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Ball Comparison for Some Efficient Fourth Order Iterative Methods Under Weak Conditions

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Abstract: We provide a ball comparison between some 4-order methods to solve nonlinear equations involving Banach space valued operators. We only use hypotheses on the first derivative, as compared to the earlier works where they considered conditions reaching up to 5-order derivative, although these derivatives do not appear in the methods. Hence, we expand the applicability of them. Numerical experiments are used to compare the radii of convergence of these methods.

Keywords: fourth order iterative methods; local convergence; banach space; radius of convergence

MSC: 65G99; 65H10; 47H17; 49M15

1. Introduction

Let $\mathbb{E}_1$, $\mathbb{E}_2$ be Banach spaces and $\mathbb{D} \subset \mathbb{E}_1$ be a nonempty and open set. Set $\mathbb{LB}(\mathbb{E}_1, \mathbb{E}_2) = \{M : \mathbb{E}_1 \to \mathbb{E}_2\}$, bounded and linear operators. A plethora of works from numerous disciplines can be phrased in the following way:

$$\lambda(x) = 0, \tag{1}$$

using mathematical modelling, where $\lambda : \mathbb{D} \to \mathbb{E}_2$ is a continuously differentiable operator in the Fréchet sense. Introducing better iterative methods for approximating a solution s_* of expression (1) is a very challenging and difficult task in general. Notice that this task is extremely important, since exact solutions of Equation (1) are available in some occasions.

We are motivated by four iterative methods given as

$$\begin{cases} y_j = x_j - \dfrac{2}{3}\lambda'(x_j)^{-1}\lambda(x_j) \\ x_{n+1} = x_j - \dfrac{1}{2}\left[\Big(3\lambda'(y_j) - \lambda'(x_j)\Big)^{-1}\Big(3\lambda'(y_j) + \lambda'(x_j)\Big)\right]\lambda'(x_j)^{-1}\lambda(x_j), \end{cases} \tag{2}$$

$$\begin{cases} y_j = x_j - \dfrac{2}{3}\lambda'(x_j)^{-1}\lambda(x_j) \\ x_{n+1} = x_j - \left[-\dfrac{1}{2}I + \dfrac{9}{8}B_j + \dfrac{3}{8}A_j\right]\lambda'(x_j)^{-1}\lambda(x_j), \end{cases} \tag{3}$$

$$\begin{cases} y_j = x_j - \dfrac{2}{3}\lambda'(x_j)^{-1}\lambda(x_j) \\ x_{n+1} = x_j - \left[I + \dfrac{1}{4}(A_j - I) + \dfrac{3}{8}(A_j - I)^2\right]\lambda'(y_j)^{-1}\lambda(x_j), \end{cases} \tag{4}$$

and

$$\begin{cases} y_j = x_j - H_j \lambda'(x_j)^{-1}\lambda(x_j) \\ x_{n+1} = z_j - \Big[3I - H_j\lambda'(x_j)^{-1}[x_j, z_j; \lambda]\Big]\lambda'(x_j)^{-1}\lambda(z_j), \end{cases} \tag{5}$$

where $H_j^0 = H^0(x_j)$, $x_0, y_0 \in \mathbb{D}$ are initial points, $H(x) = 2I + H^0(x)$, $H_j = H(X_j) \in \mathcal{LB}(\mathbb{E}_1, \mathbb{E}_1)$, $A_j = \lambda'(x_j)^{-1}\lambda'(y_j)$, $z_j = \frac{x_j+y_j}{2}$, $B_j = \lambda'(y_j)^{-1}\lambda'(x_j)$, and $[\cdot,\cdot;\lambda] : \mathbb{D}\times\mathbb{D} \to \mathcal{LB}(\mathbb{E}_1, \mathbb{E}_1)$ is a first order divided difference. These methods specialize to the corresponding ones (when $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{R}^i$, i is a natural number) studied by Nedzhibov [1], Hueso et al. [2], Junjua et al. [3], and Behl et al. [4], respectively. The 4-order convergence of them was established by Taylor series and conditions on the derivatives up to order five. Even though these derivatives of higher-order do not appear in the methods (2)–(5). Hence, the usage of methods (2)–(5) is very restricted. Let us start with a simple problem. Set $\mathbb{E}_1 = \mathbb{E}_2 = \mathbb{R}$ and $\mathbb{D} = [-\frac{5}{2}, \frac{3}{2}]$. We suggest a function $\lambda : \mathbb{A} \to \mathbb{R}$ as

$$\lambda(t) = \begin{cases} 0, & t = 0 \\ t^5 + t^3 \ln t^2 - t^4, & t \neq 0 \end{cases}.$$

Then, $s_* = 1$ is a zero of the above function and we have

$$\lambda'(t) = 5t^4 + 3t^2 \ln t^2 - 4t^3 + 2t^2,$$

$$\lambda''(t) = 20t^3 + 6t \ln t^2 - 12t^2 + 10t,$$

and

$$\lambda'''(t) = 60t^2 + 6 \ln t^2 - 24t + 22.$$

Then, the third-order derivative of function $\lambda'''(x)$ is not bounded on $\mathbb{D}$. The methods (2)–(5) cannot be applicable to such problems or their special cases that require the hypotheses on the third or higher-order derivatives of λ. Moreover, these works do not give a radius of convergence, estimations on $\|x_j - s_*\|$, or knowledge about the location of s_*. The novelty of our work is that we provide this information, but requiring only the derivative of order one, for these methods. This expands the scope of utilization of them and similar methods. It is vital to note that the local convergence results are very fruitful, since they give insight into the difficult operational task for choosing the starting points/guesses.

Otherwise with the earlier approaches: (i) We use the Taylor series and high order derivative, (ii) we do not have any clue for the choice of the starting point x_0, (iii) we have no estimate in advance about the number of iterations needed to obtain a predetermined accuracy, and (iv) we have no knowledge of the uniqueness of the solution.

The work lays out as follows: We give the convergence of these iterative schemes (2)–(5) with some main theorems in Section 2. Some numerical problems are discussed in the Section 3. The final conclusions are summarized in Section 4.

2. Local Convergence Analysis

Let us consider that $I = [0, \infty)$ and $\varphi_0 : I \to I$ be a non-decreasing and continuous function with $\varphi_0(0) = 0$.

Assume that the following equation

$$\varphi_0(t) = 1 \tag{6}$$

has a minimal positive solution ρ_0. Let $I_0 = [0, \rho_0)$. Let $\varphi : I_0 \to I$ and $\varphi_1 : I_0 \to I$ be continuous and non-decreasing functions with $\varphi(0) = 0$. We consider functions on the interval I_0 as

$$\psi_1(t) = \frac{\int_0^1 \varphi\big((1-\tau)t\big)d\tau + \frac{1}{3}\int_0^1 \varphi_1(\tau t)d\tau}{1 - \varphi_0(t)}$$

and

$$\bar{\psi}_1(t) = \psi_1(t) - 1.$$

Suppose that

$$\varphi_0(t) < 3. \tag{7}$$

Then, by (7), $\bar{\psi}_1(0) < 0$ and $\bar{\psi}_1(t) \to \infty$, as $t \to \rho_0^-$. On the basis of the classical intermediate value theorem, the function $\bar{\psi}_1(t)$ has a minimal solution R_1 in $(0, \rho_0)$. In addition, we assume

$$q(t) = 1 \tag{8}$$

has a minimal positive solution ρ_q, where

$$q(t) = \frac{1}{2}\Big(3\varphi_0(\psi_1(t)t) + \varphi_0(t)\Big).$$

Set $\rho = \min\{\rho_0,\ \rho_q\}$.

Moreover, we consider two functions ψ_2 and $\bar{\psi}_2$ on $I_1 = [0, \rho)$ as

$$\psi_2(t) = \frac{\int_0^1 \varphi\big((1-\tau)t\big)d\tau}{1 - \varphi_0(t)} + \frac{3}{2}\frac{\Big(\varphi_0\big(\psi_1(t)t\big) + \varphi_0(t)\Big)\int_0^1 \varphi_1(\tau t)d\tau}{(1 - q(t))(1 - \varphi_0(t))}$$

and

$$\bar{\psi}_2(t) = \psi_2(t) - 1.$$

Then, $\bar{\psi}_2(0) = -1$, and $\bar{\psi}_2(t) \to \infty$, with $t \to \rho^-$. We recall R_2 as the minimal solution of $\bar{\psi}_2(t) = 0$. Set

$$R = \min\{R_1,\ R_2\}. \tag{9}$$

It follows from (9) that for every $t \in [0, R)$

$$0 \le \varphi_0(t) < 1, \tag{10}$$

$$0 \le \psi_1(t) < 1, \tag{11}$$

$$0 \le q(t) < 1 \tag{12}$$

and

$$0 \le \psi_2(t) < 1 \tag{13}$$

Define by $S(s_*, r) = \{y \in \mathbb{E}_1 : \|s_* - y\| < r,\}$ and denote by $\bar{S}(s_*, r)$ the closure of $S(s_*, r)$.

The local convergence of method (2) uses the conditions (A):

(a_1) $\lambda : \mathbb{D} \to \mathbb{E}_2$ is a continuously differentiable operator in the Fréchet sense, and there exists $s_* \in \mathbb{D}$.

(a_2) There exists a function $\varphi_0 : I \to I$ non-decreasing and continuous with $\varphi_0(0) = 0$ for all $x \in \mathbb{D}$

$$\Big\|\lambda'(s_*)^{-1}\Big(\lambda'(x) - \lambda'(s_*)\Big)\Big\| \le \varphi_0(\|x - s_*\|).$$

Set $\mathbb{D}_0 = \mathbb{D} \cap S(s_*, \rho_0)$, where ρ_0 is given in (6).

(a_3) There exist functions $\varphi : I_0 \to I$, $\varphi_1 : I_0 \to I$ non-decreasing and continuous with $\varphi(0) = 0$ so that for all $x, y \in \mathbb{D}_0$

$$\Big\|\lambda'(s_*)^{-1}\Big(\lambda'(y) - \lambda'(x)\Big)\Big\| \le \varphi(\|y - x\|)$$

and

$$\Big\|\lambda'(s_*)^{-1}\lambda'(x)\Big\| \le \varphi_1(\|y - x\|)$$

(a_4) $S(s_*, R) \subset \mathbb{D}$, radii ρ_0, ρ_q as given, respectively by (6), (8) exist; the condition (7) holds, where R is defined in (9).

(a_5)
$$\int_0^1 \varphi_0(\tau R^*)d\tau < 1, \text{ for some } R^* \geq R.$$

Set $\mathbb{D}_1 = D \cap S(s_*, R^*)$.

We can now proceed with the local convergence study of Equation (2) adopting the preceding notations and the conditions (A).

Theorem 1. *Under the conditions (A) sequence $\{x_j\}$ starting at $x_0 \in S(s_*, R) - \{s_*\}$ converges to s_*, $\{x_j\} \subset S(x, R)$ so that*

$$\|y_j - s_*\| \leq \psi_1(\|x_j - s_*\|)\|x_j - s_*\| \leq \|x_j - s_*\| < R \tag{14}$$

and

$$\|x_{n+1} - s_*\| \leq \psi_2(\|x_j - s_*\|)\|x_j - s_*\| \leq \|x_j - s_*\|, \tag{15}$$

with ψ_1 and ψ_2 functions considered previously and R is given in (9). Moreover, s_* is a unique solution in the set $\mathbb{D}_1$.

Proof. We proof the estimates (14) and (15) by adopting mathematical induction. Therefore, we consider $x \in S(s_*, R) - \{s_*\}$. By (a_1), (a_2), (9), and (10), we have

$$\|\lambda'(s_*)^{-1}(\lambda'(s_*) - \lambda'(x))\| \leq \varphi_0(\|s_* - x_0\|) < \varphi_0(R) < 1, \tag{16}$$

hence $\lambda'(x)^{-1} \in \mathcal{LB}(\mathbb{E}_2, \mathbb{E}_1)$ and

$$\|\lambda'(x)^{-1}\lambda'(s_*)\| \leq \frac{1}{1 - \varphi_0(\|s_* - x_0\|)}. \tag{17}$$

The point y_0 is also exists by (17) for $n = 0$. Now, by using (a_1), we have

$$\lambda(x) = \lambda(x) - \lambda(s_*) = \int_0^1 \lambda'(s_* + \tau(x - s_*))d\tau(x - s_*). \tag{18}$$

From (a_3) and (18), we yield

$$\left\|\lambda'(s_*)^{-1}\lambda(x)\right\| \leq \int_0^1 \varphi_1(\tau\|x - s_*\|)d\tau\|x - s_*\|. \tag{19}$$

We can also write by method (2) for $n = 0$

$$y_0 - s_* = \left(x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0)\right) + \frac{1}{3}\lambda'(x_0)^{-1}\lambda(x_0). \tag{20}$$

By expressions (9), (11), (17), (19), and (20), we obtain in turn that

$$\begin{aligned} \|y_0 - s_*\| &\leq \left\|\lambda'(x_0)^{-1}\lambda'(s_*)\right\| \left\|\int_0^1 \lambda'(s_*)^{-1}\left(\lambda'(s_* + \tau(x_0 - s_*)) - \lambda'(x_0)\right)(x_0 - s_*)d\tau\right\| \\ &\quad + \frac{1}{3}\left\|\lambda'(x_0)^{-1}\lambda'(s_*)\right\| \left\|\lambda'(s_*)^{-1}\lambda(x_0)\right\| \\ &\leq \frac{\int_0^1 \varphi((1-\tau)\|x_0 - s_*\|)d\tau + \frac{1}{3}\int_0^1 \varphi(\tau\|x_0 - s_*\|)d\tau}{1 - \varphi_0(\|x_0 - s_*\|)} \\ &= \psi_1(\|x_0 - s_*\|)\|x_0 - s_*\| \leq \|x_0 - s_*\| < R, \end{aligned} \tag{21}$$

which confirms $y_0 \in S(s_*,\ R)$ and (14) for $n = 0$. We need to show that $\Big(3\lambda'(y_0) - 3\lambda'(x_0)\Big)^{-1} \in \mathcal{LB}(\mathbb{E}_2, \mathbb{E}_1)$.

In view of (a_2), (12), and (21), we have

$$\begin{aligned}
&\Big\|\big(2\lambda'(s_*)\big)^{-1}\Big[3\lambda'(y_0) - \lambda'(x_0) - 3\lambda'(s_*) + \lambda'(s_*)\Big]\Big\| \\
&\leq \frac{1}{2}\Big[3\Big\|\lambda'(s_*)^{-1}(\lambda'(y_0) - \lambda'(s_*))\Big\| + \Big\|\lambda'(s_*)^{-1}(\lambda'(x_0) - \lambda'(s_*))\Big\|\Big] \\
&\leq \frac{1}{2}\Big[\varphi_0(\|y_0 - s_*\|) + \varphi_0(\|x_0 - s_*\|)\Big] \\
&\leq \frac{1}{2}\Big[\varphi_0(\psi_1(\|x_0 - s_*\|)\|x_0 - s_*\|) + \varphi_0(\|x_0 - s_*\|)\Big] \\
&= q(\|x_0 - s_*\|) < 1,
\end{aligned} \tag{22}$$

so

$$\Big\|\Big(3\lambda'(y_0) - \lambda'(x_0)\Big)^{-1}\lambda'(s_*)\Big\| \leq \frac{1}{1 - q(\|x_0 - s_*\|)}. \tag{23}$$

Using (9), (13), (17), (a_3), (21), (23), and the second substep of method (2) (since x_1 exists by (23)), we can first write

$$\begin{aligned}
x_1 - s_* &= x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) \\
&\quad + \Big[I - \frac{1}{2}\big(3\lambda'(y_0) - \lambda'(x_0)\big)^{-1}\big(3\lambda'(y_0) + \lambda'(x_0)\big)\Big]\lambda'(x_0)^{-1}\lambda(x_0)
\end{aligned} \tag{24}$$

so

$$\begin{aligned}
\|x_1 - s_*\| &\leq \|x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0)\| + \frac{3}{2}\|\big(3\lambda'(y_0) - \lambda'(x_0)\big)^{-1}\lambda'(s_*)\| \\
&\times\Big[\|\lambda'(s_*)^{-1}(\lambda'(y_0) - \lambda'(x_0))\| + \|\lambda'(s_*)^{-1}(\lambda'(x_0) - \lambda'(s_*))^{-1}\|\Big]\|\lambda'(x_0)^{-1}\lambda(s_*)\|\|\lambda'(x_0)^{-1}\lambda(x_0)\| \\
&\leq \left[\frac{\int_0^1 \varphi((1-\tau)t)d\tau}{1 - \varphi_0(t)} + \frac{3}{2}\frac{\Big(\varphi_0(\|y_0 - s_*\|) + \varphi_0(\|x_0 - s_*\|)\Big)\int_0^1 \varphi_1(\tau\|x_0 - s_*\|)d\tau}{(1 - q(\|x_0 - s_*\|))(1 - \varphi_0(\|x_0 - s_*\|))}\right]\|x_0 - s_*\| \\
&\leq \psi_2(\|x_0 - s_*\|)\|x_0 - s_*\| \leq \|x_0 - s_*\|.
\end{aligned} \tag{25}$$

So, (15) holds and $x_1 \in S(s_*, R)$.

To obtain estimate (25), we also used the estimate

$$\begin{aligned}
&I - \frac{1}{2}\big(3\lambda'(y_0) - \lambda'(x_0)\big)^{-1}\big(3\lambda'(y_0) + \lambda'(x_0)\big) \\
&= \frac{1}{2}\big(3\lambda'(y_0) - \lambda'(x_0)\big)^{-1}\Big[2\big(3\lambda'(y_0) - \lambda'(x_0)\big) - \big(3\lambda'(y_0) + \lambda'(x_0)\big)\Big] \\
&= \frac{3}{2}\big(3\lambda'(y_0) - \lambda'(x_0)\big)^{-1}\Big[\big(\lambda'(y_0) - \lambda'(s_*)\big) + \big(\lambda'(s_*) - \lambda'(x_0)\big)\Big]
\end{aligned} \tag{26}$$

The induction for (14) and (15) can be finished, if x_m, y_m, x_{m+1} replace $x_0, y_0,\ x_1$ in the preceding estimations. Then, from the estimate

$$\|x_{m+1} - s_*\| \leq \mu\|x_m - s_*\| < R,\ \mu = \varphi_2(\|x_0 - s_*\|) \in [0,\ 1), \tag{27}$$

we arrive at $\lim_{m\to\infty} x_m = s_*$ and $x_{m+1} \in S(s_*,\ R)$. Let us consider that $K = \int_0^1 \lambda'(y_* + \tau(s_* - y_*))d\tau$ for $y^* \in \mathbb{D}_1$ with $K(y_*) = 0$. From (a_1) and (a_5), we obtain

$$\begin{aligned}
&\|\lambda'(s_*)^{-1}(\lambda'(s_*) - K)\| \leq \int_0^1 \varphi_0(\tau\|s_* - y_*\|)d\tau \\
&\leq \int_0^1 \varphi_0(\tau R)d\tau < 1.
\end{aligned} \tag{28}$$

So, $K^{-1} \in \mathcal{L}\mathbb{B}(\mathbb{E}_1, \mathbb{E}_2)$, and $s_* = y_*$ by the identity

$$0 = K(s_*) - K(y_*) = K(s_* - y_*). \tag{29}$$

□

Proof. Next, we deal with method (3) in an analogous way. We shall use the same notation as previously. Let $\varphi_0, \varphi, \varphi_1, \rho_0, \psi_1, R_1$, and $\bar{\psi}_1$, be as previously.

We assume

$$\varphi_0(\psi_1(t)t) = 1 \tag{30}$$

has a minimal solution ρ_1. Set $\rho = \min\{\rho_0, \rho_1\}$. Define functions ψ_2 and $\bar{\psi}_2$ on interval $I_2 = [0, \rho)$ by

$$\psi_2(t) = \frac{\int_0^1 \varphi((1-\tau)t)d\tau}{1-\varphi_0(t)} + \left[2 + \frac{3\Big(\varphi_0(\psi_1(t)t) + \varphi_0(t)\Big)}{8(1-\varphi_0(t))} + \frac{9\Big(\varphi_0(\psi_1(t)t) + \varphi_0(t)\Big)}{8(1-\varphi_0(\psi_1(t)t))}\right] \frac{\int_0^1 \varphi_1(\tau t)d\tau}{1-\varphi_0(t)}$$

and

$$\bar{\psi}_2(t) = \psi_2(t) - 1.$$

Then, $\bar{\psi}_2(0) = -1$ and $\bar{\psi}_2(t) \to \infty$, with $t \to \rho^-$. R_2 is known as the minimal solution of equation $\bar{\psi}_2(t) = 0$ in $(0, \rho)$, and set

$$R = \min\{R_1, R_2\}. \tag{31}$$

Replace ρ_q by ρ_1 in the conditions (A) and call the resulting conditions $(A)'$.

Moreover, we use the estimate obtained for the second substep of method (3)

$$\begin{aligned} x_1 - s_* &= x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) + \Big[\frac{3}{2}I - \frac{9}{8}B_0 - \frac{9}{16}A_0\Big]\lambda'(x_0)^{-1}\lambda(x_0) \\ &= x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) + \Big[-2I + \frac{3}{8}(I - A_0) + \frac{9}{8}(I - B_0)\Big]\lambda'(x_0)^{-1}\lambda(x_0) \\ &= x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) + \Big[-2I + \frac{3}{8}\lambda'(x_0)^{-1}\Big(\lambda'(x_0) - \lambda'(y_0)\Big) \\ &\quad + \frac{9}{8}\lambda'(y_0)^{-1}\Big(\lambda'(y_0) - \lambda'(x_0)\Big)\Big]\lambda'(x_0)^{-1}\lambda(x_0). \end{aligned} \tag{32}$$

Then, by replacing (24) by (32) in the proof of Theorem 1, we have instead of (25)

$$\begin{aligned} \|x_1 - s_*\| &= \left[\frac{\int_0^1 \varphi((1-\tau)\|s_* - x_0\|)d\tau}{1-\varphi(\|s_* - x_0\|)} + \left\{2 + \frac{3\Big(\varphi(\|y_0 - s_*\|) + \varphi_0(\|s_* - x_0\|)\Big)}{8(1-\varphi_0(\|s_* - x_0\|))}\right.\right. \\ &\quad \left.\left. + \frac{9\Big(\varphi_0(\|y_0 - s_*\|) + \varphi_0(\|s_* - x_0\|)\Big)}{8(1-\varphi_0(\|y_0 - s_*\|))}\right\} \frac{\int_0^1 \varphi_1(\|s_* - x_0\|)d\tau}{1-\varphi_0(\|s_* - x_0\|)}\right] \|s_* - x_0\| \\ &\leq \psi_2(\|s_* - x_0\|)\|s_* - x_0\| \leq \|s_* - x_0\|. \end{aligned} \tag{33}$$

The rest follows as in Theorem 1. □

Hence, we arrived at the next Theorem.

Theorem 2. *Under the conditions $(A)'$, the conclusions of Theorem 1 hold for method (3).*

Proof. Next, we deal with method (4) in the similar way. Let $\varphi_0, \varphi, \varphi_1, \rho_0, \rho_1, \rho, \psi_1, R_1$, and $\bar{\psi}_1$, be as in the case of method (3). We consider functions ψ_2 and $\bar{\psi}_2$ on I_1 as

$$\psi_2(t) = \frac{\int_0^1 \varphi((1-\tau)t)d\tau}{1-\varphi_0(t)} + \frac{\varphi_0(\psi_1(t)t) + \varphi_0(t)}{(1-\varphi_0(t))(1-\varphi_0(\psi_1(t)t))} + \frac{1}{4}\frac{(\varphi_0(\psi_1(t)t) + \varphi_0(t))}{(1-\varphi_0(t))}$$
$$+ \frac{3}{8}\left(\frac{\varphi_0(\psi_1(t)t) + \varphi_0(t)}{1-\varphi_0(t)}\right)^2$$

and

$$\bar{\psi}_2(t) = \psi_2(t) - 1.$$

The minimal zero of $\bar{\psi}_2(t) = 0$ is denoted by R_2 in $(0, \rho)$, and set

$$R = \min\{R_1, R_2\}. \tag{34}$$

Notice again that from the second substep of method (4), we have

$$\begin{aligned} x_1 - s_* &= x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) + \left[\lambda'(x_0)^{-1} - \lambda'(y_0)^{-1} - \frac{1}{4}(A_0 - I) - \frac{3}{8}(I - A_0)^2\right]\lambda(x_0) \\ &= x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) + \Big\{\lambda'(x_0)^{-1}\Big[(\lambda'(y_0) - \lambda'(s_*)) + (\lambda'(s_*) - \lambda'(x_0))\Big] \\ &- \frac{1}{4}\lambda'(x_0)^{-1}\Big[(\lambda'(y_0) - \lambda'(s_*)) + (\lambda'(s_*) - \lambda'(x_0))\Big] \\ &- \frac{3}{8}\lambda'(x_0)^{-1}\Big[(\lambda'(y_0) - \lambda'(s_*)) + (\lambda'(s_*) - \lambda'(x_0))\Big]^2\Big\}\lambda(x_0), \end{aligned} \tag{35}$$

so

$$\begin{aligned} \|x_1 - s_*\| &\leq \Bigg[\frac{\int_0^1 \varphi((1-\tau)\|s_* - x_0\|)d\tau}{1-\varphi(\|s_* - x_0\|)} + \frac{\varphi_0(\psi(\|s_* - x_0\|)\|s_* - x_0\|) + \varphi_0(\|s_* - x_0\|)}{(1-\varphi_0(\|s_* - x_0\|))(1-\varphi_0(\psi(\|s_* - x_0\|)\|s_* - x_0\|))} \\ &+ \frac{1}{4}\frac{(\varphi(\psi_1(\|s_* - x_0\|)\|s_* - x_0\|) + \varphi_0(\|s_* - x_0\|))}{(1-\varphi_0(\|s_* - x_0\|))} \\ &+ \frac{3}{8}\left(\frac{\varphi(\psi_1(\|s_* - x_0\|)\|s_* - x_0\|) + \varphi_0(\|s_* - x_0\|)}{(1-\varphi_0(\|s_* - x_0\|))}\right)^2\Bigg]\|s_* - x_0\| \\ &\leq \psi_2(\|s_* - x_0\|)\|s_* - x_0\| \leq \|s_* - x_0\|. \end{aligned} \tag{36}$$

The rest follows as in Theorem 1. □

Hence, we arrived at the next following Theorem.

Theorem 3. *Under the conditions* $(A)'$*, conclusions of Theorem 1 hold for scheme* (4).

Proof. Finally, we deal with method (5). Let $\varphi_0, \varphi, \varphi_1, \rho_0, I_0$ be as in method (2). Let also $\varphi_2 : I_0 \to I$, $\varphi_3 : I_0 \to I$, $\varphi_4 : I_0 \to I$ and $\varphi_5 : I_0 \times I_0 \to I$ be continuous and increasing functions with $\varphi_3(0) = 0$. We consider functions ψ_1 and $\bar{\psi}_1$ on I_0 as

$$\psi_1(t) = \frac{\int_0^1 \varphi((1-\tau)t)d\tau + \varphi_2(t)\int_0^1 \varphi_1(\tau t)d\tau}{1-\varphi_0(t)}$$

and

$$\bar{\psi}_1(t) = \psi_1(t) - 1.$$

Suppose that

$$\varphi_1(0)\varphi_2(0) < 1. \tag{37}$$

Then, by (6) and (37), we yield $\bar{\psi}_1(0) < 0$ and $\bar{\psi}_1(t) \to \infty$ with $t \to \rho_0^-$. R_1 is known as the minimal zero of $\bar{\psi}_1(t) = 0$ in $(0, \rho_0)$. We assume

$$\varphi_0(g(t)t) = 1, \tag{38}$$

where $g(t) = \frac{1}{2}(1 + \psi_1(t))$, has a minimal positive solution ρ_1. Set $I_1 = [0, \rho)$, where $\rho = \min\{\rho_0, \rho_1\}$. We suggest functions ψ_2 and $\bar{\psi}_2$ on I_1 as

$$\psi_2(t) = \Bigg[\frac{\int_0^1 \varphi((1-\tau)g(t)t)d\tau}{1 - \varphi_0(g(t)t)} + \frac{\big(\varphi_0(g(t)t) + \varphi_0(t)\big)\int_0^1 \varphi_1(\tau g(t)t)d\tau}{(1 - \varphi_0(g(t)))(1 - \varphi_0(t))}$$
$$+ 2\frac{\varphi_3\left(\frac{t}{2}(1 + \psi_1(t))\right)\int_0^1 \varphi_1(\tau g(t)t)d\tau}{(1 - \varphi_0(t))^2} + \frac{\varphi_4(t)\varphi_5(t, \psi_1(t)t)\int_0^1 \varphi_1(\tau g(t)t)d\tau}{(1 - \varphi_0(t))^2}\Bigg] g(t)$$

and

$$\bar{\psi}_2(t) = \psi_2(t) - 1.$$

Suppose that

$$\Big(2\varphi_3(0) + \varphi_4(0)\varphi_5(0,0)\Big)\varphi_1(0) < 1. \tag{39}$$

By (39) and the definition of I_1, we have $\bar{\psi}_2(0) < 0$, $\bar{\psi}_2(t) \to \infty$ with $t \to \rho^-$. We assume R_2 as the minimal solution of $\bar{\psi}_2(t) = 0$. Set

$$R = \min\{R_1, R_2\}. \tag{40}$$

The study of local convergence of scheme (5) is depend on the conditions (C):

(c_1) = (a_1).
(c_2) = (a_2).
(c_3) There exist functions $\varphi : I_1 \to I$, $\varphi_1 : I_0 \to I$, $\varphi_2 : I_0 \to I$, $\varphi_3 : I_0 \to I$, $\varphi_4 : I_0 \to I$, and $\varphi_5 : I_0 \times I_0 \to I$, increasing and continuous functions with $\varphi(0) = \varphi_3(0) = 0$ so for all $x, y \in \mathbb{D}_0$

$$\begin{aligned}
&\|\lambda'(s_*)^{-1}(\lambda'(y) - \lambda'(x))\| \le \varphi(\|y - x\|),\\
&\|\lambda'(s_*)^{-1}\lambda'(x)\| \le \varphi_1(\|x - s_*\|),\\
&\|I - H(x)\| \le \varphi_2(\|x - s_*\|),\\
&\|\lambda'(s_*)^{-1}([x, y; \lambda] - \lambda'(x))\| \le \varphi_3(\|y - x\|),\\
&\|H^0(x)\| \le \varphi_4(\|x - s_*\|),
\end{aligned}$$

and

$$\|\lambda'(s_*)^{-1}[x, y; \lambda]\| \le \varphi_5(\|x - s_*\|, \|y - s_*\|),$$

(c_4) $S(s_*, R) \subseteq \mathbb{D}$, ρ_0, ρ_1 given, respectively by (6), (38) exist, (37) and (38) hold, and R is defined in (40).
(c_5) = (a_5).

Then, using the estimates

$$
\begin{aligned}
\|y_0 - s_*\| &= \|x_0 - s_* - \lambda'(x_0)^{-1}\lambda(x_0) + (I - H_0)\lambda'(x_0)^{-1}\lambda(x_0)\| \\
&\leq \frac{\int_0^1 \varphi((1-\tau)\|x_0 - s_*\|)d\tau\|x_0 - s_*\|}{1 - \varphi_0(\|x_0 - s_*\|)} + \|I - H_0\|\|\lambda'(x_0)^{-1}\lambda'(s_*)\|\|\lambda'(s_*)^{-1}\lambda(x_0)\| \\
&\leq \left[\frac{\int_0^1 \varphi((1-\tau)\|x_0 - s_*\|)d\tau + \varphi_2(\|x_0 - s_*\|)\int_0^1 \varphi_1(\tau\|x_0 - s_*\|)d\tau}{1 - \varphi_0(\|x_0 - s_*\|)}\right]\|x_0 - s_*\| \\
&\leq \psi_1(\|x_0 - s_*\|)\|x_0 - s_*\| \leq \|x_0 - s_*\| < R,
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
\|x_1 - s_*\| &= \|z_0 - s_* - \lambda'(z_0)^{-1}\lambda(z_0) + \lambda'(z_0)^{-1}(\lambda'(x_0) - \lambda'(z_0))\lambda'(x_0)^{-1}\lambda(z_0)\| \\
&\quad + 2\lambda'(x_0)^{-1}([x_0, z_0; \lambda] - \lambda'(x_0))\lambda'(x_0)^{-1}\lambda(z_0) + H_j^0\lambda'(x_0)^{-1}[x_0, z_0; \lambda]\lambda'(x_0)^{-1}\lambda(z_0)\| \\
&\leq \left[\frac{\int_0^1 \varphi((1-\tau)g(\|x_0 - s_*\|)\|x_0 - s_*\|)d\tau}{1 - \varphi_0(g(\|x_0 - s_*\|)\|x_0 - s_*\|)}\right. \\
&\quad + \frac{(\varphi_0(\|x_0 - s_*\|) + \varphi_0(g(\|x_0 - s_*\|)\|x_0 - s_*\|))\int_0^1 \varphi_1(\tau g(\|x_0 - s_*\|)\|x_0 - s_*\|)d\tau}{(1 - \varphi_0(g(\|x_0 - s_*\|)\|x_0 - s_*\|))(1 - \varphi_0(\|x_0 - s_*\|))} \\
&\quad + 2\frac{\varphi_3\left(\frac{(1+\psi_1(\|x_0 - s_*\|))\|x_0 - s_*\|}{2}\right)\int_0^1 \varphi_1(\tau g(\|x_0 - s_*\|)\|x_0 - s_*\|)d\tau}{(1 - \varphi_0(\|x_0 - s_*\|))^2} \\
&\quad \left. + \frac{\varphi_4(\|x_0 - s_*\|)\varphi_5(\|x_0 - s_*\|, \|y_0 - s_*\|)\int_0^1 \varphi_1(\tau g(\|x_0 - s_*\|)\|x_0 - s_*\|)d\tau}{(1 - \varphi_0(\|x_0 - s_*\|))^2}\right]\|z_0 - s_*\| \\
&\leq \psi_2(\|x_0 - s_*\|)\|x_0 - s_*\| \leq \|x_0 - s_*\|.
\end{aligned} \tag{42}
$$

Here, recalling that $z_0 = \frac{x_0 + y_0}{2}$, we also used the estimates

$$
\begin{aligned}
\|z_0 - s_*\| = \left\|\frac{x_0 + y_0}{2} - s_*\right\| &\leq \frac{1}{2}(\|x_0 - s_*\| + \|y_0 - s_*\|) \\
&\leq \frac{1}{2}(1 + \psi_1(\|x_0 - s_*\|))\|x_0 - s_*\|,
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\alpha &= \lambda'(z_0)^{-1} - \lambda'(x_0)^{-1} = \lambda'(z_0)^{-1}\left[(\lambda'(x_0) - \lambda'(s_*)) + (\lambda'(s_*) - \lambda'(z_0))\right]\lambda'(x_0)^{-1}, \\
\beta &= (-2I + H_0\lambda'(x_0)^{-1}[x_0, z_0; \lambda])\lambda'(x_0)^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma &= -2I + (2I + H_0^0)\lambda'(x_0)^{-1}[x_0, z_0; \lambda] \\
&= -2I + 2I\lambda'(x_0)^{-1}[x_0, z_0; \lambda] + 2H_0^0\lambda'(x_0)^{-1}[x_0, z_0; \lambda] \\
&= 2\lambda'(x_0)^{-1}([x_0, z_0; \lambda] - \lambda'(x_0)) + H_0^0\lambda'(x_0)^{-1}[x_0, z_0; \lambda]
\end{aligned}
$$

to obtain (41) and (42). □

Hence, we arrived at the next following Theorem.

Theorem 4. *Under the conditions* (C), *the conclusions of Theorem 1 hold for method* (5).

3. Numerical Applications

We test the theoretical results on many examples. In addition, we use five examples and out of them: The first one is a counter example where the earlier results are applicable; the next three are real life problems, e.g., a chemical engineering problem, an electron trajectory in the air gap among two parallel surfaces problem, and integral equation of Hammerstein problem, which are displayed in

Examples 1–5. The last one compares favorably (5) to the other three methods. Moreover, the solution to corresponding problem are also listed in the corresponding example which is correct up to 20 significant digits. However, the desired roots are available up to several number of significant digits (minimum one thousand), but due to the page restriction only 30 significant digits are displayed.

We compare the four methods namely (2)–(5), denoted by NM, HM, JM, and BM, respectively on the basis of radii of convergence ball and the approximated computational order of convergence $\rho = \frac{\log\left[\|x_{(j+1)}-x_{(j)}\|/\|x_{(j)}-x_{(j-1)}\|\right]}{\log\left[\|x_{(j)}-x_{(j-1)}\|/\|x_{(j-1)}-x_{(j-2)}\|\right]}$, $j = 2, 3, 4, \ldots$ (for the details please see Cordero and Torregrosa [5]) $(ACOC)$. We have included the radii of ball convergence in the following Tables 1–6 except, the Table 4 that belongs to the values of abscissas t_j and weights w_j. We use the *Mathematica* 9 programming package with multiple precision arithmetic for computing work.

We choose in all examples $H^0(x) = 0$ and $H(x) = 2I$, so $\varphi_2(t) = 1$ and $\varphi_4(t) = 0$. The divided difference is $[x, y; \lambda] = \int_0^1 \lambda'(y + \theta(x-y))d\theta$. In addition, we choose the following stopping criteria (i) $\|x_{j+1} - x_j\| < \epsilon$ and (ii) $\|\lambda(x_j)\| < \epsilon$, where $\epsilon = 10^{-250}$.

Example 1. *Set $\mathbb{X} = \mathbb{Y} = \mathbb{R}$. We suggest a function λ on $\mathbb{D} = [-\frac{1}{\pi}, \frac{2}{\pi}]$ as*

$$\lambda(x) = \begin{cases} 0, & x = 0 \\ x^5 \sin(1/x) + x^3 \log(\pi^2 x^2), & x \neq 0 \end{cases}.$$

But, $\lambda'''(x)$ is unbounded on Ω at $x = 0$. The solution of this problem is $s_* = \frac{1}{\pi}$. The results in Nedzhibov [1], Hueso et al. [2], Junjua et al. [3], and Behl et al. [4] cannot be utilized. In particular, conditions on the 5th derivative of λ or may be even higher are considered there to obtain the convergence of these methods. But, we need conditions on λ' according to our results. In additon, we can choose

$$H = \frac{80 + 16\pi + (\pi + 12\log 2)\pi^2}{2\pi + 1}, \quad \varphi_1(t) = 1 + Ht, \quad \varphi_0(t) = \varphi(t) = Ht,$$

$$\varphi_5(s, t) = \frac{1}{2}(\varphi_1(s) + \varphi_1(t)) \quad \text{and} \quad \varphi_3(t) = \frac{1}{2}\varphi_2(t).$$

The distinct radius of convergence, number of iterations n, and COC (ρ) are mentioned in Table 1.

Table 1. Comparison on the basis of different radius of convergence for Example 1.

Schemes	R_1	R_2	R	x_0	n	ρ
NM	0.011971	0.010253	0.010253	0.30831	4	4.0000
HM	0.011971	0.01329	0.011971	0.32321	4	4.0000
JM	0.011971	0.025483	0.011971	0.32521	4	4.0000
BM	0	0	0	-	-	-

Equation (39) is violated with these choices of φ_i. This is the reason that R is zero in the method BM. Therefore, our results hold only, if $x_0 = s_*$.

Example 2. *The function*

$$\lambda_2(x) = x^4 - 1.674 - 7.79075x^3 + 14.7445x^2 + 2.511x. \tag{44}$$

appears in the conversion to ammonia of hydrogen-nitrogen [6,7]. The function λ_2 has 4 zeros, but we choose $s_* \approx 3.9485424455620457727 + 0.3161235708970163733i$. Moreover, we have

$$\varphi_0(t) = \varphi(t) = 40.6469t, \quad \varphi_1(t) = 1 + 40.6469t, \quad \varphi_3(t) = \frac{1}{2}\varphi_2(t), \quad \text{and} \quad \varphi_5(s, t) = \frac{1}{2}(\varphi_1(s) + \varphi_1(t)).$$

The distinct radius of convergence, number of iterations n, and COC (ρ) are mentioned in Table 2.

Table 2. Comparison on the basis of different radius of convergence for Example 2.

Schemes	R_1	R_2	R	x_0	n	ρ
NM	0.0098841	0.0048774	0.0048774	$3.953 + 0.3197i$	4	4.0000
HM	0.0098841	0.016473	0.016473	$3.9524 + 0.32i$	4	4.0000
JM	0.0098841	0.0059094	0.0059094	$3.9436 + 0.3112i$	4	4.0000
BM	0	0	0	-	-	-

Equation (39) is violated with these choices of φ_i. This is the reason that R is zero in the method BM. Therefore, our results hold only, if $x_0 = s_*$.

Example 3. *An electron trajectory in the air gap among two parallel surfaces is formulated given as*

$$x(t) = x_0 + \left(v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) + e\frac{E_0}{m\omega^2}\big(\cos(\omega t + \alpha) + \sin(\omega + \alpha)\big), \tag{45}$$

where e, m, x_0, v_0, and $E_0 \sin(\omega t + \alpha)$ are the charge, the mass of the electron at rest, the position, velocity of the electron at time t_0, and the RF electric field among two surfaces, respectively. For particular values of these parameters, the following simpler expression is provided:

$$f_3(x) = x + \frac{\pi}{4} - \frac{1}{2}\cos(x). \tag{46}$$

The solution of function f_3 is $s_* \approx -0.309093271541794952741986808924$. Moreover, we have

$$\varphi(t) = \varphi_0(t) = 0.5869t, \;\; \varphi_1(t) = 1 + 0.5869t, \;\; \varphi_3(t) = \frac{1}{2}\varphi_2(t) \;\; \text{and} \;\; \varphi_5(s,t) = \frac{1}{2}\big(\varphi_1(s) + \varphi_1(t)\big).$$

The distinct radius of convergence, number of iterations n, and COC (ρ) are mentioned in Table 3.

Table 3. Comparison on the basis of different radius of convergence for Example 3.

Schemes	R_1	R_2	R	x_0	n	ρ
NM	0.678323	0.33473	0.33473	0.001	4	4.0000
HM	0.678323	1.13054	0.678323	-0.579	4	4.0000
JM	0.678323	0.40555	0.40555	0.091	5	4.0000
BM	0	7.60065×10^{-18}	0	-	-	-

Equation (39) is violated with these choices of φ_i. This is the reason that R is zero in the method BM. Therefore, our results hold only, if $x_0 = s_*$.

Example 4. *Considering mixed Hammerstein integral equation Ortega and Rheinbolt [8], as*

$$x(s) = 1 + \frac{1}{5}\int_0^1 U(s,t)x(t)^3 dt, \;\; x \in C[0,1], \; s,t \in [0,1], \tag{47}$$

where the kernel U is

$$U(s,t) = \begin{cases} s(1-t), s \leq t, \\ (1-s)t, t \leq s. \end{cases}$$

We phrase (47) by using the Gauss-Legendre quadrature formula with $\int_0^1 \phi(t)dt \simeq \sum_{k=1}^{10} w_k \phi(t_k)$, where t_k and w_k are the abscissas and weights respectively. Denoting the approximations of $x(t_i)$ with x_i $(i = 1,2,3,...,10)$, then we yield the following 8×8 system of nonlinear equations

$$5x_i - 5 - \sum_{k=1}^{10} a_{ik} x_k^3 = 0, \;\; i = 1,2,3...,10,$$

$$a_{ik} = \begin{cases} w_k t_k(1-t_i), & k \le i, \\ w_k t_i(1-t_k), & i < k. \end{cases}$$

The values of t_k and w_k can be easily obtained from Gauss-Legendre quadrature formula when $k = 8$ mentioned in Table 4.

Table 4. Values of abscissas t_j and weights w_j.

j	t_j	w_j
1	0.01304673574141413996101799	0.03333567215434406879678440
2	0.06746831665550774463395165	0.07472567457529029657288816
3	0.16029521585048779688283632	0.10954318125799102199776746
4	0.28330230293537640460036703	0.13463335965499817754561346
5	0.42556283050918439455758700	0.14776211235737643508694649
6	0.57443716949081560544241300	0.14776211235737643508694649
7	0.71669769706462359539963297	0.13463335965499817754561346
8	0.83970478414951220311716368	0.10954318125799102199776746
9	0.93253168334449225536604834	0.07472567457529029657288816
10	0.98695326425858586003898201	0.03333567215434406879678440

The required approximate root is $s_* \approx (1.001377, \ldots, 1.006756, \ldots, 1.014515, \ldots, 1.021982, \ldots, 1.026530, \ldots, 1.026530, \ldots, 1.021982, \ldots, 1.014515, \ldots, 1.006756, \ldots, 1.001377, \ldots)^T$. *Moreover, we have*

$$\varphi_0(t) = \varphi(t) = \frac{3}{20}t, \;\; \varphi_1(t) = 1 + \frac{3}{20}t, \;\; \varphi_3(t) = \frac{1}{2}\varphi_2(t) \;\; \text{and} \;\; \varphi_5(s,t) = \frac{1}{2}\big(\varphi_1(s) + \varphi_1(t)\big).$$

The distinct radius of convergence, number of iterations n, and COC (ρ) are mentioned in Table 5.

Table 5. Comparison on the basis of different radius of convergence for Example 4.

Schemes	R_1	R_2	R	x_0	n	ρ
NM	2.6667	1.3159	1.3159	(1,1,...,1)	4	4.0000
HM	2.6667	4.4444	2.6667	(1.9,1.9,...,1.9)	5	4.0000
JM	2.6667	1.5943	1.5943	(2.1,2.1,...,2.1)	5	4.0000
BM	0	0	0	-	-	-

Equation (39) is violated with these choices of φ_i. This is the reason that R is zero in the method BM. Therefore, our results hold only, if $x_0 = s_*$.

Example 5. *We consider a boundary value problem from [8], which is defined as follows:*

$$t'' = \frac{1}{2}t^3 + 3t' - \frac{3}{2-x} + \frac{1}{2}, \;\; t(0) = 0, \;\; t(1) = 1. \tag{48}$$

We assume the following partition on $[0,\ 1]$

$$x_0 = 0 < x_1 < x_2 < \cdots < x_j, \;\; \text{where} \;\; x_{j+1} = x_j + h, \; h = \frac{1}{j}.$$

We discretize this BVP (48) by

$$t'_i \approx \frac{t_{i+1} - t_{i-1}}{2h}, \;\; t''_i \approx \frac{t_{i-1} - 2t_i + t_{i+1}}{h^2}, \;\; i = 1,\ 2,\ \ldots,\ j-1.$$

Then, we obtain a $(k-1) \times (k-1)$ order nonlinear system, given by

$$t_{i+1} - 2t_i + t_{i-1} - \frac{h^2}{2}t_i^3 - \frac{3}{2-x_i}h^2 - 3\frac{t_{i+1} - t_{i-1}}{2}h - \frac{1}{h^2} = 0, \; i = 1, 2, \ldots, j-1,$$

where $t_0 = t(x_0) = 0,\ t_1 = t(x_1),\ \ldots,\ t_{j-1} = t(x_{j-1}),\ t_j = t(x_j) = 1$ *and initial approximation* $t_0^{(0)} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$. *In particular, we choose* $k = 6$ *so that we can obtain a* 5×5 *nonlinear system. The required solution of this problem is*

$$\bar{x} \approx \left(0.09029825\ldots,\ 0.1987214\ldots,\ 0.3314239\ldots,\ 0.4977132\ldots,\ 0.7123306\ldots\right)^T.$$

The distinct radius of convergence, number of iterations n, and COC (ρ) *are mentioned in Table 6.*

Table 6. Convergence behavior of distinct fourth-order methods for Example 5.

Methods	j	$\|F(x^{(j)})\|$	$\|x^{(j+1)} - x^{(j)}\|$	ρ
	1	8.1 (−6)	2.0 (−4)	
MM	2	1.0 (−23)	3.1 (−23)	
	3	9.1 (−95)	2.4 (−94)	
	4	3.7 (−379)	9.0 (−379)	3.9996
	1	7.8 (−6)	1.9 (−5)	
HM	2	7.6 (−24)	2.4 (−23)	
	3	2.7 (−95)	7.2 (−95)	
	4	2.6 (−381)	6.3 (−381)	3.9997
	1	7.8 (−6)	1.9 (−5)	
JM	2	7.6 (−24)	2.4 (−23)	
	3	2.7 (−95)	7.2 (−95)	
	4	2.6 (−381)	6.3 (−381)	3.9997
	1	7.2 (−6)	1.7 (−5)	
BM	2	4.2 (−24)	1.3 (−23)	
	3	1.9 (−96)	5.2 (−96)	
	4	5.6 (−386)	1.4 (−385)	3.9997

4. Conclusions

The convergence order of iterative methods involves Taylor series, and the existence of high order derivatives. Consequently, upper error bounds on $\|x_j - s_*\|$ and uniqueness results are not reported with this technique. Hence, the applicability of these methods is limited to functions with high order derivatives. To address these problems, we present local convergence results based on the first derivative. Moreover, we compare methods (2)–(5). Notice that our convergence criteria are sufficient but not necessary. Therefore, if e.g., the radius of convergence for the method (5) is zero, that does not necessarily imply that the method does not converge for a particular numerical example. Our method can be adopted in order to expand the applicability of other methods in an analogous way.

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