8) and (10) that

$$G_\beta\,(x,t) = \frac{1}{\Gamma\,(\beta-1)} t(1-x)x^{\beta-2}\,(1-t)^{\beta-2}\int_0^1 (1-(\frac{x-t+st(1-x)}{x(1-t)})^{\beta-2})ds. \tag{13}$$

Now, using the fact that

$$(\beta-2)(1-(\frac{x-t+st(1-x)}{x(1-t)})) \leq 1-(\frac{x-t+st(1-x)}{x(1-t)})^{\beta-2} \leq 1-(\frac{x-t+st(1-x)}{x(1-t)}), \tag{14}$$

we deduce from (13) that

$$\begin{aligned}
\Gamma\,(\beta-1)\, G_\beta\,(x,t) &\leq t(1-x)x^{\beta-3}\,(1-t)^{\beta-3}\int_0^1 t(1-x)(1-s)ds \\
&\leq \frac{1}{2}t^2(1-x)^2x^{\beta-3}\,(1-t)^{\beta-3}.
\end{aligned}$$

Similarly, using again (13) and (14), we obtain

$$\Gamma\,(\beta-1)\, G_\beta\,(x,t) \geq \frac{(\beta-2)}{2}t^2(1-x)^2x^{\beta-3}\,(1-t)^{\beta-3}.$$

□

Throughout this paper, for $2 < \beta \leq 3$ and $\varphi \in C([0,1])$, we denote by

$$G_\beta\varphi(x) = \int_0^1 G_\beta(x,t)\,\varphi(t)dt, \text{ for } x \in [0,1], \tag{15}$$

where $G_\beta(x,t)$ is given by (8).

Lemma 3. *Let $0 < \alpha \leq 1, 2 < \beta \leq 3$ and $\varphi \in C([0,1])$. Then the following assertions hold:*

$$\left\|G_\beta\varphi\right\| \leq K_\beta \left\|\varphi\right\| \text{ and } \left\|I^\alpha(G_\beta\varphi)\right\| \leq M_{\alpha,\beta}\left\|\varphi\right\|, \tag{16}$$

where

$$K_\beta := \frac{4}{\beta^2\Gamma(\beta+1)}\left(\frac{\beta-2}{\beta}\right)^{\beta-2}, \tag{17}$$

and

$$M_{\alpha,\beta} := \frac{\omega^{\alpha+\beta-2}}{\beta^2\Gamma(\alpha+\beta+1)}\left(\left(1+\sqrt{\frac{(\alpha+\beta-1)(1-\alpha)}{\beta-1}}\right)^2+\frac{\alpha(\alpha+\beta)}{\beta-1}\right), \tag{18}$$

with $\omega := \frac{\alpha+\beta-1}{\beta} - \frac{1}{\beta}\sqrt{\frac{(\alpha+\beta-1)(1-\alpha)}{\beta-1}}$.

Proof. Let $\varphi \in C([0,1])$. By (15), we have for $x \in [0,1]$

$$\left|G_\beta\varphi(x)\right| \leq \left\|\varphi\right\| \int_0^1 G_\beta(x,t)\,dt. \tag{19}$$

Using Lemma 2, we obtain

$$\begin{aligned}
\int_0^1 G_\beta(x,t)\,dt &= \frac{1}{\Gamma(\beta)}\int_0^1 G(x,t)\,dt + \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}\,dt \\
&= \frac{(\beta-1)}{\Gamma(\beta)}(1-x)x^{\beta-2}\int_0^1 t(1-t)^{\beta-2}\,dt \\
&\quad -\frac{1}{\Gamma(\beta)}x^{\beta-1}\int_0^1 (1-t)^{\beta-1}\,dt + \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}\,dt \\
&= \frac{1}{\Gamma(\beta+1)}((1-x)x^{\beta-2} - x^{\beta-1} + x^\beta) \\
&= \frac{1}{\Gamma(\beta+1)}(1-x)^2x^{\beta-2} := \theta(x).
\end{aligned} \tag{20}$$

By simple computation we obtain

$$\|\theta\| = \max_{x\in[0,1]}|\theta(x)| = \theta\left(\frac{\beta-2}{\beta}\right) = K_\beta. \tag{21}$$

Hence from (19) and (21), we get the first inequality in (16).

Now, using Definition 1 and (20), we obtain for $x \in [0,1]$

$$\begin{aligned}
\left|I^{\alpha}(G_{\beta}\varphi)(x)\right| &\leq \frac{\|\varphi\|}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^x (x-t)^{\alpha-1}(1-t)^2 t^{\beta-2}dt \\
&= \frac{\|\varphi\|}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^x (x-t)^{\alpha-1}(t^{\beta}-2t^{\beta-1}+t^{\beta-2})dt \\
&= \frac{\|\varphi\|}{\Gamma(\beta+1)}\Big(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}x^{\alpha+\beta}-2\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}x^{\alpha+\beta-1} \\
&\quad +\frac{\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)}x^{\alpha+\beta-2}\Big) \\
&= \frac{\|\varphi\|}{\Gamma(\alpha+\beta)}\psi(x),
\end{aligned} \tag{22}$$

where

$$\psi(x) = \frac{1}{\alpha+\beta}x^{\alpha+\beta} - \frac{2}{\beta}x^{\alpha+\beta-1} + \frac{\alpha+\beta-1}{\beta(\beta-1)}x^{\alpha+\beta-2}.$$

Observe that

$$\begin{aligned}
\psi'(x) &= x^{\alpha+\beta-3}\Big(x^2 - 2\frac{(\alpha+\beta-1)}{\beta}x + \frac{(\alpha+\beta-1)(\alpha+\beta-2)}{\beta(\beta-1)}\Big) \\
&= x^{\alpha+\beta-3}(x-\omega)(x-\overline{\omega}),
\end{aligned}$$

where $\omega = \frac{(\alpha+\beta-1)}{\beta} - \frac{1}{\beta}\sqrt{\frac{(\alpha+\beta-1)(1-\alpha)}{\beta-1}}$ and $\overline{\omega} = \frac{(\alpha+\beta-1)}{\beta} + \frac{1}{\beta}\sqrt{\frac{(\alpha+\beta-1)(1-\alpha)}{\beta-1}}$.

Since $\omega \in (0,1]$ and $\overline{\omega} \geq 1$, it follows that $\psi'(x) \geq 0$ on $[0,\omega]$ and $\psi'(x) \leq 0$ on $[\omega, 1]$. Hence

$$\|\psi\| = \psi(\omega). \tag{23}$$

By combining (22) and (23), we obtain the second inequality in (16). $\square$

3. Main Results

Let $0 < \alpha \leq 1$ and $2 < \beta \leq 3$. For each real number $M > 0$, denote by

$$\mathcal{D}_M = \{(x,u,v) \in \mathbb{R}^3 : \ 0 \leq x \leq 1, \ |u| \leq MM_{\alpha,\beta}, \ |v| \leq MK_{\beta}\},$$

where K_β and $M_{\alpha,\beta}$ are respectively given by (17) and (18).

By $B[O,M]$, we denote the closed ball centered at O with radius M in the space $C([0,1])$.

3.1. Existence and Uniqueness of a Solution

Theorem 1. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a continuous function and assume that there exist numbers $M, L_1, L_2 \geq 0$ such that*

(i) $|f(x,u,v)| \leq M$ *for any* $(x,u,v) \in \mathcal{D}_M$.

(ii) $|f(x,u_2,v_2) - f(x,u_1,v_1)| \leq L_1|u_2-u_1| + L_2|v_2-v_1|$,

for any $(x,u_i,v_i) \in \mathcal{D}_M$, $i = 1,2$.

(iii) $q := L_1 M_{\alpha,\beta} + L_2 K_\beta < 1$.

Then the boundary value problem (4) has a unique solution $u \in C([0,1])$ satisfying

$$\|u\| \leq MM_{\alpha,\beta} \text{ and } \|D^{\alpha}u\| \leq MK_{\beta}. \tag{24}$$

Proof. Consider the operator $T : C([0,1]) \to C([0,1])$ defined for $\varphi \in C([0,1])$ by

$$T\varphi(x) = f(x, I^{\alpha}(G_\beta\varphi)(x), G_\beta\varphi(x)), \ x \in [0,1], \tag{25}$$

where $G_\beta\varphi$ is defined by (15) and I^α is the Riemann–Liouville fractional integral operator given by Definition 1.

We shall investigate problem (4) via the operator equation (25).

Observe that if φ is a fixed point of the operator T, then by Lemma 1, (15) and Lemma 2,

$$u(x) := I^\alpha(G_\beta\varphi)(x), \tag{26}$$

is a solution of problem (4) and vice versa.

We claim that T is a contraction operator from $B[O,M]$ into itself.

First, we show that the operator T maps $B[O,M]$ into itself.

Indeed, since φ is continuous and by Proposition 1 (i) the Green' s function $G_\beta(x,t)$ is continuous on $[0,1]\times[0,1]$, it is not difficult to check that $T\varphi$ is continuous on $[0,1]$.

Now, for any $\varphi \in B[O,M]$, we have by Lemma 3

$$\|G_\beta\varphi\| \le MK_\beta \text{ and } \|I^\alpha(G_\beta\varphi)\| \le MM_{\alpha,\beta}. \tag{27}$$

Hence, for $x\in[0,1]$, we have $(x, I^\alpha(G_\beta\varphi)(x), G_\beta\varphi(x)) \in \mathcal{D}_M$. Therefore, from assumption (i), it follows that $\|T\varphi\| \le M$. Therefore, the operator T maps $B[O,M]$ into itself.

Secondly, we prove that $T : B[O,M] \to B[O,M]$ is a contraction operator. Indeed, for any $\varphi_1, \varphi_2 \in B[O,M]$, by using assumption (ii) and Lemma 3, we obtain for $x\in[0,1]$,

$$\begin{aligned}
|T\varphi_2(x) - T\varphi_1(x)| &= \left|f(x, I^\alpha(G_\beta\varphi_2)(x), G_\beta\varphi_2(x)) - f(x, I^\alpha(G_\beta\varphi_1)(x), G_\beta\varphi_1(x))\right| \\
&\le L_1 \left\|I^\alpha(G_\beta\varphi_2) - I^\alpha(G_\beta\varphi_1)\right\| + L_2 \left\|G_\beta\varphi_2 - G_\beta\varphi_1)\right\| \\
&= L_1 \left\|I^\alpha(G_\beta(\varphi_2-\varphi_1))\right\| + L_2 \left\|G_\beta(\varphi_2-\varphi_1)\right\| \\
&\le L_1 M_{\alpha,\beta} \|\varphi_2-\varphi_1\| + L_2 K_\beta \|\varphi_2-\varphi_1\| \\
&= q \|\varphi_2-\varphi_1\|,
\end{aligned}$$

where q is defined in assumption (iii).

Therefore, T is a contraction operator in $B[O,M]$. Hence, it has a unique fixed point φ in $B[O,M]$.

Therefore, problem (4) has a unique solution $u\in C([0,1])$ given by (26). The estimates (24) follow from Lemma 3 and the fact that $\|\varphi\| \le M$.

The the proof is completed. □

Next, we present a particular case of Theorem 1. To this end, denote

$$\mathcal{D}_M^+ = \{(x,u,v)\in\mathbb{R}^3 :\ 0\le x\le 1,\ 0\le u\le MM_{\alpha,\beta},\ 0\le v\le MK_\beta\}.$$

Corollary 1. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a continuous function and assume that there exists numbers $M, L_1, L_2 \ge 0$ such that*

(i) $0 \le f(x,u,v) \le M$ *for any* $(x,u,v) \in \mathcal{D}_M^+$.

(ii) $|f(x,u_2,v_2) - f(x,u_1,v_1)| \le L_1|u_2-u_1| + L_2|v_2-v_1|$,

for any $(x,u_i,v_i)\in\mathcal{D}_M^+$, $i=1,2$.

(iii) $q := L_1M_{\alpha,\beta} + L_2K_\beta < 1$.

Then the boundary value problem (4) has a unique nonnegative solution $u\in C([0,1])$ satisfying

$$0 \le u(x) \le MM_{\alpha,\beta} \text{ and } 0 \le D^\alpha u \le MK_\beta. \tag{28}$$

3.2. Iterative Method and Examples

Consider the following iterative process.

$$\begin{cases} \text{Let } \varphi_0 \in B[O,M], \\ \varphi_{k+1}(x) := T\varphi_k(x) = f(x, I^\alpha(G_\beta\varphi_k)(x), G_\beta\varphi_k(x)), \text{ for } k = 0,1,...;\ x \in [0,1]. \end{cases} \tag{29}$$

Theorem 2. *Assume that hypotheses of Theorem 1 are satisfied. The sequence $(\varphi_k)_{k\geq 0}$ converges with the rate of geometric progression and we have*

$$\left\| I^\alpha(G_\beta\varphi_k) - u \right\| \leq M_{\alpha,\beta} \frac{q^k}{1-q} \left\| \varphi_1 - \varphi_0 \right\|, \tag{30}$$

where u is the exact solution of problem (4) and q is given in assumption (iii) in Theorem 1.

Proof. It is known by the Banach contracting mapping principle that the sequence $(\varphi_k)_{k\geq 0}$ converges with the rate of geometric progression and we have

$$\left\| \varphi_k - \varphi \right\| \leq \frac{q^k}{1-q} \left\| \varphi_1 - \varphi_0 \right\|, \tag{31}$$

where φ is the unique fixed point of the operator T in $B[O,M]$.

Using this fact and Lemma 3, we obtain

$$\begin{aligned} \left\| I^\alpha(G_\beta\varphi_k) - u \right\| &= \left\| I^\alpha(G_\beta\varphi_k) - I^\alpha(G_\beta\varphi) \right\| \\ &= \left\| I^\alpha(G_\beta(\varphi_k - \varphi)) \right\| \\ &\leq M_{\alpha,\beta} \left\| \varphi_k - \varphi \right\| \\ &\leq M_{\alpha,\beta} \frac{q^k}{1-q} \left\| \varphi_1 - \varphi_0 \right\|. \end{aligned}$$

The proof is completed. □

Proposition 2. (*Monotony*)*Assume that hypotheses of Theorem 1 are satisfied. In addition, we assume that the function $f(x,u,v)$ is nondecreasing in u and v for any $(x,u,v) \in \mathcal{D}_M$. Let $\varphi_0, \psi_0 \in B[O,M]$ be initial approximations such that $\varphi_0(x) \leq \psi_0(x)$, for all $x \in [0,1]$. Then*

(*i*) *for all $k \in \mathbb{N}$ and $x \in [0,1]$,*

$$I^\alpha(G_\beta\varphi_k)(x) \leq I^\alpha(G_\beta\psi_k)(x). \tag{32}$$

(*ii*) *Suppose further that for all $(x,u,v) \in \mathcal{D}_M$*

$$\varphi_0(x) \leq f(x,u,v) \leq \psi_0(x). \tag{33}$$

Then the sequences $(I^\alpha(G_\beta\varphi_k))_{k\geq 0}$ and $(I^\alpha(G_\beta\psi_k))_{k\geq 0}$ converge to the unique solution u of problem (4) and

$$I^\alpha(G_\beta\varphi_k) \leq I^\alpha(G_\beta\varphi_{k+1}) \leq u \leq I^\alpha(G_\beta\psi_{k+1}) \leq I^\alpha(G_\beta\psi_k). \tag{34}$$

In particular, if $\varphi_0 \geq 0$ (resp. $\psi_0 \leq 0$), then u is nonnegative (resp. nonpositive) solution.

Proof. (*i*) We claim that for all $k \in \mathbb{N}$, we have

$$\varphi_k(x) \leq \psi_k(x), \text{ on } [0,1]. \tag{35}$$

We proceed by induction. From hypothesis, the inequality is clear for $k = 0$. For a given $k \in \mathbb{N}$, assume that $\varphi_k(x) \leq \psi_k(x)$.

Since the Green function is nonnegative, we deduce from (15) and Definition 1 that

$$G_\beta \varphi_k \leq G_\beta \psi_k \text{ and } I^\alpha(G_\beta \varphi_k) \leq I^\alpha(G_\beta \psi_k).$$

Combining this fact and that the function $f(x,u,v)$ is nondecreasing in u and v, we obtain

$$\varphi_{k+1}(x) := f(x, I^\alpha(G_\beta\varphi_k)(x), G_\beta\varphi_k(x)) \leq f(x, I^\alpha(G_\beta\psi_k)(x), G_\beta\psi_k(x)) = \psi_{k+1}(x).$$

So our claim is proved.

Using (35), (15) and Definition 1 we get inequality in (32)

(ii) From Theorem 2, we know that the sequences $(I^\alpha(G_\beta\varphi_k))_{k\geq 0}$ and $(I^\alpha(G_\beta\psi_k))_{k\geq 0}$ converge to the unique solution u of problem (4).

We claim that the sequence $(\varphi_k)_{k\geq 0}$ is nondecreasing.

Indeed, since for $x \in [0,1]$, we have $(x, I^\alpha(G_\beta\varphi_0)(x), G_\beta\varphi_0(x)) \in \mathcal{D}_M$, we deduce from (33) that

$$\varphi_0(x) \leq f(x, I^\alpha(G_\beta\varphi_0)(x), G_\beta\varphi_0(x)) = \varphi_1(x).$$

Assume that $\varphi_k(x) \leq \varphi_{k+1}(x)$. From (15), Definition 1 and the monotony of the function f, we deduce that

$$\varphi_{k+1}(x) = f(x, I^\alpha(G_\beta\varphi_k)(x), G_\beta\varphi_k(x)) \leq f(x, I^\alpha(G_\beta\varphi_{k+1})(x), G_\beta\varphi_{k+1}(x)) = \varphi_{k+2}(x).$$

Hence the sequence $(\varphi_k)_{k\geq 0}$ is nondecreasing.

Therefore, by using again (15) and Definition 1, it follows that the sequence $(I^\alpha(G_\beta\varphi_k))_{k\geq 0}$ is nondecreasing.

Since the sequence $(I^\alpha(G_\beta\varphi_k))_{k\geq 0}$ converges to u, we obtain

$$I^\alpha(G_\beta\varphi_k) \leq I^\alpha(G_\beta\varphi_{k+1}) \leq u$$

Similarly, we prove that the sequence $(I^\alpha(G_\beta\psi_k))_{k\geq 0}$ is nonincreasing and that

$$u \leq I^\alpha(G_\beta\psi_{k+1}) \leq I^\alpha(G_\beta\psi_k).$$

So inequalities in (34) are proved.

Finally, from (34), we have

$$I^\alpha(G_\beta\varphi_0) \leq u \leq I^\alpha(G_\beta\psi_0).$$

This implies that if $\varphi_0 \geq 0$ (resp. $\psi_0 \leq 0$), then u is nonnegative (resp. nonpositive) solution.

This completes the proof. □

Example 1. *Consider the following boundary value problem:*

$$\begin{cases} D^{\frac{5}{2}}(D^{\frac{1}{2}}u)(x) = xu(x) + x^2(D^{\frac{1}{2}}u(x))^2 + 2x + 1, & x \in (0,1), \\ u(0) = D^{\frac{1}{2}}u(0) = D^{\frac{1}{2}}u(1) = (D^{\frac{1}{2}}u)'(1) = 0. \end{cases} \tag{36}$$

In this case $K_{\frac{5}{2}} = 8.6123 \times 10^{-2}$, $M_{\frac{1}{2},\frac{5}{2}} = 5.4279 \times 10^{-2}$ *and* $f(x,u,v) = xu + x^2v^2 + 2x + 1$. *So condition* (i) *in Theorem 1 will be satisfied if we choose* $M > 0$ *such that*

$$MM_{\frac{1}{2},\frac{5}{2}} + M^2K_{\frac{5}{2}}^2 + 3 \leq M.$$

It is easy to verify that $M = 4$ *is an example of suitable choice.*

Since

$$f'_u = x \text{ and } f'_v = 2x^2 v,$$

it follows that for any $(x,u,v) \in \mathcal{D}_4 = \{(x,u,v), 0 \le x \le 1, |u| \le 4M_{\frac{1}{2},\frac{5}{2}}, |v| \le 4K_{\frac{5}{2}}\}$,

$$|f'_u| \le 1 \text{ and } |f'_v| \le 8K_{\frac{5}{2}} \le 1.$$

Hence, $L_1 = 1$ *and* $L_2 = 1$ *satisfy the condition* (ii) *in Theorem 1. Also, we have* $q := L_1 M_{\frac{1}{2},\frac{5}{2}} + L_2 K_{\frac{5}{2}} = M_{\frac{1}{2},\frac{5}{2}} + K_{\frac{5}{2}} < 1$.

Thus by Theorem 1, problem (36) has a unique solution, and the iterative method converges.

In Figure 4, we present the approximation of the unique solution of problem (36) with $u_k(x) := I^{\frac{1}{2}}(G_{\frac{5}{2}}\varphi_k)(x)$ *and* $\varphi_0(x) := 2x + 1$.

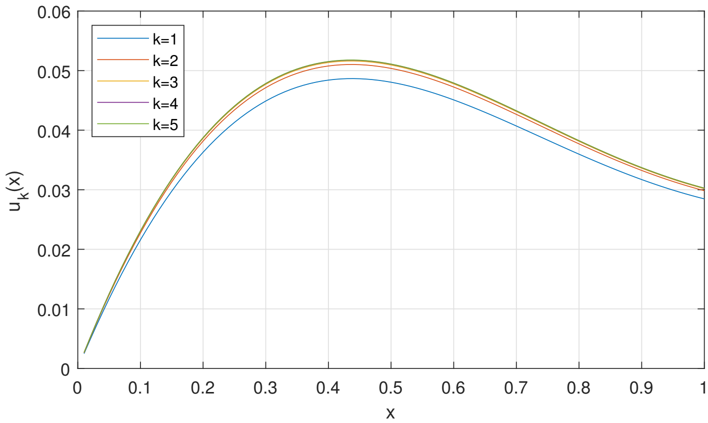


Figure 4. The approximation of the solution of problem (36).

Example 2. *Consider the following boundary value problem:*

$$\begin{cases} D^{\frac{8}{3}}(u')(x) = -3u^2(u'(x))^2 + 3u(x) + 4u'(x) + \sin(\pi x), & x \in (0,1), \\ u(0) = u'(0) = u'(1) = u''(1) = 0. \end{cases} \tag{37}$$

In this example, $K_{\frac{8}{3}} = 5.5637 \times 10^{-2}$, $M_{1,\frac{8}{3}} = 2.1030 \times 10^{-2}$ *and* $f(x,u,v) = -3u^2v^2 + 3u + 4v + \sin(\pi x)$.

As in Example 1, we verify that all conditions of Theorem 1 are satisfied with $M = 3$, $L_1 = 4$ *and* $L_2 = 5$. *Hence problem (37) has a unique solution, and the iterative method converges. Moreover, since in* $\mathcal{D}_3$ *we have* $f'_u \ge 0$ *and* $f'_v \ge 0$, *the function* $f(x,u,v)$ *is nondecreasing in both* u *and* v. *Take the initial approximation* $\varphi_0 = f(x,0,0) = \sin(\pi x) \ge 0$, $0 \le x \le 1$. *By the positivity of the Green's function and Lemma 3, we have*

$$0 \le v_0 := G_{\frac{8}{3}}\varphi_0 \le K_{\frac{8}{3}} \text{ and } 0 \le u_0 := I^1(G_{\frac{8}{3}}\varphi_0) \le M_{1,\frac{8}{3}}.$$

Therefore form the iterative process (29), we obtain

$$\begin{aligned} \varphi_1(x) &= f(x, u_0(x), v_0(x)) \\ &= -3u_0^2 v_0^2 + 3u_0 + 4v_0 + \sin(\pi x) \\ &= 3u_0(1 - u_0 v_0^2) + 4v_0 + \sin(\pi x) \\ &\ge \sin(\pi x) = \varphi_0. \end{aligned}$$

By Proposition 2, $(u_k := I^\alpha(G_\beta \varphi_k))_{k \ge 0}$ *is a nonnegative increasing sequence which converges to the unique nonnegative solution* u. *Some iterations are depicted in Figure 5.*

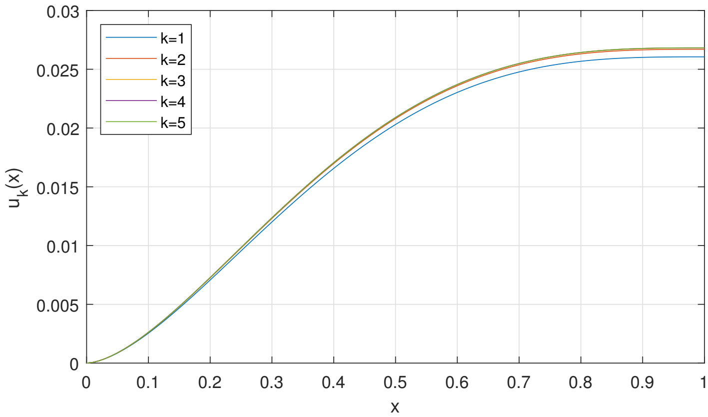


Figure 5. The approximation of the solution of problem (37).

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