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Some Inequalities for g-Frames in Hilbert C^*-Modules

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Abstract: In this paper, we obtain new inequalities for g-frames in Hilbert C^*-modules by using operator theory methods, which are related to a scalar $\lambda \in \mathbb{R}$ and an adjointable operator with respect to two g-Bessel sequences. It is demonstrated that our results can lead to several known results on this topic when suitable scalars and g-Bessel sequences are chosen.

Keywords: Hilbert C^*-module; g-frame; g-Bessel sequence; adjointable operator

MSC: 46L08; 42C15; 47B48; 46H25

1. Introduction

Since their appearance in the literature [1] on nonharmonic Fourier series, frames for Hilbert spaces have been a useful tool and applied to different branches of mathematics and other fields. For details on frames, the reader can refer to the papers [2–11]. The author in [12] extended the concept of frames to bounded linear operators and thus gave us the notion of g-frames, which possess some properties that are quite different from those of frames (see [13,14]).

In the past decade, much attention has been paid to the extension of frame and g-frame theory from Hilbert spaces to Hilbert C^*-modules, and some significant results have been presented (see [15–23]). It should be pointed out that, due to the essential differences between Hilbert spaces and Hilbert C^*-modules and the complex structure of the C^*-algebra involved in a Hilbert C^*-module, the problems on frames and g-frames for Hilbert C^*-modules are expected to be more complicated than those for Hilbert spaces. Also, increasingly more evidence is indicating that there is a close relationship between the theory of wavelets and frames and Hilbert C^*-modules in many aspects. This suggests that the discussion of frame and g-frame theory in Hilbert C^*-modules is interesting and important.

The authors in [24] provided a surprising inequality while further discussing the remarkable identity for Parseval frames derived from their research on effective algorithms to compute the reconstruction of a signal, which was later generalized to the situation of general frames and dual frames [25]. Those inequalities have already been extended to several generalized versions of frames in Hilbert spaces [26–28]. Moreover, the authors in [29–31] showed that g-frames in Hilbert C^*-modules have their inequalities based on the work in [24,25]; it is worth noting that the inequalities given in [30] are associated with a scalar in $[0,1]$ or $[\frac{1}{2},1]$. In this paper, we establish several new inequalities for g-frames in Hilbert C^*-modules, where a scalar λ in $\mathbb{R}$, the real number set, and an adjointable operator with respect to two g-Bessel sequences are involved. Also, we show that some corresponding results in [29,31] can be considered a special case of our results.

We continue with this section for a review of some notations and definitions.

This paper adopts the following notations: $\mathbb{J}$ and $\mathcal{A}$ are, respectively, a finite or countable index set and a unital C^*-algebra; $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{K}_j$'s ($j \in \mathbb{J}$) are Hilbert C^*-modules over $\mathcal{A}$ (or simply Hilbert $\mathcal{A}$-modules), setting $\langle f, f\rangle = |f|^2$ for any $f \in \mathcal{H}$. The family of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ is designated $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$, which is abbreviated to $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$.

A sequence $\Lambda = \{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)\}_{j\in\mathbb{J}}$ denotes a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ if there are real numbers $0 < C \leq D < \infty$ satisfying

$$C\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle, \quad \forall f \in \mathcal{H}. \tag{1}$$

If only the second inequality in Equation (1) is required, then Λ is said to be a g-Bessel sequence.

For a given g-frame $\Lambda = \{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)\}_{j\in\mathbb{J}}$, there is always a positive, invertible, and self-adjoint operator in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, which we call the g-frame operator of Λ, defined by

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f. \tag{2}$$

For any $\mathbb{I} \subset \mathbb{J}$, let $\mathbb{I}^c$ be the complement of $\mathbb{I}$. We define a positive and self-adjoint operator in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ related to $\mathbb{I}$ and a g-frame $\Lambda = \{\Lambda_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)\}_{j\in\mathbb{J}}$ in the following form

$$S_{\mathbb{I}}^\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_{\mathbb{I}}^\Lambda f = \sum_{j\in\mathbb{I}} \Lambda_j^* \Lambda_j f. \tag{3}$$

Recall that a g-Bessel $\Gamma = \{\Gamma_j \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)\}_{j\in\mathbb{J}}$ is an alternate dual g-frame of Λ if, for every $f \in \mathcal{H}$, we have $f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Gamma_j f$.

Let $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ and $\Gamma = \{\Gamma_j\}_{j\in\mathbb{J}}$ be g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$. We observe from the Cauchy–Schwarz inequality that the operator

$$S_{\Gamma\Lambda} : \mathcal{H} \to \mathcal{H}, \quad S_{\Gamma\Lambda} f = \sum_{j\in\mathbb{J}} \Gamma_j^* \Lambda_j f \tag{4}$$

is well defined, and a direct calculation shows that $S_{\Gamma\Lambda} \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$.

2. The Main Results

The following result for operators is used to prove our main results.

Lemma 1. *Suppose that $U, V, L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and that $U + V = L$. Then, for any $\lambda \in \mathbb{R}$, we have*

$$U^*U + \frac{\lambda}{2}(V^*L + L^*V) = V^*V + (1 - \frac{\lambda}{2})(U^*L + L^*U) + (\lambda - 1)L^*L \geq (\lambda - \frac{\lambda^2}{4})L^*L.$$

Proof. On the one hand, we obtain

$$U^*U + \frac{\lambda}{2}(V^*L + L^*V) = U^*U + \frac{\lambda}{2}((L^* - U^*)L + L^*(L - U)) = U^*U - \frac{\lambda}{2}(U^*L + L^*U) + \lambda L^*L.$$

On the other hand, we have

$$\begin{aligned}
&V^*V + (1 - \frac{\lambda}{2})(U^*L + L^*U) + (\lambda - 1)L^*L \\
&\quad = (L^* - U^*)(L - U) + (U^*L + L^*U) - \frac{\lambda}{2}(U^*L + L^*U) + (\lambda - 1)L^*L \\
&\quad = L^*L - (U^*L + L^*U) + U^*U + (U^*L + L^*U) - \frac{\lambda}{2}(U^*L + L^*U) + (\lambda - 1)L^*L \\
&\quad = U^*U - \frac{\lambda}{2}(U^*L + L^*U) + \lambda L^*L = (U - \frac{\lambda}{2}L)^*(U - \frac{\lambda}{2}L) + (\lambda - \frac{\lambda^2}{4})L^*L \geq (\lambda - \frac{\lambda^2}{4})L^*L.
\end{aligned}$$

This completes the proof. □

Theorem 1. *Let $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$. Suppose that $\Gamma = \{\Gamma_j\}_{j\in\mathbb{J}}$ and $\Theta = \{\Theta_j\}_{j\in\mathbb{J}}$ are two g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$, and that the operator $S_{\Gamma\Lambda}$ is defined in Equation (4). Then, for any $\lambda \in \mathbb{R}$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\right|^2+\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j S_{\Gamma\Lambda}f\rangle=\left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2+\sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)S_{\Gamma\Lambda}f,\Lambda_j f\rangle\\
&\geq(\lambda-\frac{\lambda^2}{4})\sum_{j\in\mathbb{J}}\langle\Lambda_j f,(\Gamma_j-\Theta_j)S_{\Gamma\Lambda}f\rangle+(1+\frac{\lambda}{2}-\frac{\lambda^2}{4})\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j S_{\Gamma\Lambda}f\rangle\\
&-\frac{\lambda}{2}\sum_{j\in\mathbb{J}}\langle\Theta_j S_{\Gamma\Lambda}f,\Lambda_j f\rangle.
\end{aligned}\tag{5}$$

Proof. We let

$$Uf=\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f \quad\text{and}\quad Vf=\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f \tag{6}$$

for each $f \in \mathcal{H}$. Then, $U, V \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ and, further,

$$Uf+Vf=\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f+\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f=\sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j f=S_{\Gamma\Lambda}f.$$

By Lemma 1, we get

$$\begin{aligned}
&|Uf|^2+\frac{\lambda}{2}(\langle Vf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Vf\rangle)\\
&=|Vf|^2+(1-\frac{\lambda}{2})(\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle)+(\lambda-1)|S_{\Gamma\Lambda}f|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
|Uf|^2&=|Vf|^2+(1-\frac{\lambda}{2})(\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle)+(\lambda-1)|S_{\Gamma\Lambda}f|^2\\
&\quad-\frac{\lambda}{2}(\langle Vf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Vf\rangle)\\
&=|Vf|^2+\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle-\frac{\lambda}{2}(\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle)\\
&\quad-\frac{\lambda}{2}(\langle Vf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Vf\rangle)+(\lambda-1)|S_{\Gamma\Lambda}f|^2\\
&=|Vf|^2+\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle-\frac{\lambda}{2}(\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle Vf,S_{\Gamma\Lambda}f\rangle)\\
&\quad-\frac{\lambda}{2}(\langle S_{\Gamma\Lambda}f,Uf\rangle+\langle S_{\Gamma\Lambda}f,Vf\rangle)+(\lambda-1)|S_{\Gamma\Lambda}f|^2\\
&=|Vf|^2+\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle-\lambda|S_{\Gamma\Lambda}f|^2+(\lambda-1)|S_{\Gamma\Lambda}f|^2\\
&=|Vf|^2+\langle Uf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle-\langle Uf,S_{\Gamma\Lambda}f\rangle-\langle Vf,S_{\Gamma\Lambda}f\rangle.
\end{aligned}$$

It follows that

$$|Uf|^2+\langle Vf,S_{\Gamma\Lambda}f\rangle=|Vf|^2+\langle S_{\Gamma\Lambda}f,Uf\rangle, \tag{7}$$

from which we arrive at

$$\left|\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\right|^2+\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j S_{\Gamma\Lambda}f\rangle=\left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2+\sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)S_{\Gamma\Lambda}f,\Lambda_j f\rangle.$$

We are now in a position to prove the inequality in Equation (5).

Again by Lemma 1,

$$
\begin{aligned}
|Uf|^2 &\geq (\lambda - \frac{\lambda^2}{4})|S_{\Gamma\Lambda}f|^2 - \frac{\lambda}{2}(\langle Vf, S_{\Gamma\Lambda}f\rangle + \langle S_{\Gamma\Lambda}f, Vf\rangle) \\
&= (\lambda - \frac{\lambda^2}{4})\langle Uf, S_{\Gamma\Lambda}f\rangle + (\lambda - \frac{\lambda^2}{4})\langle Vf, S_{\Gamma\Lambda}f\rangle - \frac{\lambda}{2}\langle Vf, S_{\Gamma\Lambda}f\rangle - \frac{\lambda}{2}\langle S_{\Gamma\Lambda}f, Vf\rangle \\
&= (\lambda - \frac{\lambda^2}{4})\langle Uf, S_{\Gamma\Lambda}f\rangle + (\frac{\lambda}{2} - \frac{\lambda^2}{4})\langle Vf, S_{\Gamma\Lambda}f\rangle - \frac{\lambda}{2}\langle S_{\Gamma\Lambda}f, Vf\rangle. \qquad (8)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\sum_{j\in\mathbb{J}}(\Gamma_j - \Theta_j)^*\Lambda_j f\right|^2 + \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_{\Gamma\Lambda}f\rangle = |Uf|^2 + \langle Vf, S_{\Gamma\Lambda}f\rangle \\
&\quad\geq (\lambda - \frac{\lambda^2}{4})\langle Uf, S_{\Gamma\Lambda}f\rangle + (1 + \frac{\lambda}{2} - \frac{\lambda^2}{4})\langle Vf, S_{\Gamma\Lambda}f\rangle - \frac{\lambda}{2}\langle S_{\Gamma\Lambda}f, Vf\rangle \\
&\quad= (\lambda - \frac{\lambda^2}{4})\sum_{j\in\mathbb{J}}\langle\Lambda_j f, (\Gamma_j - \Theta_j)S_{\Gamma\Lambda}f\rangle + (1 + \frac{\lambda}{2} - \frac{\lambda^2}{4})\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_{\Gamma\Lambda}f\rangle - \frac{\lambda}{2}\sum_{j\in\mathbb{J}}\langle\Theta_j S_{\Gamma\Lambda}f, \Lambda_j f\rangle
\end{aligned}
$$

for any $f \in \mathcal{H}$. □

Corollary 1. *Suppose that $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ with g-frame operator S_Λ and that $\tilde{\Lambda}_j = \Lambda_j S_\Lambda^{-1}$ for each $j \in \mathbb{J}$. Then, for any $\lambda \in \mathbb{R}$, for all $\mathbb{I} \subset \mathbb{J}$ and all $f \in \mathcal{H}$, we have*

$$
\begin{aligned}
\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle + \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}^c}f, \tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}^c}f\rangle &= \sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}}f, \tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}}f\rangle \\
&\geq (\lambda - \frac{\lambda^2}{4})\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + (1 - \frac{\lambda^2}{4})\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle.
\end{aligned}
$$

Proof. Taking $\Gamma_j = \Lambda_j S_\Lambda^{-\frac{1}{2}}$ for any $j \in \mathbb{J}$, then it is easy to see that $S_{\Gamma\Lambda} = S_\Lambda^{\frac{1}{2}}$. For each $j \in \mathbb{J}$, let

$$
\Theta_j = \begin{cases} \Gamma_j, & j \in \mathbb{I}, \\ 0, & j \in \mathbb{I}^c. \end{cases}
$$

Now, for each $f \in \mathcal{H}$,

$$
\begin{aligned}
\left|\sum_{j\in\mathbb{J}}(\Gamma_j - \Theta_j)^*\Lambda_j f\right|^2 &= \left|\sum_{j\in\mathbb{I}^c} S_\Lambda^{-\frac{1}{2}}\Lambda_j^*\Lambda_j f\right|^2 = |S_\Lambda^{-\frac{1}{2}}S^{\Lambda}_{\mathbb{I}^c}f|^2 = \langle S_\Lambda^{-\frac{1}{2}}S^{\Lambda}_{\mathbb{I}^c}f, S_\Lambda^{-\frac{1}{2}}S^{\Lambda}_{\mathbb{I}^c}f\rangle \\
&= \langle S^{\Lambda}_{\mathbb{I}^c}f, S_\Lambda^{-1}S^{\Lambda}_{\mathbb{I}^c}f\rangle = \langle S_\Lambda S_\Lambda^{-1}S^{\Lambda}_{\mathbb{I}^c}f, S_\Lambda^{-1}S^{\Lambda}_{\mathbb{I}^c}f\rangle \\
&= \sum_{j\in\mathbb{J}}\langle\Lambda_j S_\Lambda^{-1}S^{\Lambda}_{\mathbb{I}^c}f, \Lambda_j S_\Lambda^{-1}S^{\Lambda}_{\mathbb{I}^c}f\rangle = \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}^c}f, \tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}^c}f\rangle. \qquad (9)
\end{aligned}
$$

Since $|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f|^2 = |\sum_{j\in\mathbb{I}}\Gamma_j^*\Lambda_j f|^2 = |\sum_{j\in\mathbb{I}} S_\Lambda^{-\frac{1}{2}}\Lambda_j^*\Lambda_j f|^2$, a replacement of $\mathbb{I}^c$ by $\mathbb{I}$ in the last item of Equation (9) leads to

$$
\left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2 = \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}}f, \tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}}f\rangle. \qquad (10)
$$

We also have

$$
\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_{\Gamma\Lambda}f\rangle = \sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle, \quad \sum_{j\in\mathbb{J}}\langle(\Gamma_j - \Theta_j)S_{\Gamma\Lambda}f, \Lambda_j f\rangle = \sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle. \qquad (11)
$$

Hence, the conclusion follows from Theorem 1. □

Let $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ be a Parseval g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$; then, $S_\Lambda = \mathrm{Id}_{\mathcal{H}}$. Thus, for any $\mathbb{I} \subset \mathbb{J}$,

$$\sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}^c} f, \tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}^c} f\rangle = \sum_{j\in\mathbb{J}}\langle\Lambda_j S^{\Lambda}_{\mathbb{I}^c} f, \Lambda_j S^{\Lambda}_{\mathbb{I}^c} f\rangle = |S^{\Lambda}_{\mathbb{I}^c} f|^2 = \Big|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big|^2.$$

Similarly,

$$\sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}} f, \tilde{\Lambda}_j S^{\Lambda}_{\mathbb{I}} f\rangle = \Big|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big|^2.$$

This fact, together with Corollary 1, yields

Corollary 2. *Suppose that $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ is a Parseval g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$. Then, for any $\lambda \in \mathbb{R}$, for all $\mathbb{I} \subset \mathbb{J}$ and all $f \in \mathcal{H}$, we have*

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle + \Big|\sum_{j\in\mathbb{I}^c}\Lambda_j^*\Lambda_j f\Big|^2 &= \sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + \Big|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\Big|^2 \\ &\geq (\lambda - \frac{\lambda^2}{4})\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + (1 - \frac{\lambda^2}{4})\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle.\end{aligned}$$

Corollary 3. *Suppose that $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ with an alternate dual g-frame $\Gamma = \{\Gamma_j\}_{j\in\mathbb{J}}$. Then, for any $\lambda \in \mathbb{R}$, for all $\mathbb{I} \subset \mathbb{J}$ and all $f \in \mathcal{H}$, we have*

$$\begin{aligned}\Big|\sum_{j\in\mathbb{I}}\Gamma_j^*\Lambda_j f\Big|^2 + \sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Gamma_j f\rangle &= \Big|\sum_{j\in\mathbb{I}^c}\Gamma_j^*\Lambda_j f\Big|^2 + \sum_{j\in\mathbb{I}}\langle\Gamma_j f, \Lambda_j f\rangle \\ &\geq (\lambda - \frac{\lambda^2}{4})\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Gamma_j f\rangle + (1 + \frac{\lambda}{2} - \frac{\lambda^2}{4})\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Gamma_j f\rangle - \frac{\lambda}{2}\sum_{j\in\mathbb{I}^c}\langle\Gamma_j f, \Lambda_j f\rangle.\end{aligned}$$

Proof. We conclude first that $S_{\Gamma\Lambda} = \mathrm{Id}_{\mathcal{H}}$. Now, the result follows immediately from Theorem 1 if, for any $\mathbb{I} \subset \mathbb{J}$, we take $\Theta_j = \begin{cases}\Gamma_j, & j \in \mathbb{I}^c, \\ 0, & j \in \mathbb{I}.\end{cases}$ □

Remark 1. *Theorems 4.1 and 4.2 in [31] can be obtained if we take $\lambda = 1$, respectively, in Corollaries 1 and 2.*

Theorem 2. *Let $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$. Suppose that $\Gamma = \{\Gamma_j\}_{j\in\mathbb{J}}$ and $\Theta = \{\Theta_j\}_{j\in\mathbb{J}}$ are two g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ and that the operator $S_{\Gamma\Lambda}$ is defined in Equation (4). Then, for any $\lambda \in \mathbb{R}$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned}\Big|\sum_{j\in\mathbb{J}}(\Gamma_j - \Theta_j)^*\Lambda_j f\Big|^2 + \Big|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\Big|^2 \geq (\lambda - \frac{\lambda^2}{2})\Big|\sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j f\Big|^2 - (1-\lambda)\sum_{j\in\mathbb{J}}\langle(\Gamma_j - \Theta_j)S_{\Gamma\Lambda} f, \Lambda_j f\rangle \\ + (1-\lambda)\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_{\Gamma\Lambda} f\rangle.\end{aligned}$$

Moreover, if U^*V is positive, where U and V are given in Equation (6), then

$$\left|\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\right|^2+\left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2$$
$$\leq \sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)S_{\Gamma\Lambda}f,\Lambda_j f\rangle+\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j S_{\Gamma\Lambda}f\rangle. \tag{12}$$

Proof. Combining Equation (7) with Lemma 1, we obtain

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\right|^2+\left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2\\
&\quad=|Uf|^2+|Vf|^2=2|Vf|^2+\langle S_{\Gamma\Lambda}f,Uf\rangle-\langle Vf,S_{\Gamma\Lambda}f\rangle\\
&\quad\geq(2-\frac{\lambda^2}{2})|S_{\Gamma\Lambda}f|^2-(2-\lambda)(\langle S_{\Gamma\Lambda}f,Uf\rangle+\langle Uf,S_{\Gamma\Lambda}f\rangle)+\langle S_{\Gamma\Lambda}f,Uf\rangle-\langle Vf,S_{\Gamma\Lambda}f\rangle\\
&\quad=(2-\frac{\lambda^2}{2})|S_{\Gamma\Lambda}f|^2-(2-\lambda)\langle S_{\Gamma\Lambda}f,Uf\rangle-(2-\lambda)\langle Uf,S_{\Gamma\Lambda}f\rangle\\
&\qquad-(2-\lambda)\langle Vf,S_{\Gamma\Lambda}f\rangle+(1-\lambda)\langle Vf,S_{\Gamma\Lambda}f\rangle+\langle S_{\Gamma\Lambda}f,Uf\rangle\\
&\quad=(2-\frac{\lambda^2}{2})|S_{\Gamma\Lambda}f|^2-(1-\lambda)\langle S_{\Gamma\Lambda}f,Uf\rangle-(2-\lambda)|S_{\Gamma\Lambda}f|^2+(1-\lambda)\langle Vf,S_{\Gamma\Lambda}f\rangle\\
&\quad=(\lambda-\frac{\lambda^2}{2})|S_{\Gamma\Lambda}f|^2-(1-\lambda)\langle S_{\Gamma\Lambda}f,Uf\rangle+(1-\lambda)\langle Vf,S_{\Gamma\Lambda}f\rangle\\
&\quad=(\lambda-\frac{\lambda^2}{2})\left|\sum_{j\in\mathbb{J}}\Gamma_j^*\Lambda_j f\right|^2-(1-\lambda)\sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)S_{\Gamma\Lambda}f,\Lambda_j f\rangle+(1-\lambda)\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j S_{\Gamma\Lambda}f\rangle
\end{aligned}$$

for any $f\in\mathcal{H}$. We next prove Equation (12). Since U^*V is positive, we see from Equation (7) that

$$|Uf|^2=|Vf|^2+\langle S_{\Gamma\Lambda}f,Uf\rangle-\langle Vf,S_{\Gamma\Lambda}f\rangle=\langle S_{\Gamma\Lambda}f,Uf\rangle-\langle Vf,Uf\rangle\leq\langle S_{\Gamma\Lambda}f,Uf\rangle$$

for each $f\in\mathcal{H}$. A similar discussion gives $|Vf|^2\leq\langle Vf,S_{\Gamma\Lambda}f\rangle$. Thus,

$$\begin{aligned}
\left|\sum_{j\in\mathbb{J}}(\Gamma_j-\Theta_j)^*\Lambda_j f\right|^2+\left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2&=|Uf|^2+|Vf|^2\leq\langle S_{\Gamma\Lambda}f,Uf\rangle+\langle Vf,S_{\Gamma\Lambda}f\rangle\\
&=\sum_{j\in\mathbb{J}}\langle(\Gamma_j-\Theta_j)S_{\Gamma\Lambda}f,\Lambda_j f\rangle+\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Theta_j S_{\Gamma\Lambda}f\rangle.
\end{aligned}$$

□

Corollary 4. *Let $\Lambda=\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ with g-frame operator S_Λ, and $\tilde{\Lambda}_j=\Lambda_j S_\Lambda^{-1}$ for each $j\in\mathbb{J}$. Then, for any $\lambda\in\mathbb{R}$, for all $\mathbb{I}\subset\mathbb{J}$ and all $f\in\mathcal{H}$, we have*

$$\begin{aligned}
&(\lambda-\frac{\lambda^2}{2})\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Lambda_j f\rangle-(1-\lambda)\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f,\Lambda_j f\rangle+(1-\lambda)\sum_{j\in\mathbb{I}}\langle\Lambda_j f,\Lambda_j f\rangle\\
&\qquad\leq\sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda}f,\tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda}f\rangle+\sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S_{\mathbb{I}^c}^{\Lambda}f,\tilde{\Lambda}_j S_{\mathbb{I}^c}^{\Lambda}f\rangle\leq\sum_{j\in\mathbb{J}}\langle\Lambda_j f,\Lambda_j f\rangle.
\end{aligned}$$

Proof. For every $j\in\mathbb{J}$, taking $\Gamma_j=\Lambda_j S_\Lambda^{-\frac{1}{2}}$ and $\Theta_j=\begin{cases}\Gamma_j, & j\in\mathbb{I},\\ 0, & j\in\mathbb{I}^c,\end{cases}$ then the operators U and V defined in Equation (6) can be expressed as $U=S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}^c}^{\Lambda}$ and $V=S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}}^{\Lambda}$, respectively. Hence, $U^*V=S_{\mathbb{I}^c}^{\Lambda}S_\Lambda^{-1}S_{\mathbb{I}}^{\Lambda}$. Since $S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}}^{\Lambda}S_\Lambda^{-\frac{1}{2}}$ and $S_\Lambda^{-\frac{1}{2}}S_{\mathbb{I}^c}^{\Lambda}S_\Lambda^{-\frac{1}{2}}$ are positive and commutative, it follows that

$$0 \leq S_{\Lambda}^{-\frac{1}{2}} S_{\mathbb{I}^c}^{\Lambda} S_{\Lambda}^{-\frac{1}{2}} S_{\Lambda}^{-\frac{1}{2}} S_{\mathbb{I}}^{\Lambda} S_{\Lambda}^{-\frac{1}{2}} = S_{\Lambda}^{-\frac{1}{2}} S_{\mathbb{I}^c}^{\Lambda} S_{\Lambda}^{-1} S_{\mathbb{I}}^{\Lambda} S_{\Lambda}^{-\frac{1}{2}},$$

and, consequently, $S_{\mathbb{I}^c}^{\Lambda} S_{\Lambda}^{-1} S_{\mathbb{I}}^{\Lambda} \geq 0$. Note also that

$$\left|\sum_{j\in\mathbb{J}} \Gamma_j^* \Lambda_j f\right|^2 = \left|S_{\Lambda}^{-\frac{1}{2}} \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f\right|^2 = |S_{\Lambda}^{\frac{1}{2}} f|^2 = \langle S_\Lambda f, f\rangle = \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Lambda_j f\rangle.$$

Now, the result follows by combining Theorem 2 and Equations (9)–(11). □

Theorem 3. *Let $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ with g-frame operator S_Λ. Suppose that $\Gamma = \{\Gamma_j\}_{j\in\mathbb{J}}$ and $\Theta = \{\Theta_j\}_{j\in\mathbb{J}}$ are two g-Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ and that the operator $S_{\Gamma\Lambda}$ is defined in Equation (4). Then, for any $\lambda \in \mathbb{R}$ and any $f \in \mathcal{H}$, we have*

$$\begin{aligned}\sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Theta_j S_{\Lambda}^{\frac{1}{2}} f\rangle - \left|\sum_{j\in\mathbb{J}} \Theta_j^* \Lambda_j f\right|^2 \leq \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Theta_j (S_{\Lambda}^{\frac{1}{2}} - S_{\Gamma\Lambda}) f\rangle - \frac{\lambda}{2} \sum_{j\in\mathbb{J}} \langle \Lambda_j f, (\Gamma_j - \Theta_j) S_{\Gamma\Lambda} f\rangle \\ + (1 - \frac{\lambda}{2}) \sum_{j\in\mathbb{J}} \langle (\Gamma_j - \Theta_j) S_{\Gamma\Lambda} f, \Lambda_j f\rangle + \frac{\lambda^2}{4} \left|\sum_{j\in\mathbb{J}} \Gamma_j^* \Lambda_j f\right|^2.\end{aligned}$$

Moreover, if U^*V is positive, where U and V are given in Equation (6), then

$$\sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Theta_j S_{\Lambda}^{\frac{1}{2}} f\rangle - \left|\sum_{j\in\mathbb{J}} \Theta_j^* \Lambda_j f\right|^2 \geq \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Theta_j (S_{\Lambda}^{\frac{1}{2}} - S_{\Gamma\Lambda}) f\rangle.$$

Proof. Combining Equations (7) and (8) leads to

$$\begin{aligned}&\sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Theta_j S_{\Lambda}^{\frac{1}{2}} f\rangle - \left|\sum_{j\in\mathbb{J}} \Theta_j^* \Lambda_j f\right|^2 = \langle S_{\Lambda}^{\frac{1}{2}} V f, f\rangle - |Vf|^2 \\ &\leq \langle S_{\Lambda}^{\frac{1}{2}} V f, f\rangle - (\lambda - \frac{\lambda^2}{4}) \langle Uf, S_{\Gamma\Lambda} f\rangle - (\frac{\lambda}{2} - \frac{\lambda^2}{4}) \langle Vf, S_{\Gamma\Lambda} f\rangle \\ &\quad + \frac{\lambda}{2} \langle S_{\Gamma\Lambda} f, Vf\rangle - \langle Vf, S_{\Gamma\Lambda} f\rangle + \langle S_{\Gamma\Lambda} f, Uf\rangle \\ &= \langle Vf, (S_{\Lambda}^{\frac{1}{2}} - S_{\Gamma\Lambda}) f\rangle - (\frac{\lambda}{2} - \frac{\lambda^2}{4})(\langle Uf, S_{\Gamma\Lambda} f\rangle + \langle Vf, S_{\Gamma\Lambda} f\rangle) - \frac{\lambda}{2} \langle Uf, S_{\Gamma\Lambda} f\rangle \\ &\quad + \frac{\lambda}{2}(\langle S_{\Gamma\Lambda} f, Vf\rangle + \langle S_{\Gamma\Lambda} f, Uf\rangle) + (1 - \frac{\lambda}{2}) \langle S_{\Gamma\Lambda} f, Uf\rangle \\ &= \langle Vf, (S_{\Lambda}^{\frac{1}{2}} - S_{\Gamma\Lambda}) f\rangle - (\frac{\lambda}{2} - \frac{\lambda^2}{4}) |S_{\Gamma\Lambda} f|^2 \\ &\quad - \frac{\lambda}{2} \langle Uf, S_{\Gamma\Lambda} f\rangle + \frac{\lambda}{2} |S_{\Gamma\Lambda} f|^2 + (1 - \frac{\lambda}{2}) \langle S_{\Gamma\Lambda} f, Uf\rangle \\ &= \langle Vf, (S_{\Lambda}^{\frac{1}{2}} - S_{\Gamma\Lambda}) f\rangle + \frac{\lambda^2}{4} |S_{\Gamma\Lambda} f|^2 - \frac{\lambda}{2} \langle Uf, S_{\Gamma\Lambda} f\rangle + (1 - \frac{\lambda}{2}) \langle S_{\Gamma\Lambda} f, Uf\rangle \\ &= \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Theta_j (S_{\Lambda}^{\frac{1}{2}} - S_{\Gamma\Lambda}) f\rangle - \frac{\lambda}{2} \sum_{j\in\mathbb{J}} \langle \Lambda_j f, (\Gamma_j - \Theta_j) S_{\Gamma\Lambda} f\rangle \\ &\quad + (1 - \frac{\lambda}{2}) \sum_{j\in\mathbb{J}} \langle (\Gamma_j - \Theta_j) S_{\Gamma\Lambda} f, \Lambda_j f\rangle + \frac{\lambda^2}{4} \left|\sum_{j\in\mathbb{J}} \Gamma_j^* \Lambda_j f\right|^2, \quad \forall f \in \mathcal{H}.\end{aligned}$$

Suppose that U^*V is positive; then, $|Vf|^2 \leq \langle Vf, S_{\Gamma\Lambda}f\rangle$. Now, the "Moreover" part follows from the following inequality:

$$\begin{aligned}\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_\Lambda^{\frac{1}{2}} f\rangle - \left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2 &= \langle S_\Lambda^{\frac{1}{2}} Vf, f\rangle - |Vf|^2 \geq \langle S_\Lambda^{\frac{1}{2}} Vf, f\rangle - \langle Vf, S_{\Gamma\Lambda}f\rangle \\ &= \langle Vf, (S_\Lambda^{\frac{1}{2}} - S_{\Gamma\Lambda})f\rangle = \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j(S_\Lambda^{\frac{1}{2}} - S_{\Gamma\Lambda})f\rangle.\end{aligned}$$

□

Corollary 5. *Let $\Lambda = \{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in\mathbb{J}}$ with g-frame operator S_Λ. Then, for any $\lambda \in \mathbb{R}$, for all $\mathbb{I} \subset \mathbb{J}$ and all $f \in \mathcal{H}$, we have*

$$\begin{aligned}0 &\leq \sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle - \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda} f, \tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda} f\rangle \\ &\leq (1-\lambda)\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + \frac{\lambda^2}{4}\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle.\end{aligned}$$

Proof. For each $j \in \mathbb{J}$, let Γ_j and Θ_j be the same as in the proof of Corollary 4. By Theorem 3, we have

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle - \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda} f, \tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda} f\rangle &= \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_\Lambda^{\frac{1}{2}} f\rangle - \left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2 \\ \leq -\frac{\lambda}{2}\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + (1-\frac{\lambda}{2})\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + \frac{\lambda^2}{4}\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle \\ = (1-\lambda)\sum_{j\in\mathbb{I}^c}\langle\Lambda_j f, \Lambda_j f\rangle + \frac{\lambda^2}{4}\sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Lambda_j f\rangle.\end{aligned}$$

By Theorem 3 again,

$$\begin{aligned}\sum_{j\in\mathbb{I}}\langle\Lambda_j f, \Lambda_j f\rangle - \sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda} f, \tilde{\Lambda}_j S_{\mathbb{I}}^{\Lambda} f\rangle &= \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j S_\Lambda^{\frac{1}{2}} f\rangle - \left|\sum_{j\in\mathbb{J}}\Theta_j^*\Lambda_j f\right|^2 \\ &\geq \sum_{j\in\mathbb{J}}\langle\Lambda_j f, \Theta_j(S_\Lambda^{\frac{1}{2}} - S_{\Gamma\Lambda})f\rangle = 0,\end{aligned}$$

and the proof is finished. □

Remark 2. *Taking $\lambda = 1$ in Corollaries 4 and 5, we can obtain Theorem 2.4 in [29].*

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