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Connectedness and Path Connectedness of Weak Efficient Solution Sets of Vector Optimization Problems via Nonlinear Scalarization Methods

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Abstract: The connectedness and path connectedness of the solution sets to vector optimization problems is an important and interesting study in optimization theories and applications. Most papers involving the direction established the connectedness and connectedness for the solution sets of vector optimization problems or vector equilibrium problems by means of the linear scalarization method rather than the nonlinear scalarization method. The aim of the paper is to deal with the connectedness and the path connectedness for the weak efficient solution set to a vector optimization problem by using the nonlinear scalarization method. Firstly, the union relationship between the weak efficient solution set to the vector optimization problem and the solution sets to a series of parametric scalar minimization problems, is established. Then, some properties of the solution sets of scalar minimization problems are investigated. Finally, by using the union relationship, the connectedness and the path connectedness for the weak efficient solution set of the vector optimization problem are obtained.

Keywords: vector optimization problem; nonlinear scalarization; connectedness; path connectedness

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1. Introduction

Whether the decision is made by a team or an individual, it usually involves several conflicting goals. Problems in the real world must be solved optimally according to criteria, which leads to the development of vector (multi-criteria) optimization problems. Vector optimization theory is widely used in many fields such as economic management, financial insurance, engineering design, transportation, environmental protection, decision-making science and so on. The properties of solution sets are a very important research direction in optimization theories and applications; a lot of research results have been obtained on this aspect. Among the properties of solution sets, the connectedness that can provide the possibility of continuously moving from one solution to any other solution is of considerable interest (see, for example, [1–10]).

It is well known that the scalarization method is one effective approach to deal with the connectedness of the solution sets to vector optimization problems, vector variational inequalities and vector equilibrium problems. Recently, by means of the linear scalarization method, the authors in [11–15] established the connectedness of the solution set to the class of vector optimization, weak vector variational inequalities and weak vector equilibrium problems. However, to the best of our knowledge, there are very few results on the path connectedness of the solution sets of vector optimization problems. Very recently, in terms of the linear scalarization method, Han and Huang [16] investigated the path connectedness of the weak efficient solution set for a generalized vector quasi-equilibrium problem. Xu and Zhang [17] established the path connectedness for the solution set of a strong vector equilibrium problem in terms of a nonlinear separation theorem under some assumptions.

Most papers mentioned above established the connectedness and the path connectedness for the weak efficient solution sets of vector optimization problems or vector equilibrium problems by means of the linear scalarization method rather than the nonlinear scalarization method. Naturally, one question is raised: how to investigate the connectedness and the path connectedness of the weak solution sets of vector optimization problems by using the nonlinear scalarization method? Therefore, the aim of this paper is to establish the connectedness and path connectedness of the weak efficient solution set for a vector optimization problem via the nonlinear scalarization method.

The rest of the paper is organized as follows. In Section 2, some basic definitions and necessary lemmas are recalled. In Section 3, a union relationship between the weak efficient solution set of a vector optimization problem and the solution sets of a series of parametric scalar minimization problems is established without any convexity assumptions of objective function. In Section 4, by using the union relationship, the connectedness and the path connectedness of the weak efficient solution set for the vector optimization problem are obtained. As applications, a strategic game with vector payoffs is given. In Section 5, we give the conclusions of the paper.

2. Preliminaries

Throughout this paper, let Λ, X and Y be topological vector spaces. Assume that $C \subseteq Y$ is a closed, convex and pointed cone with nonempty interior. Let D be a nonempty subset of Y. Denote the interior, the convex hull and the closure of D by $\mathrm{int}\, D$, $\mathrm{conv}\, D$ and $\mathrm{cl}\, D$, respectively. Let Y^* be the topological dual space of Y and C^* be defined by

$$C^* := \{l \in Y^* : l(c) \geq 0,\ \forall\, c \in C\}.$$

A nonempty convex subset B of the convex cone C is called a base if $C = \mathrm{conv}\,(B) = \bigcup\{\lambda x : \lambda \geq 0, x \in B\}$ and $0 \notin \mathrm{cl}\, B$. Let $e \in \mathrm{int}\, C$ and

$$B^* := \{l \in C^* : l(e) = 1\}.$$

Let A be a nonempty subset of X and $F : X \to Y$. In this paper, we consider the following vector optimization problem:

$$(\mathrm{VOP}) \quad \min f(x), \quad x \in A.$$

Definition 1. *$\bar{x} \in A$ is called a weak efficient solution for VOP, written as $\bar{x} \in \mathrm{WE}(f, A)$, iff*

$$f(A) \cap (f(\bar{x}) - \mathrm{int}\, C) = \emptyset.$$

Definition 2. *Let $\phi : Y \to \mathbb{R}$ be a real-valued function [18].*

(i) *The function ϕ is called monotone increasing on Y iff, for each $y_1, y_2 \in Y$, one has*

$$y_1 - y_2 \in C \Rightarrow \phi(y_1) \geq \phi(y_2).$$

(ii) *The function ϕ is called strictly monotone increasing on Y iff, for each $y_1, y_2 \in Y$, one has*

$$y_1 - y_2 \in \mathrm{int}\, C \Rightarrow \phi(y_1) > \phi(y_2).$$

Remark 1. *It is easy to see that, if ϕ is strictly monotone increasing and continuous, then ϕ is monotone increasing.*

Definition 3. *A set-valued mapping G [19]: $\Lambda \rightrightarrows Y$ is said to be lower semicontinuous (l.s.c, for short) at $\lambda_0 \in \Lambda$ iff for any open set V with $G(\lambda_0) \cap V \neq \emptyset$, there exists a neighbourhood U of λ_0 such that $G(\lambda) \cap V \neq \emptyset$, for all $\lambda \in U$. We say that G is l.s.c on Λ, if it is l.s.c at each points of $\lambda \in \Lambda$.*

Definition 4. *Let A be a convex subset of X and $\varphi : A \to Y$ [20].*

(i) *φ is called a properly quasiconvex function on A iff, for each $x_1, x_2 \in A$ and for any $\lambda \in [0,1]$, one has*

$$\textit{either } \varphi(\lambda x_1 + (1-\lambda)x_2) \in \varphi(x_1) - C \textit{ or } \varphi(\lambda x_1 + (1-\lambda)x_2) \in \varphi(x_2) - C.$$

(ii) *φ is called a strictly proper quasiconvex function on A iff, for each $x_1, x_2 \in A$ and for any $\lambda \in (0,1)$, one has*

$$\textit{either } \varphi(\lambda x_1 + (1-\lambda)x_2) \in \varphi(x_1) - \operatorname{int} C \textit{ or } \varphi(\lambda x_1 + (1-\lambda)x_2) \in \varphi(x_2) - \operatorname{int} C.$$

Definition 5. *Let X be a topological linear space and A be a nonempty subset of X [21]. A set-valued mapping $T : A \rightrightarrows X$ is said to be a KKM mapping iff, for any finite subset $\{y_1, ..., y_m\}$ of A, we have*

$$\operatorname{conv}(\{y_1, ..., y_m\}) \subseteq \bigcup_{i=1}^{m} T(y_i).$$

Definition 6. *A topological linear space M is said to be connected iff there do not exist nonempty open subset $V_i \subseteq M, i = 1,2$, such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = M$ [22]. M is said to be path connected (or arcwise connected) iff $\forall x, y \in M$ there exists a continuous mapping $\gamma : [0,1] \to M$, such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Definition 7. *For given $e \in \operatorname{int} C$ and $q \in Y$ [23], the nonlinear scalarization function $\eta_e(\cdot, q) : Y \to \mathbb{R}$ is defined by:*

$$\eta_e(y, q) = \sup\{t \in \mathbb{R} : y \in te + q + C\}, \ \ y \in Y.$$

Proposition 1. *For fixed $e \in \operatorname{int} C$ and any $q \in Y$ [23], one has*

(i) *$\eta_e(y, q) > t \Leftrightarrow y \in te + q + \operatorname{int} C$;*
(ii) *$\eta(y, q) \geq t \Leftrightarrow y \in te + q + C$;*
(iii) *$\eta(\cdot, q)$ is a continuous, concave function and strictly monotone increasing on Y;*
(iv) *$\eta(\cdot, \cdot)$ is continuous on $Y \times Y$.*

Proposition 2. *For any $q \in Y$ [23], one has*

$$\eta_e(y, q) = \min\{l(y) - l(q) : l \in B^*\}, \ \ y \in Y.$$

Lemma 1. *Assume that $\{A_\gamma : \gamma \in \Gamma\}$ is a family of connected sets in topological space Φ [24]. If $\bigcap_{\gamma \in \Gamma} A_\gamma \neq \emptyset$, then $\bigcup_{\gamma \in \Gamma} A_\gamma$ is a connected set in topological space Φ.*

Lemma 2. *Let A be a nonempty subset of a Hausdorff topological vector space X and T [25]: $A \rightrightarrows X$ be a KKM mapping with closed values. If there exists $y_0 \in A$ such that $T(y_0)$ is compact, then $\bigcap_{y \in A} T(y) \neq \emptyset$.*

Lemma 3. *Let A be a paracompact Hausdorff path connected space and let Y be a Banach space [2]. Assume that*

(i) *$F : A \rightrightarrows Y$ is a lower semicontinuous set-valued mapping;*
(ii) *For each $x \in A$, $F(x)$ is nonempty, closed and convex.*

Then, $F(A)$ is a path connected set.

3. Scalarization for VOP

In this section, we consider the following scalar minimization problem with the parametric q, which induced by the nonlinear scalarization function $\eta_e(\cdot, q)$.

$$(P_q) \quad \min \eta_e(f(x), q).$$

Definition 8. *A point $\bar{x} \in A$ is called a solution for P_q, written as $\bar{x} \in S(q)$, iff*

$$\eta_e(f(\bar{x}), q) \leq \eta_e(f(x), q), \quad \forall x \in A.$$

Theorem 1. *One has*

$$WE(f, A) = \bigcup_{q \in Y} S(q).$$

Proof. For any $\bar{x} \in \bigcup_{q \in Y} S(q)$, there exists $q \in Y$ such that $\bar{x} \in S(q)$. Then we have

$$\eta_e(f(\bar{x}), q) \leq \eta_e(f(x), q), \quad \forall x \in A. \tag{1}$$

Assume that $\bar{x} \notin WE(f, A)$. Then there exists $x' \in A$ such that $f(x') \cap (f(\bar{x}) - \operatorname{int} C) \neq \emptyset$, i.e.,

$$f(x') \in f(\bar{x}) - \operatorname{int} C. \tag{2}$$

It follows from the strict monotonicity of the function η that

$$\eta_e(f(x'), q) < \eta_e(f(\bar{x}), q), \tag{3}$$

which contradicts (1). Therefore, $\bar{x} \in WE(f, A)$.

Next, we claim that

$$WE(f, A) \subseteq \bigcup_{q \in Y} S(q). \tag{4}$$

Let $\bar{x} \in WE(f, A)$. Then $f(x) \cap (f(\bar{x}) - \operatorname{int} C) = \emptyset, \forall x \in A$, that is,

$$f(\bar{x}) \notin f(x) + \operatorname{int} C, \quad \forall x \in A. \tag{5}$$

With the help of Proposition 1 (i), we can obtain that

$$\eta_e(f(\bar{x}), f(x)) \leq 0, \quad \forall x \in A. \tag{6}$$

In terms of Proposition 2, we have

$$\eta_e(f(x), f(x)) = \min\{l(f(x)) - l(f(x)) : l \in B^*\} = 0, \quad \forall x \in A. \tag{7}$$

This implies

$$\eta_e(f(\bar{x}), f(x)) \leq \eta_e(f(x), f(x)), \quad \forall x \in A. \tag{8}$$

This shows that $\bar{x} \in S(f(x))$ and so $WE(f, A) \subseteq \bigcup_{q \in Y} S(q)$. Therefore, $WE(f, A) = \bigcup_{q \in Y} S(q)$. □

Remark 2. *Theorem 1 gives the union relationship between the weak efficient solution set of VOP and the solution sets of a series for scalar minimization problems (P_q) without any convexity assumptions on the objective function and the feasible set. Hence, the result improves the corresponding ones in [14–16].*

4. Connectedness and Path Connectedness of VOP

In this section, we shall apply the union relationship established in Section 3 to study the connectedness and the path connectedness of $WE(f, A)$ for VOP.

Lemma 4. *Suppose that A is a closed subset of X and f is continuous on A. Then, for any $q \in Y$, $S(q)$ is a closed set.*

Proof. Let $\{x_n\} \subseteq S(q)$ with $x_n \to x_0$. Then, we have

$$\eta_e(f(x_n), q) \leq \eta_e(f(x), q), \quad \forall\, x \in A. \tag{9}$$

Now, we need to prove that $x_0 \in S(q)$. Indeed, it follows from the closedness of A that $x_0 \in A$. Then, by (9), the continuity of $\eta_e(\cdot, q)$ and f, we have $\eta_e(f(x_0), q) \leq m$. This implies that

$$\eta_e(f(x_0), q) \leq \eta_e(f(x), q), \quad \forall\, x \in A. \tag{10}$$

So, $x_0 \in S(q)$ and for any $q \in Y$, $S(q)$ is a closed set. □

Lemma 5. *Suppose that A is a nonempty convex set of X and $f : A \to Y$ is a properly quasiconvex function. Then, for any $q \in Y$, $S(q)$ is convex.*

Proof. For any $q \in Y$, we let $x_i \in S(q),\ i = 1, 2$ and $\lambda \in [0, 1]$. Then

$$\eta_e(f(x_i), q) \leq \eta_e(f(x), q),\ \forall\, x \in A,\ i = 1, 2. \tag{11}$$

It is clear that $x(\lambda) := \lambda x_1 + (1 - \lambda)x_2 \in A$ for each $\lambda \in [0, 1]$ because of the convexity of A. As f is a properly quasiconvex function on A, one has

$$f(x(\lambda)) \in f(x_i) - C,\ \text{either } i = 1 \text{ or } i = 2,\ \forall\, \lambda \in [0, 1]. \tag{12}$$

Since $\eta_e(\cdot, q)$ is monotone increasing for each $q \in Y$, and by (11), we obtain

$$\eta_e(f(x(\lambda)), q) \leq \eta_e(f(x_i), q) \leq \eta_e(f(x), q),\ \forall x \in A,\ \text{either } i = 1 \text{ or } i = 2. \tag{13}$$

That is, $x(\lambda) \in S(q)$ for each $\lambda \in [0, 1]$. Hence, $S(q)$ is convex. □

Lemma 6. *Assume that*

(i) *A is a compact and convex subset of X;*
(ii) *f is continuous on A;*
(iii) *f is properly quasiconvex on A.*

Then, $\bigcap_{q \in Y} S(q)$ is nonempty.

Proof. For each $x \in A$, define

$$T(x) = \{\bar{x} \in A : f(x) - f(\bar{x}) \in C\}. \tag{14}$$

Clearly, $x \in T(x)$ and so it is nonempty for each $x \in A$. Since the continuity of f, it is easy to get that $T(x) \subseteq A$ is a closed set. Furthermore, by the compactness of A, we obtain $T(x)$ is compact.

We now claim that $T : A \rightrightarrows A$ is a KKM mapping. Indeed, if not, then there exists a finite subset $\{x_1, ..., x_m\} \subseteq A$ and $x_0 \in \text{conv}(\{x_1, ..., x_m\})$ such that

$$x_0 \notin T(x_i), \quad \forall\, i = 1, 2, ..., m. \tag{15}$$

This shows that

$$f(x_i) - f(x_0) \notin C, \quad \forall\, i = 1, 2, ..., m. \tag{16}$$

Since $x_0 \in \operatorname{conv}(\{x_1, ..., x_m\})$, there exist $\lambda_i \geq 0$ $(i = 1, 2, ..., m)$ with $\sum_{i=1}^{m} \lambda_i = 1$, such that $x_0 = \sum_{i=1}^{m} \lambda_i x_i$. Noting that f is properly quasiconvex on A, there exists $i_0 \in \{1, 2, ..., m\}$ such that

$$f(x_{i_0}) \in f(\sum_{i=1}^{m} \lambda_i x_i) + C = f(x_0) + C. \tag{17}$$

This contradicts with (17). Therefore, T is a KKM mapping and $\bigcap_{x \in A} T(x) \neq \emptyset$ by Lemma 2.

Let $\bar{x} \in \bigcap_{x \in A} T(x)$, then for any $x \in A$, we have $f(x) - f(\bar{x}) \in C$. It follows from the monotonicity of $h_e(\cdot, q)$ that for any $x \in A$, $\eta_e(f(\bar{x}), q) \leq \eta_e(f(x), q)$, that is, $\bar{x} \in S(q)$. By the arbitrariness of $\bar{x} \in \bigcap_{x \in A} T(x)$, we have $\bigcap_{x \in A} T(x) \subseteq \bigcap_{q \in Y} S(q)$. Hence, $\bigcap_{q \in Y} S(q)$ is nonempty. □

Lemma 7. *Assume that*

(i) *A is a compact subset of X;*
(ii) *f is continuous on A;*
(iii) *f is strictly proper quasiconvex on A.*

Then, $S(\cdot)$ is l.s.c on Y.

Proof. Assume that there exists $q_0 \in Y$ such that $S(\cdot)$ is not l.s.c on q_0. Then there exist $x_0 \in S(q_0)$, a neighborhood W_0 of $0 \in X$ and a sequence $\{q_n\}$ with $q_n \to q_0$, such that

$$(x_0 + W_0) \cap S(q_n) = \emptyset. \tag{18}$$

There are two cases to be considered.

Case 1. $S(q_0)$ is singleton. Let $x_n \in S(q_n)$. We have

$$\eta_e(f(x_n), q) \leq \eta_e(f(x), q), \quad \forall\, x \in A. \tag{19}$$

Clearly, $x_n \in A$. By the compactness of A, without loss of generality, we can assume that $x_n \to \bar{x}$. Now, we claim that $\bar{x} \in S(q_0)$. Indeed, if not, then there exists $x_0' \in A$ such that

$$\eta_e(f(x_0'), q_0) < \eta_e(f(\bar{x}), q_0). \tag{20}$$

Since $\eta_e(\cdot, \cdot)$ and f respectively are continuous on $Y \times Y$ and A, it follows from (20) that there exists $N \in \mathbb{N}$, such that

$$\eta_e(f(x_0'), q_n) < \eta_e(f(x_n), q_n), \quad \forall\, n \geq N. \tag{21}$$

This contradicts (19). Therefore, $\bar{x} \in S(q_0)$. As $S(q_0)$ is singleton, it follows that $\bar{x} = x_0$ and so $x_n \to x_0$. Hence, $x_n \in (x_0 + W_0) \cap S(q_n)$ for n large enough, which contradicts (18).

Case 2. $S(q_0)$ is not singleton. Without loss of generality, we assume that $x_0, x' \in S(q_0)$ with $x_0 \neq x'$, that is,

$$\eta_e(f(x_0), q_0) \leq \eta_e(f(x), q_0), \quad \forall\, x \in A, \tag{22}$$

and

$$\eta_e(f(x'), q_0) \leq \eta_e(f(x), q_0), \quad \forall\, x \in A. \tag{23}$$

Since f is strictly proper quasiconvex on A, for any $\lambda \in (0, 1)$, one has

$$\text{either } f(x(\lambda)) \in f(x_0) - \operatorname{int} C \text{ or } f(x(\lambda)) \in f(x') - \operatorname{int} C, \tag{24}$$

where $x(\lambda) = \lambda x' + (1-\lambda)x_0$. It is easy to see that there exists $\lambda_0 \in (0,1)$ such that

$$x(\lambda_0) \in x_0 + W_0. \tag{25}$$

It follows from the strict monotonicity of $\eta_e(\cdot, q_0)$ and (24) that

$$\eta_e(f(x(\lambda_0)), q_0) < \eta_e(\text{either } f(x_0) \text{ or } f(x'), q_0). \tag{26}$$

Combining with (22) and (23), we can obtain that

$$\eta_e(f(x(\lambda_0)), q_0) < \eta_e(f(x), q_0), \quad \forall\, x \in A. \tag{27}$$

It follows from (18) and (25) that $x(\lambda_0) \notin S(q_n)$. Hence, there exists $\tilde{x} \in A$ such that

$$\eta_e(f(\tilde{x}), q_n) < \eta_e(f(x(\lambda_0)), q_n). \tag{28}$$

In terms of the continuity of $\eta_e(\cdot,\cdot)$, one has

$$\eta_e(f(\tilde{x}), q_0) \leq \eta_e(f(x(\lambda_0)), q_0). \tag{29}$$

This is a contradiction with (27). Therefore, $S(\cdot)$ is l.s.c on Y. □

Theorem 2. *Assume the following conditions are satisfied:*

(i) *A is a compact and convex subset of X;*
(ii) *f is continuous on A;*
(iii) *f is properly quasiconvex on A.*

Then, $WE(f, A)$ is a connected set.

Proof. It follows from Lemmas 5 and 6 that for any $q \in Y$, $S(q)$ is a convex and nonempty set. Obviously, $S(q)$ is connected. By Lemma 1 and Theorem 1, we can see that

$$WE(f, A) = \bigcup_{q \in Y} S(q)$$

is a connected set. □

Theorem 3. *Suppose that the following conditions are satisfied:*

(i) *A is a compact and convex set of X;*
(ii) *f is continuous on A;*
(iii) *f is strictly proper quasiconvex on A.*

Then, $WE(f, A)$ is a path connected set.

Proof. By means of Lemmas 4–6, we have that for any $q \in Y$, $S(q)$ is a closed, convex and nonempty set. With the help of Theorem 1, we can see that

$$WE(f, A) = \bigcup_{q \in Y} S(q).$$

Therefore, it follows from Lemmas 3 and 7 that $WE(f, A)$ is path connected. □

Remark 3. *Qiu and Yang [6] prove the connectedness of the set of approximate solutions by using the upper semicontinuity of the solution sets of the following scalar minimization problem*

$$(\tilde{P}_q) \ \ \min \xi_e(f(x), q),$$

where $\xi_e(f(x), q) = \inf\{t \in \mathbb{R} : f(x) \in te + q - C\}$ *is the Gerstewitz function defined in [18,26]. Theorem 2 in this paper is different from Theorem 5.1 in [6]. In fact, on the one hand, we derive the connectedness by using the nonlinear scalarization function* η_e *that is different of the Gerstewitz function. On the other hand, Theorem 2 of this paper is obtained without the upper semicontinuity of the solution sets of the scalar minimization problem* P_q*. Furthermore, we also establish the path connectedness of the weak efficient solution set in Theorem 3, which is not established in [6].*

Now, we give the following example to illustrate that Theorem 2 and Theorem 3.

Example 1. *Let* $X = \mathbb{R}$ *and* $A = [-1, 1]$*. Let* $C = \mathbb{R}^2_+$*,* $Y = \mathbb{R}^2$ *and* $e = (1, 1)$*. Define* $f : A \to Y$ *as follows:*

$$f(x) = \begin{cases} (x^2 + \sin x + 2, x), & x \in [0, 1], \\ (2, 0), & x \in [-1, 0). \end{cases}$$

It is easy to see that the constraint set $A = [-1, 1]$ *is a compact and convex set. The function* f *is continuous on* A*. And for each* $x_1, x_2 \in A$*,* $\lambda \in (0, 1)$*, we always have either* $f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - \operatorname{int} C$ *or* $f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - \operatorname{int} C$*. That is, all assumptions in Theorems 2 and 3 are satisfied. It follows from a direct computation that* $WE(f, A) = [-1, 0]$ *is connected. By Proposition 2, we know that*

$$\begin{aligned} \eta_e(y, q) &= \min_{l \in B^*} \{l(y - q)\} \\ &= \min_{0 \le \alpha \le 1, 0 \le \beta \le 1, \alpha + \beta = 1} \{l(y - q)\} \\ &= \min_{0 \le \alpha \le 1} \{(1 - \alpha)(y_1 - q_1) + \alpha(y_2 - q_2)\} \\ &= \min_{0 \le \alpha \le 1} \{y_1 - q_1 + \alpha(y_2 - y_1 + q_1 - q_2)\} \\ &= \begin{cases} y_1 - q_1, & y_2 - y_1 + q_1 - q_2 \ge 0, \\ y_2 - q_2, & y_2 - y_1 + q_1 - q_2 < 0. \end{cases} \end{aligned}$$

By a direct computation, we see that $S(q) = [-1, 0]$ *for any* $q \in Y$*. Hence,* $WE(f, A) = \bigcup_{q \in Y} S(q)$ *is connected and it is also path connected.*

Remark 4. *Now we give an example in economics. This is a strategic game with vector payoffs described in [27]. The bicriteria strategic game is a tuple* $\Psi = \langle N, (\Omega_i)_{i \in N}, (\psi_i)_{i \in N} \rangle$*, where N is the set of players,* Ω_i *is the strategy set for player* $i \in N$*,* Ω *is the Cartesian product* $\prod_{i \in N} \Omega_i$ *of the strategy sets* $(\Omega_i)_{i \in N}$*, and* $\psi_i : \Omega \to \mathbb{R}^2$ *is the utility function for player i. By [27],* $\hat{x} \in \prod_{i \in N} \Omega_i$ *is called a weak Pareto efficient solution of the game if, for each* $i \in N$*,* $\hat{x}_i \in WPB(\hat{x}_{-i})$*, where* $\hat{x}_{-i} \in \Omega_{-i} := \prod_{j \in N \setminus \{i\}} \Omega_j$*. Here, the set* $WPB(\hat{x}_{-i})$ *of weak Pareto best answer to* $\hat{x}_{-i}$ *is the set of the weak efficient solution* x_i *to the following bi-objective optimization problem (for short, BOOP)*

$$\min_{R^2_+} \psi_i(x_i, \hat{x}_{-i}), \ \ s.t. \ \ x_i \in \Omega_i.$$

Obviously, $\hat{x} \in \Omega$ *is a weak Pareto solution of the bicriteria strategic game* Ψ *if and only if for each* $i \in N$*,* $\hat{x}_i \in \Omega_i$ *is a solution of (VOP) with* $f(x) := \psi_i(x_i, \hat{x}_{-i})$*,* $A := \Omega_i$*,* $C = \mathbb{R}^2_+$*.*

Now, we give the following example to illustrate the above economic problem.

Example 2. *Let $N = 1,2$. Let $C = \mathbb{R}^2_+$ and $\Omega_1 = [0,5]$ and $\Omega_2 = [2,6]$. Let $\psi_i(x) = (x_i, x_i - x_{-i})$, $i = 1,2$. It follows from a direct computation that for each $y_i \in \Omega_i$, $\psi_1(x) - \psi_1(y_1) = (x_1, x_1 - x_2) - (y_1, y_1 - x_2) = (x_1 - y_1, x_1 - y_1)$ and $\psi_2(x) - \psi_2(y_2) = (x_2, x_2 - x_1) - (y_2, y_2 - x_1) = (x_2 - y_2, x_2 - y_2)$. Then, one has $WPB(\hat{x}_{-1}) = \{5\}$, $WPB(\hat{x}_{-2}) = \{6\}$. Hence, $WPB = \{(5,6)\}$ is connected and it is also path connected.*

Now, we check that the function $f_i(x) = \psi_i(x) = (x_i, x_i - x_{-i})$ $(i = 1,2)$ satisfy the assumptions in Theorems 2 and 3. It is easy to see that the functions $f_i(x)$ $(i = 1,2)$ are linear functions. Therefore, f is continuous and strictly proper quasiconvex on Ω_i $(i = 1,2)$.

5. Conclusions

In this paper, we firstly established the union relationship between the weak efficient solution set to VOP and the solution sets to a series of parametric scalar minimization problems P_q. Then, we applied the union relationship to obtain the connectedness and the path connectedness of VOP under suitable assumptions. The method may be viewed as a refinement and improvement of the linear scalarization method used in [6,14–16]. However, we find that the parametric set Y of q in the union relationship of Theorem 1 is too large. Moreover, by the nonlinear scalarization method, the union relationship can be established only for the weak efficient solution set. Therefore, alteration of the parametric set Y of q by other parametric sets and the study for the connectedness of efficient solution sets of vector optimization problems, is a good direction for us.

It would be also interesting to investigate the connectedness and the path connectedness of Robust efficient solution sets to vector optimization problems under uncertain data by means of the nonlinear scalarization method. Support vector machine (SVM) and extreme learning machine (ELM) have gained increasing interest from various research fields recently (see, for example, [28–30]). If we can combine the knowledge of SVM and ELM with VOP, it may constitute very valuable research in the future.

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