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A Study of Determinants and Inverses for Periodic Tridiagonal Toeplitz Matrices with Perturbed Corners Involving Mersenne Numbers

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Abstract: In this paper, we study periodic tridiagonal Toeplitz matrices with perturbed corners. By using some matrix transformations, the Schur complement and matrix decompositions techniques, as well as the Sherman-Morrison-Woodbury formula, we derive explicit determinants and inverses of these matrices. One feature of these formulas is the connection with the famous Mersenne numbers. We also propose two algorithms to illustrate our formulas.

Keywords: determinant; inverse; Mersenne number; periodic tridiagonal Toeplitz matrix; Sherman-Morrison-Woodbury formula

1. Introduction

Mersenne numbers are ubiquitous in combinatorics, group theory, chaos, geometry, physics, etc. [1]. They are generated by the following recurrence [2]:

$$M_{n+1} = 3M_n - 2M_{n-1} \qquad \text{where} \quad M_0 = 0,\ M_1 = 1,\ n \geq 1; \tag{1}$$

$$M_{-(n+1)} = \frac{3}{2}M_{-n} - \frac{1}{2}M_{-(n-1)} \ \text{where} \quad M_0 = 0,\ M_{-1} = -\frac{1}{2},\ n \geq 1. \tag{2}$$

The Binet formula says that the nth Mersenne number $M_n = 2^n - 1$ [3]. One application we would like to mention is that Nussbaumer [4] applied number theoretical transform closely related to Mersenne number to deal with problems of digital filtering and convolution of discrete signals.

In this paper, we study some basic quantities (determinants and inverses) associated with the periodic tridiagonal Toeplitz matrix with perturbed corners of type 1, which is defined as follows

$$\mathbb{A} = \begin{pmatrix} \alpha_1 & 2\hbar & 0 & \cdots & 0 & \gamma_1 \\ 0 & -3\hbar & \ddots & \ddots & & 0 \\ 0 & \hbar & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2\hbar & 0 \\ 0 & & \ddots & \ddots & -3\hbar & 2\hbar \\ \alpha_n & 0 & \cdots & 0 & \hbar & \gamma_n \end{pmatrix}_{n\times n}, \tag{3}$$

where $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \hbar$ are complex numbers with $\hbar \neq 0$. Let $\hat{I}_n$ be the $n \times n$ "reverse unit matrix", which has ones along the secondary diagonal and zeros elsewhere. A matrix of the form $\mathbb{B} := \hat{I}_n \mathbb{A} \hat{I}_n$ is called a periodic tridiagonal Toeplitz matrix with perturbed corners of type 2, we say that $\mathbb{B}$ is induced by $\mathbb{A}$. It is readily seen that $\mathbb{A}$ is a periodic tridiagonal Toeplitz matrix with perturbed corners of type 1 if and only if its transpose $\mathbb{A}^T$ is a periodic tridiagonal Toeplitz matrix with perturbed corners of type 2.

Tridiagonal matrices appear not only in pure linear algebra, but also in many practical applications, such as, parallel computing [5], computer graphics [6], fluid mechanics [7,8], chemistry [9], and partial differential equations [10–15]. Taking linear hyperbolic equation as an example, some scholars have studied some matrices in discretized partial differential equations. Chan and Jin [16] discussed a linear hyperbolic equation considered by Holmgren and Otto [17] in one-dimensional and two-dimensional cases. Here we restate the linear hyperbolic equation in the two-dimensional case,

$$\frac{\partial u(x_1,x_2,t)}{\partial t}+v_1\frac{\partial u(x_1,x_2,t)}{\partial x_1}+v_2\frac{\partial u(x_1,x_2,t)}{\partial x_2}=g,$$

where $0 < x_1,\ x_2 \le 1,\ t > 0,\ u(x_1,0,t) = f(x_1-at),\ ,\ u(0,x_1,t) = f(x_2-at),\ u(x_1,x_2,t) = f(x_1+x_2), g=(v_1+v_2-a)f'$. Here v_1, v_2, and a are positive constants and f is a scalar function with derivative f'. Denote s_1, s_2, k as the two spatial steps and time step respectively. For simplicity, assume that $v_1=v_2=v$ and $s_1=s_2=\mathrm{s}$. The linear hyperbolic equation discretized based on trapezoidal rule in time and center difference in two spaces, respectively. It's coefficient matrix is a tridiagonal matrix with perturbed last row:

$$\wp=\begin{pmatrix} 2 & \oslash & 0 & \cdots & \cdots & \cdots & 0 \\ -\oslash & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & -\oslash & 2 & \oslash \\ 0 & \cdots & \cdots & \cdots & 0 & -2\oslash & 2+2\oslash \end{pmatrix}_{n\times n},$$

where $\oslash = vk/\mathrm{s}$. On the other hand, some parallel computing algorithms are also designed for solving tridiagonal systems on graphics processing unit (GPU), which are parallel cyclic reduction [18] and partition methods [19]. Recently, Yang et al. [20] presented a parallel solving method which mixes direct and iterative methods for block-tridiagonal equations on CPU-GPU heterogeneous computing systems, while Myllykoski et al. [21] proposed a generalized graphics processing unit implementation of partial solution variant of the cyclic reduction (PSCR) method to solve certain types of separable block tridiagonal linear systems. Compared to an equivalent CPU implementation that utilizes a single CPU core, PSCR method indicated up to 24-fold speedups.

On the other hand, many studies have been conducted for tridiagonal matrices or periodic tridiagonal matrices, especially for their determinants and inverses [22–30]. Two decades ago, Wittenburg [31] studied the inverse of tridiagonal toeplitz and periodic matrices and applied them to elastostatics and vibration theory. Recently, El-Mikkawy and Atlan [32] proposed a symbolic algorithm based on the Doolittle LU factorization and Jia et al. put forward some algorithms [33–35] based on block diagonalization technique for k-tridiagonal matrix. In 2018, Tim and Emrah [36] used backward continued fractions to derive the LU factorization of periodic tridiagonal matrix and then derived an explicit formula for its inverse. Furthermore, some scholars were attracted by the fact that one could view periodic tridiagonal Toeplitz matrices as a special case of periodic tridiagonal matrices. Shehawey [37] generalized Huang and McColl's [38] work and put forward the inverse formula for periodic tridiagonal Toeplitz matrices.

The rest of the paper is organized as follows: Section 2 describes the detailed derivations of the determinants and inverses of periodic tridiagonal Toeplitz matrices with perturbed corners through matrix transformations, Schur complement and matrix decomposition with the Sherman-Morrison-Woodbury formula [39]. Specifically, the formulas on representation of the determinants and inverses of these typies matrices in the form of products of Mersenne numbers and some initial values. Furthermore, the properties of the periodic tridiagonal Toeplitz matrices with perturbed corners of type 2 can also be obtained. Section 3 presents the numerical results to test the effectiveness of our theoretical results. The final conclusions are given in Section 4.

2. Determinants and Inverses

In this section, we derive explicit formulas for the determinants and inverses of a periodic tridiagonal Toeplitz matrix with perturbed corners. Main effort is made to work out those for periodic tridiagonal Toeplitz matrix with perturbed corners of type 1, since the results for type 2 matrices would follow immediately.

Theorem 1. *Let* $\mathbb{A} = (a_{i,j})_{i,j=1}^{n}$ $(n \geq 3)$ *be an* $n \times n$ *periodic tridiagonal Toeplitz matrix with perturbed corners of type* 1*. Then*

$$\det \mathbb{A} = (-\hbar)^{n-2}\left\{\left[2M_{n-2}\alpha_1 - 4(M_{n-3}+1)\alpha_n\right]\hbar + M_{n-1}(\alpha_1\gamma_n - \alpha_n\gamma_1)\right\}, \tag{4}$$

where M_i $(i = n-3,\ n-2,\ n-1)$ *is the ith Mersenne number.*

Proof. Define the circulant matrix

$$\epsilon = (\epsilon_{i,j})_{i,j=1}^{n}, \tag{5}$$

where

$$\epsilon_{i,j} = \begin{cases} 1, & i = n, j = 1, \\ 1, & j = i+1, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, ϵ is invertible, and

$$\det \epsilon = (-1)^{n-3}. \tag{6}$$

Multiply $\mathbb{A}$ by ϵ from right and then partition $\mathbb{A}\epsilon$ into four blocks:

$$\mathbb{A}\epsilon = \left(\begin{array}{cc|cccccc}
\gamma_1 & \alpha_1 & 2\hbar & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & -3\hbar & 2\hbar & 0 & & & \vdots \\ \hline
0 & 0 & \hbar & -3\hbar & 2\hbar & 0 & & \vdots \\
\vdots & \vdots & 0 & \hbar & -3\hbar & 2\hbar & \ddots & \vdots \\
\vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \vdots & \vdots & \vdots & & \ddots & \ddots & 2\hbar \\
2\hbar & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & -3\hbar \\
\gamma_n & \alpha_n & 0 & 0 & \cdots & \cdots & 0 & \hbar
\end{array}\right)
= \left(\begin{array}{c|c} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \hline \mathbb{A}_{21} & \mathbb{A}_{22} \end{array}\right). \tag{7}$$

Since $\mathbb{A}_{22}$ is upper triangular, its determinant is clear which is

$$\det \mathbb{A}_{22} = \hbar^{n-2}. \tag{8}$$

As we assume $\hbar \neq 0$, so $\mathbb{A}_{22}$ is invertible. It is known (see, e.g., ([29], Lemma 2.5)) that $\mathbb{A}_{22}^{-1} = (\ddot{a}_{i,j})_{i,j=1}^{n}$ where

$$\ddot{a}_{i,j} = \begin{cases} \frac{M_{j-i+1}}{\hbar}, & i \leq j, \\ 0, & i > j, \end{cases}$$

and M_i is the ith Mersenne number.

Next, taking the determinants for both sides of (7) and by (see, e.g., ([40], p. 10)), we get

$$\det(\mathbb{A}\epsilon) = \det \mathbb{A}_{22} \det(\mathbb{A}_{11} - \mathbb{A}_{12}\mathbb{A}_{22}^{-1}\mathbb{A}_{21}). \tag{9}$$

Therefore

$$\det \mathbb{A} = \frac{\det \mathbb{A}_{22} \det(\mathbb{A}_{11} - \mathbb{A}_{12}\mathbb{A}_{22}^{-1}\mathbb{A}_{21})}{\det \epsilon}. \tag{10}$$

To find $\det \mathbb{A}$, we need to evaluate the determinant of $(\mathbb{A}_{11} - \mathbb{A}_{12}\mathbb{A}_{22}^{-1}\mathbb{A}_{21})$. From (7) we have

$$\mathbb{A}_{11} - \mathbb{A}_{12}\mathbb{A}_{22}^{-1}\mathbb{A}_{21} = \begin{pmatrix} \gamma_1 - 2M_{n-2}\gamma_n - 4M_{n-3}\hbar & \alpha_1 - 2M_{n-2}\alpha_n \\ M_{n-1}\gamma_n + 2M_{n-2}\hbar & M_{n-1}\alpha_n \end{pmatrix},$$

and so

$$\det\left(\mathbb{A}_{11} - \mathbb{A}_{12}\mathbb{A}_{22}^{-1}\mathbb{A}_{21}\right) = [4(M_{n-3}+1)\alpha_n - 2M_{n-2}\alpha_1]\hbar - M_{n-1}(\alpha_1\gamma_n - \alpha_n\gamma_1). \tag{11}$$

Finally, applying (6), (8), and (11) to (10), we get the determinant of $\mathbb{A}$, which completes the proof. □

Theorem 2. *Let $\mathbb{A} = (a_{i,j})_{i,j=1}^{n} (n \geq 3)$ be a nonsingular periodic tridiagonal Toeplitz matrix with perturbed corners of type* 1. *Then $\mathbb{A}^{-1} = (\breve{a}_{i,j})_{i,j=1}^{n}$, where*

$$\breve{a}_{i,j} = \begin{cases} \frac{2M_{n-2}\hbar + M_{n-1}\gamma_n}{\psi}, & i=1, j=1, \\ \frac{4M_{n-3}\hbar - \gamma_1 + 2M_{n-2}\gamma_n}{\psi}, & i=1, j=2, \\ \frac{(M_{n-2}+1)\alpha_n}{-\psi}, & i=2, j=1, \\ \frac{2M_{n-3}\alpha_1\hbar + M_{n-2}(\alpha_1\gamma_n - \alpha_n\gamma_1)}{-\psi\hbar}, & i=2, j=2, \\ \frac{3(M_{n-3}+1)\alpha_n}{-\psi}, & i=3, j=1, \\ \frac{(M_{n-3}-1)\alpha_1\hbar + (M_{n-2}+1)\alpha_n\hbar + M_{n-3}(\alpha_1\gamma_n - \alpha_n\gamma_1)}{-\psi\hbar}, & i=3, j=2, \\ 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2} + \frac{1}{\hbar}, & i \in \{2,3\}, j = i+1, \\ 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2}, & \begin{cases} i \in \{1,2,3\}, i+2 \leq j \leq n; \\ 3 \leq j \leq i \leq n, \end{cases} \\ \frac{3}{2}\breve{a}_{i-1,j} - \frac{1}{2}\breve{a}_{i-2,j}, & \begin{cases} j \in \{1,2\}, 4 \leq i \leq n; \\ 4 \leq i < j \leq n, \end{cases} \end{cases} \tag{12}$$

$$\psi = 2M_{n-2}\alpha_1\hbar - (M_{n-1}+1)\alpha_n\hbar + M_{n-1}(\alpha_1\gamma_n - \alpha_n\gamma_1), \tag{13}$$

and M_i $(i = n-3,\ n-2,\ n-1)$ is the ith Mersenne number.

Proof. Let $\mathbb{A}^{-1} = (\breve{a}_{i,j})_{i,j=1}^{n}$ and the identity matrix $I_n = (e_{i,j})_{i,j=1}^{n}$, where

$$e_{i,j} = \begin{cases} 1, \ i = j, \\ 0, \ otherwise. \end{cases} \tag{14}$$

For a nonsingular $\mathbb{A}$,

$$\mathbb{A}^{-1}\mathbb{A} = \mathbb{A}\mathbb{A}^{-1} = I_n. \tag{15}$$

According to (15), we get

$$e_{i,j} = 2\breve{a}_{i,j-1}\hbar - 3\breve{a}_{i,j}\hbar + \breve{a}_{i,j+1}\hbar, \qquad 1 \le i \le n, \ 2 \le j \le n-1, \tag{16}$$

$$e_{i,j} = \breve{a}_{i-1,j}\hbar - 3\breve{a}_{i,j}\hbar + 2\breve{a}_{i+1,j}\hbar, \qquad 3 \le i \le n-1, \ 1 \le j \le n. \tag{17}$$

Based on (14),we get from (16) that

$$\breve{a}_{i,j} = 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2}, \begin{cases} i \in \{1,2,3\}, i+2 \le j \le n; \\ 3 \le j \le i \le n, \end{cases} \tag{18}$$

and $\breve{a}_{i,i+1} = 3\breve{a}_{i,i} - 2\breve{a}_{i,i-1} + \frac{1}{\hbar}$ for $i = 2,3$.

Similarly, from (17), we get that

$$\breve{a}_{i,j} = \frac{3\breve{a}_{i-1,j}}{2} - \frac{\breve{a}_{i-2,j}}{2}, \begin{cases} j \in \{1,2\}, 4 \le i \le n; \\ 4 \le i < j \le n. \end{cases} \tag{19}$$

Therefore, based on the above analysis, we need to determine six initial values, that is, $\breve{a}_{i,j}$ $(i \in \{1,2,3\}, j \in \{1,2\})$, for the recurrence relations (18) and (19) in order to compute the inverse of $\mathbb{A}$. The rest of the proof is devoted to evaluating these particular entries of $\mathbb{A}^{-1}$.

We decompose $\mathbb{A}$ as follows:

$$\mathbb{A} = \hbar\Delta + FG, \tag{20}$$

where $\Delta = 3T_{M,n}^{-1}, F = \left(f_1^T, f_2^T\right), G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ with

$$\begin{aligned} f_1 &= (\alpha_1 + \frac{2M_n\hbar}{M_{n+1}}, -\hbar, 0, \cdots, 0, \alpha_n - \frac{2\hbar}{M_{n+1}})_{1\times n}, \\ f_2 &= (\gamma_1 - \frac{(M_n+1)\hbar}{M_{n+1}}, 0, \cdots, 0, \gamma_n + \frac{2M_n\hbar}{M_{n+1}})_{1\times n}, \\ g_1 &= (1, 0, \cdots, 0)_{1\times n}, \\ g_2 &= (0, \cdots, 0, 1)_{1\times n}, \end{aligned}$$

and M_i the ith Mersenne number as before.

It could be verified that $\Delta^{-1} = \frac{1}{3}(t_{ij})_{i,j=1}^{n}$, where

$$t_{ij} = \begin{cases} M_{j-i+1}, & 1 \le i \le j \le n, \\ -2M_{j-i-1}, & 1 \le j < i \le n, \end{cases}$$

and M_{-m} is given in (2) for $m = 1, 2, \ldots$.

Applying the Sherman-Morrison-Woodbury formula (see, e.g., ([39] p. 50)) to (20) gives

$$\mathbb{A}^{-1} = (\hbar\Delta + FG)^{-1} = \frac{1}{\hbar}\Delta^{-1} - \frac{1}{\hbar^2}\Delta^{-1}F(I_n + \frac{1}{\hbar}G\Delta^{-1}F)^{-1}G\Delta^{-1}. \tag{21}$$

Now we compute each component on the right side of (21).

Multiplying respectively Δ^{-1} by G and F from left and right,

$$G\Delta^{-1} = \frac{1}{3}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{22}$$

$$\Delta^{-1}F = \frac{1}{3}\begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix}, \tag{23}$$

where η_1 and η_2 are row vectors, ξ_1 and ξ_2 are column vectors,

$$\begin{aligned}
\eta_1 &= (M_j)_{j=1}^n, \\
\eta_2 &= (-2M_{j-n-1})_{j=1}^n, \\
\xi_1^T &= (\xi_{1,1} - 3\hbar, \xi_{2,1}, \xi_{3,1}, \cdots, \xi_{n,1}), \\
\xi_{i,1} &= M_{n-i+1}\alpha_n - 2M_{-i}\alpha_1, \;\; i = 1, 2, \cdots, n, \\
\xi_2 &= \left(M_{n-i+1}\gamma_n - 2M_{-i}\gamma_1 + 2M_{n-i}\hbar\right)_{i=1}^n.
\end{aligned}$$

Then multiplying (23) by $\frac{G}{\hbar}$ from the left, further adding I_n and computing the inverse of the matrix

$$(I_n + \frac{G}{\hbar}\Delta^{-1}F)^{-1} = \frac{3\hbar}{h}\begin{pmatrix} -2M_{-n}\gamma_1 + \gamma_n + 3\hbar & -(\gamma_1 + M_n\gamma_n + 2M_{n-1}\hbar) \\ 2M_{-n}\alpha_1 - \alpha_n & \alpha_1 + M_n\alpha_n \end{pmatrix},$$

where $h = M_{n+1}\left[M_{1-n}(\alpha_1\gamma_n - \alpha_n\gamma_1) + M_{2-n}\alpha_1\hbar - \alpha_n\hbar\right]$. Multiplying the pervious formula $\left(I_n + \frac{G}{\hbar}\Delta^{-1}F\right)^{-1}$ by $\Delta^{-1}F$ from the left and by $G\Delta^{-1}$ from the right, respectively, yields

$$\Delta^{-1}F\left(I_n + \frac{1}{\hbar}G\Delta^{-1}F\right)^{-1}G\Delta^{-1} = (k_{ij})_{i,j=1}^n, \tag{24}$$

where

$$\begin{aligned}
k_{1j} &= \frac{\theta_j'\hbar^3 + (\theta_j''\gamma_1 + \theta_j'''\gamma_n)\hbar^2}{M_{n+1}\psi} + \frac{M_j\hbar}{3}, && 1 \le j \le n, \\
k_{ij} &= \frac{(\alpha_1\eta_{ij}' + \alpha_n\eta_{ij}'')\hbar^2 + (\alpha_1\gamma_n - \alpha_n\gamma_1)\eta_{ij}'''\hbar}{3M_{n+1}\psi}, && 2 \le i \le n, 1 \le j \le n, \\
\psi &= 2M_{n-2}\alpha_1\hbar - (M_{n-1}+1)\alpha_n\hbar + M_{n-1}(\alpha_1\gamma_n - \alpha_n\gamma_1), \\
\theta_j' &= 3M_j(M_{n-1}+1) - M_{n-1}M_{n-j+1}(M_j+1), && 1 \le j \le n, \\
\theta_j'' &= M_nM_j - M_{n-j+1}(M_{j-1}+1), && 1 \le j \le n, \\
\theta_j''' &= M_j(M_{n-1}+1) - M_nM_{n-j+1}(M_{j-1}+1), && 1 \le j \le n, \\
\eta_{ij}' &= 2M_nM_{n-i}M_j - 3M_iM_j(M_{n-i}+1) + M_nM_{i-1}M_{n-j+1}(M_{j-i+1}+1), && 2 \le i \le n, 1 \le j \le n, \\
\eta_{ij}'' &= M_{i-1}M_{n+j-1}(M_{n+j-1}+1) - M_{n-i+2}M_j(M_{n-1}+1), && 2 \le i \le n, 1 \le j \le n, \\
\eta_{ij}''' &= M_{n+1}[M_{n-i}M_j + M_{i-1}M_{n-j+1}(M_{j-i}+1)], && 2 \le i \le n, 1 \le j \le n.
\end{aligned}$$

From (21) and (24), we have

$$(\breve{a}_{i,j})_{i,j=1}^{n} = \frac{1}{\hbar}\Delta^{-1} - \frac{1}{\hbar^2}(k_{ij})_{i,j=1}^{n}, \tag{25}$$

where

$$\breve{a}_{i,j} = \frac{M_{j-i+1}}{3\hbar} - \frac{k_{i,j}}{\hbar^2}, \qquad 1 \le i \le j \le n, \tag{26}$$

$$\breve{a}_{i,j} = -\frac{2M_{j-i-1}}{3\hbar} - \frac{k_{i,j}}{\hbar^2}, \qquad 1 \le j < i \le n. \tag{27}$$

By (26) we compute,

$$\begin{aligned}
\breve{a}_{1,1} &= \frac{2M_{n-2}\hbar + M_{n-1}\gamma_n}{\psi}, \\
\breve{a}_{1,2} &= \frac{4M_{n-3}\hbar - \gamma_1 + 2M_{n-2}\gamma_n}{\psi}, \\
\breve{a}_{2,2} &= \frac{2M_{n-3}\alpha_1\hbar + M_{n-2}(\alpha_1\gamma_n - \alpha_n\gamma_1)}{-\psi\hbar}.
\end{aligned}$$

By (27) we compute,

$$\begin{aligned}
\breve{a}_{2,1} &= \frac{(M_{n-2}+1)\alpha_n}{-\psi}, \\
\breve{a}_{3,1} &= \frac{3(M_{n-3}+1)\alpha_n}{-\psi}, \\
\breve{a}_{3,2} &= \frac{(M_{n-3}-1)\alpha_1\hbar + (M_{n-2}+1)\alpha_n\hbar + M_{n-3}(\alpha_1\gamma_n - \alpha_n\gamma_1)}{-\psi\hbar}.
\end{aligned}$$

This completes the proof. □

Remark 1. *Formulas (26) and (27) would give an analytic formula for $\mathbb{A}^{-1}$. However, there is a big advantage of (12) from computational consideration as we shall see from Section 3.*

The next two theorems are parallel results for type 1 matrices.

Theorem 3. *Let $\mathbb{A}$ be a periodic tridiagonal Toeplitz matrix with perturbed corners of type 1 and $\mathbb{B}$ be a periodic tridiagonal Toeplitz matrix with perturbed corners of type 2, which is induced by $\mathbb{A}$. Then*

$$\det \mathbb{B} = (-\hbar)^{n-2}\left\{\left[2M_{n-2}\alpha_1 - 4(M_{n-3}+1)\alpha_n\right]\hbar + M_{n-1}(\alpha_1\gamma_n - \alpha_n\gamma_1)\right\}.$$

Proof. Since $\det \mathbb{B} = \det \hat{I}_n \det \mathbb{A} \det \hat{I}_n$, we obtain this conclusion by using Theorem 1 and $\det \hat{I}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 4. *Let $\mathbb{A}$ be a periodic tridiagonal Toeplitz matrix with perturbed corners of type 1 and $\mathbb{B}$ be a periodic tridiagonal Toeplitz matrix with perturbed corners of type 2, which is induced by $\mathbb{A}$. Then*

$$\mathbb{B}^{-1} = (\breve{a}_{n+1-i,n+1-j})_{i,j=1}^{n},$$

where $\breve{a}_{i,j}$ is the same as (12).

Proof. It follows immediately from $\mathbb{B}^{-1} = \hat{I}_n^{-1}\mathbb{A}^{-1}\hat{I}_n^{-1} = \hat{I}_n\mathbb{A}^{-1}\hat{I}_n$ and Theorem 2. □

3. Algorithms

In this section, we give two algorithms for finding the determinant and inverse of a periodic tridiagonal Toeplitz matrix with perturbed corners of type 1, which is called $\mathbb{A}$. Besides, we analyze these algorithms to illustrate our theoretical results.

Firstly, based on Theorem 1, we give an algorithm for computing determinant of $\mathbb{A}$ as following:

Based on Algorithm 1, we make a comparison of the total number operations for determinant of $\mathbb{A}$ between LU decomposition and Algorithm 1 in Table 1. Specifically, we get that the total number operation for the determinant of $\mathbb{A}$ is $2n+11$, which can be reduced to $O(logn)$ (see, [41] pp. 226–227).

Table 1. Comparison of the total number operations for determinant of $\mathbb{A}$.

Algorithms	Number Operations
LU decomposition algorithm	$13n-15$
Algorithm 1	$2n+11$

Algorithm 1: The determinant of a periodic tridiagonal Toeplitz matrix with perturbed corners of type 1

Step 1: Input $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \hbar$, order n and generate Mersenne numbers M_i $(i = n-3,\ n-2,\ n-1)$ by (1).

Step 2: Calculate and output the determinant of $\mathbb{A}$ by (4).

Next, based on Theorem 2, we give an algorithm for computing inverse of $\mathbb{A}$ as following:

Algorithm 2: The inverse of a periodic tridiagonal Toeplitz matrix with perturbed corners of type 1

Step 1: Input $\alpha_1, \alpha_n, \gamma_1, \gamma_n, \hbar$, order n and generate Mersenne numbers M_i $(i = n-3,\ n-2,\ n-1)$ by (1).

Step 2: Calculate ψ by (13) and six initial values $\breve{a}_{1,1}, \breve{a}_{1,2}, \breve{a}_{2,1}, \breve{a}_{2,2}, \breve{a}_{3,1}, \breve{a}_{3,2}$ by (12).

Step 3: Calculate the remaining elements of the inverse:

$$\begin{aligned}
\breve{a}_{2,3} &= 3\breve{a}_{2,2} - 2\breve{a}_{2,1} + \frac{1}{\hbar}, \\
\breve{a}_{3,4} &= 3\breve{a}_{3,2} - 2\breve{a}_{3,1} + \frac{1}{\hbar}, \\
\breve{a}_{i,j} &= 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2}, \ \ i \in \{1,2,3\}, i+2 \le j \le n, \\
\breve{a}_{i,j} &= 3\breve{a}_{i,j-1} - 2\breve{a}_{i,j-2}, \ \ i \in \{1,2,3\}, 3 \le j \le i \le n, \\
\breve{a}_{i,j} &= \frac{3}{2}\breve{a}_{i-1,j} - \frac{1}{2}\breve{a}_{i-2,j}, \ j \in \{1,2\}, 4 \le i \le n, \\
\breve{a}_{i,j} &= \frac{3}{2}\breve{a}_{i-1,j} - \frac{1}{2}\breve{a}_{i-2,j}, \ 4 \le i < j \le n.
\end{aligned}$$

Step 4: Output the inverse $\mathbb{A}^{-1} = (\breve{a}_{i,j})_{i,j=1}^{n}$.

To test the effectiveness of Algorithm 2, we compare the total number of operations for the inverse of $\mathbb{A}$ between LU decomposition and Algorithm 2 in Table 2. The total number operation of LU decomposition is $\frac{5n^3}{6} + 3n^2 + \frac{91n}{6} - 21$, whereas that of Algorithm 2 is $\frac{7n^2}{2} - \frac{3n}{2} + 30$.

Table 2. Comparison of the total number operations for inverse of $\mathbb{A}$.

Algorithms	Number Operations
LU decomposition algorithm	$\frac{5n^3}{6}+3n^2+\frac{91n}{6}-21$
Algorithm 2	$\frac{7n^2}{2}-\frac{3n}{2}+30$

4. Discussion

In this paper, explicit determinants and inverses of periodic tridiagonal Toeplitz matrices with perturbed corners are represented by the famous Mersenne numbers. This helps to reduce the total number of operations during the calculation process. Some recent research related to our present work can be found in [42–48]. Among them, Qi et al. presented some closed formulas for the Horadam polynomials in terms of a tridiagonal determinant and derived closed formulas for the generalized Fibonacci polynomials, the Lucas polynomials, the Pell-Lucas polynomials, and the Chebyshev polynomials of the first kind in terms of tridiagonal determinants.

5. Conclusions

Mersenne numbers are remarkably wide-spread in many diverse areas of the mathematical, biological, physical, chemical, engineering, and statistical sciences. In this paper, we present explicit formulas for the determinants and inverses of periodic tridiagonal Toeplitz matrices with perturbed corners. The representation of the determinant in the form of products of the Mersenne numbers and some initial values from matrix transformations and Schur complement. For the inverse, our main approaches include the use of matrix decomposition with the Sherman-Morrison-Woodbury formula. Especially, the inverse is just determined by six initial values. To test our method's effectiveness, we propose two algorithms for finding the determinant and inverse of periodic tridiagonal Toeplitz matrices with perturbed corners and compare the total number of operations for the two basic quantities between different algorithms. After comparison, we draw a conclusion that our algorithms are superior to LU decomposition to some extent.

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