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The Effect of Prudence on the Optimal Allocation in Possibilistic and Mixed Models

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Abstract: In this paper, several portfolio choice models are studied: a purely possibilistic model in which the return of the risky is a fuzzy number, and four models in which the background risk appears in addition to the investment risk. In these four models, risk is a bidimensional vector whose components are random variables or fuzzy numbers. Approximate formulas of the optimal allocation are obtained for all models, expressed in terms of some probabilistic or possibilistic moments, depending on the indicators of the investor preferences (risk aversion, prudence).

Keywords: prudence; optimal allocation; possibilistic moments

1. Introduction

The standard portfolio choice problem [1–3] considers the determination of the optimal proportion of the wealth an agent invests in a risk-free asset and in a risky asset. The study of this probabilistic model is usually done in the classical expected utility theory. The optimal allocation of a risky asset appears as the solution of a maximization problem. By Taylor approximations, several forms of the solution have been found, depending on different moments of the return of the risky asset, as well as some on indicators of the investors's risk preferences. In the form of the solution from [4] Chapter 2 or [5] Chapter 5, the mean value, the variance, and the Arrow–Pratt index of the investor's utility function appear. The approach from [6–8] led to forms of the approximate solution which depend on the first three moments, the Arrow–Pratt index r_u , and the prudence index P_u [9]. The solution found in [10] is expressed according to the first four moments and the indicators of risk aversion, prudence, and temperance of the utility function. Another form of the solution in which the first four moments appear can be found in [11].

All the above models are probabilistic, the risk is represented by random variables, and the attitude of the agent towards risk is expressed by notions and properties which use the probabilistic indicators (expected value, variance, covariance, moments, etc.). The probabilistic modeling does not cover all uncertainty situations in which risk appears (e.g., when the information is not extracted from a sufficiently large volume of data). Possibility theory, initiated by Zadeh in [12] can model different situations: “while probability theory offers a quantitative model for randomness and indecisiveness, possibility theory offers a qualitative model of incomplete knowledge” ([13], p. 277).

In possibility theory, risk is modeled by the notion of possibilistic distribution [14–16]. Fuzzy numbers are the most important class of possibilistic distribution [17]. They generalize real numbers, and by Zadeh's extension principle [12], the operations with real numbers can be extended to operations with fuzzy numbers. So, the set of fuzzy numbers is endowed with a rich algebraic structure, very close to the set of real numbers, and their possibilistic indicators (possibilistic expected value, possibilistic variance, possibilistic moments, etc.) have important mathematical properties [14–19]. Fuzzy numbers are also capable of modelling a large scope of risk situations ([14–16,20–25]). For this, most studies on possibilistic risk have been done in the framework offered by fuzzy numbers,

although there exist approaches on possibilistic risk in contexts larger than that offered by fuzzy numbers. For example, in [26] there is a treatment of risk aversion in an abstract framework including fuzzy numbers, random fuzzy numbers, type-2 fuzzy sets, random type-2 fuzzy sets, etc.)

In this paper, several portfolio choice models are studied: a purely possibilistic model, in which the return of the risky asset is represented by a fuzzy number [14,15], and four more models, in which a probabilistic or possibilistic background risk appears. In the formulation of the maximization problem for the first model, the possibilistic expected utility from [16], definition 4.2.7, is used. In the case of the other four models, the notion of bidimensional possibilistic expected utility ([16], p. 60) or the bidimensional mixed expected utility ([16], p. 79) is used. The approximate solutions of these two models are expressed by the possibilistic moments associated with a random variable, a fuzzy number ([14,15,24,25]), and by the indicators on the investor risk preferences.

In the first part of Section 2 the definitions of possibilistic expected utility (cf. [16]) and possibilistic indicators of a fuzzy number (expected value, variance, moments) are presented. The second part of the section contains the definition of a mixed expected utility associated with a mixed vector, a bidimensional utility function, and a weighting function ([16]).

Section 3 is concerned with the possibilistic standard portfolio-choice model, whose construction is inspired by the probabilistic model of [10]. The return of the risky asset is here a fuzzy number, while in [10] it is a random variable. The total utility function of the model is written as a possibilistic expected value. The maximization problem of the model and the first-order conditions are formulated, from which its optimal solution is determined.

Section 4 is dedicated to the optimal asset allocation in the framework of the possibilistic portfolio model defined in the previous section. Using a second-order Taylor approximation, a formula for the approximate calculation of the maximization problem solution is found. In the component of the formula appear the first three possibilistic moments, the Arrow–Pratt index, and the prudence indices of the investor's utility function. The general formula is particularized for triangular fuzzy numbers and HARA (hyperbolic absolute risks aversion) and CRRA (constant relative risk aversion) utility functions.

In Section 5 four moments are defined in which the background risk appears in addition to the investment risk. In these models, risk is represented by a bidimensional vector whose components are either random variables or fuzzy numbers. The agent will have a unidimensional utility function, but the total utility function will be:

- a bidimensional probabilistic expected utility, when both components are random variables;
- a bidimensional possibilistic expected utility ([16], p. 60), when both components are fuzzy numbers;
- a mixed expected utility ([16], p. 79), when a component is a random variable, and the other is a fuzzy number.

Section 6 is dedicated to the determination of an approximate calculation formula for the solution of the optimization problems of the four models with background risk from the previous section. We will study in detail only the model in which the investment risk is a fuzzy number and the background risk is a random variable. For the other three cases, only the approximate calculation formulas of the solutions are presented. The proofs are presented in an Appendix A.

2. Preliminaries

In this section we recall some notions and results on the possibilistic expected utility, mixed expected utility (cf. [16]), and some possibilistic indicators associated with fuzzy numbers (cf. [14,18,19,24,25,27]). For the definition and arithmetical properties of the fuzzy numbers, we refer to [14–16].

2.1. Possibilistic Expected Utility

The classic risk theory is usually developed in the framework of expected utility (EU). The main concept of EU theory is the probabilistic expected utility $E(u(X))$ associated with a utility function u (representing the agent) and a random variable X (representing the risk).

In case of a possibilistic risk EU theory, the agent will be represented by a utility function u , and the risk by a fuzzy number A . Besides these, we will consider a weighting function f . The level-sets $[A]^\gamma$, $\gamma \in [0, 1]$ mean a gradualism of risk. By the appearance of f in the definition of possibilistic expected utility and the possibilistic indicators, a weighting of this gradualism is done (by [14], p. 27, “different weighting functions can give different importances to level-sets of possibility distributions”).

Thus, we fix a mathematical context consisting of:

- a utility function u of class $\mathcal{C}^2$,
- a fuzzy number A whose level sets are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$,
- a weighting function $f : [0, 1] \rightarrow \mathbf{R}$. (f is a non-negative and increasing function that satisfies $\int_0^1 f(\gamma) d\gamma = 1$).

The possibilistic expected utility associated with the triple (u, A, f) is

$$E_f(u(A)) = \frac{1}{2} \int_0^1 [u(a_1(\gamma)) + u(a_2(\gamma))] f(\gamma) d\gamma. \quad (1)$$

In the interpretation from [14], p. 27, the possibilistic expected utility can be viewed as the result of the following process: on each γ -level set $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, one considers the uniform distribution. Then, $E_f(u(A))$ is defined as the f -weighted average of the probabilistic expected values of these uniform distributions.

The following possibilistic indicators associated with a fuzzy number A and a weighting function f are particular cases of (1).

- Possibilistic expected value [18,19]:

$$E_f(A) = \frac{1}{2} \int_0^1 [a_1(\gamma) + a_2(\gamma)] f(\gamma) d\gamma, \quad (2)$$

(u is the identity function of $\mathbf{R}$).

- Possibilistic variance [18,27]:

$$Var_f(A) = \frac{1}{2} \int_0^1 [(u(a_1(\gamma)) - E_f(A))^2 + (u(a_2(\gamma)) - E_f(A))^2] f(\gamma) d\gamma, \quad (3)$$

(for $u(x) = (x - E_f(x))^2$, $x \in \mathbf{R}$).

- The n -th order possibilistic moment [24,25]:

$$M(A^n) = \frac{1}{2} \int_0^1 [u^n(a_1(\gamma)) + u^n(a_2(\gamma))] f(\gamma) d\gamma. \quad (4)$$

Proposition 1. Let $g : \mathbf{R} \rightarrow \mathbf{R}$, $h : \mathbf{R} \rightarrow \mathbf{R}$ be two utility functions, $a, b \in \mathbf{R}$ and $u = ag + bh$. Then, $E_f(u(A)) = aE_f(g(A)) + bE_f(h(A))$.

Corollary 1. $E_f(a + bh(A)) = a + bE_f(h(A))$.

2.2. Mixed Expected Utility

In the financial-economic world, as in the social world, there may be complex situations of uncertainty with multiple risk parameters. In the papers on probabilistic risk, such phenomena

are conceptualized by the notion of a random vector (all risk parameters are random variables). However, there can be situations considered as “hybrid”, in which some parameters are random variables and others are fuzzy numbers. This is the notion of the mixed vector, which together with a multidimensional and a weighting function, are the basic entities of the mixed EU theory.

In order to treat a risk problem within a mixed EU theory, it is necessary to have a concept of expected utility.

Since two risk parameters appear in the portfolio choice model with background risk from the paper, we will present the definition of mixed expected utility in the bidimensional case.

A bidimensional mixed vector has the form (A, X) , where A is a fuzzy number and X is a random variable. We will denote by $M(X)$ the expected value of X . If $g : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous function, then $M(g(X))$ is the probabilistic expected utility of X with respect to g .

Let $u : \mathbf{R}^2 \rightarrow \mathbf{R}$ be a bidimensional utility function of class $\mathcal{C}^2$, (A, X) a mixed vector, and $f : [0, 1] \rightarrow \mathbf{R}$ a weighting function. Assume that the level sets of the fuzzy number A are $[A]^\gamma = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. For any $\gamma \in [0, 1]$, we consider the probabilistic expected values $M(u(a_i(\gamma), X))$, $i = 1, 2$.

The mixed expected utility associated with the triple $(u, (A, X), f)$ is:

$$E_f(u(A, X)) = \frac{1}{2} \int_0^1 [M(u(a_1(\gamma), X)) + M(u(a_2(\gamma), X))] f(\gamma) d\gamma. \quad (5)$$

In the definition of $E_f(u(A, X))$, we distinguish the following steps:

- In the first step, the possibilistic risk is parametrized by the decomposition of A in its level sets $[a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$.
- In the second step, for each level γ one considers the parametrized probabilistic utilities $M(u(a_1(\gamma), X))$ and $M(u(a_2(\gamma), X))$.
- In the third step, the mixed expected utility $E_f(u(A, X))$ is obtained as the f -weighted average of the family of means

$$\left(\frac{1}{2} [M(u(a_1(\gamma), X)) + M(u(a_2(\gamma), X))] \right)_{\gamma \in [0, 1]}.$$

Remark 1. If $a \in \mathbf{R}$ then $E_f(u(a, X)) = M(u(a, X))$.

Proposition 2. Let g, h be two bidimensional utility functions, $a, b \in \mathbf{R}$ and $u = ag + bh$. Then, $E_f(u(A, X)) = aE_f(g(A, X)) + bE_f(h(A, X))$.

Propositions 1 and 2 express the linearity of possibilistic expected value and mixed expected utility with respect to the utility functions which appear in the definitions of these two operators.

Corollary 2. If A is a fuzzy number and Z is a random variable, then $E_f(AZ) = M(Z)E_f(A)$ and $E_f(A^2Z) = M(Z)E_f(A^2)$.

3. Possibilistic Standard Model

In this section we will present a possibilistic portfolio choice model in which the return of the risky asset is a fuzzy number. Investing an initial wealth between a risk-free asset (bonds) and a risky asset (stocks), an agent seeks to determine that money allocation in the risky asset such that their winnings are maximum.

In defining the total utility of the model, we will use the possibilistic expected utility introduced in the previous section.

We consider an agent (characterized by a utility function u of class $\mathcal{C}^2$, increasing and concave) which invests a wealth w_0 in a risk-free asset and in a risky asset. The agent invests the amount α in a risky asset and $w_0 - \alpha$ in a risk-free asset. Let r be the return of the risk-free asset and x a value of

the return of the risky asset. We denote by $w = w_0(1 + r)$ the future wealth of the risk-free strategy. The portfolio value $(w_0 - \alpha, \alpha)$ will be (according to [4], pp. 65–66):

$$(w_0 - \alpha)(1 + r) + \alpha(1 + x) = w + \alpha(x - r). \quad (6)$$

The probabilistic investment model from [4] Chapter 4 or [5] Chapter 5 starts from the hypothesis that the return of the risky asset is a random variable X_0 . Then, x is a value of X_0 and (6) leads to the following maximization problem:

$$\max_{\alpha} M[u(w + \alpha(X_0 - r))]. \quad (7)$$

By denoting $X = X_0 - r$ the excess return, the model (7) becomes:

$$\max_{\alpha} M[u(w + \alpha X)]. \quad (8)$$

If we make the assumption that the return of the risky asset is a fuzzy number B_0 , then x will be a value of B_0 . To describe the possibilistic model resulting from such a hypothesis, we fix a weighting function $f : [0, 1] \rightarrow \mathbf{R}$. The expression (6) suggests to us the following optimization problem:

$$\max_{\alpha} E_f[u(w + \alpha(B_0 - r))]. \quad (9)$$

By denoting with $B = B_0 - r$ the excess return, the problem (8) becomes:

$$\max_{\alpha} E_f[u(w + \alpha B)]. \quad (10)$$

There is a similarity between the optimization problem (8) and the optimization problem (10). Between the two optimization problems, there are two fundamental differences:

- In (8) there is a probabilistic risk X , and in (10) there is a possibilistic risk A .
- Problem (8) is formulated in terms of a probabilistic expected utility operator $M(u(\cdot))$, while (10) is formulated using the possibilistic expected utility operator $E_f(u(\cdot))$.

Assume that the level sets of the fuzzy number B are $[B]^{\gamma} = [b_1(\gamma), b_2(\gamma)]$, $\gamma \in [0, 1]$. According to (1), the total utility function of the model (10) will have the following form:

$$V(\alpha) = E_f[u(w + \alpha B)] = \frac{1}{2} \int_0^1 [u(w + \alpha b_1(\gamma)) + u(w + \alpha b_2(\gamma))] f(\gamma) d\gamma.$$

Deriving twice, one obtains:

$$V''(\alpha) = \frac{1}{2} \int_0^1 [b_1^2(\gamma) u''(w + \alpha b_1(\gamma)) + b_2^2(\gamma) u''(w + \alpha b_2(\gamma))] f(\gamma) d\gamma.$$

Since $u'' \leq 0$, it follows $V''(\alpha) \leq 0$, thus V is concave.

We assume everywhere in this paper that the portfolio risk is small, thus analogously with [5] (Section 5.2), we can take the possibilistic excess return B as $B = k\mu + A$, where $\mu > 0$ and A is a fuzzy number with $E_f(A) = 0$. Of course $E_f(B) = k\mu$ in that case. The total utility $V(\alpha)$ will be written:

$$V(\alpha) = E_f[u(w + \alpha(k\mu + A))]. \quad (11)$$

Assuming that the level sets of A are $[A]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$, the expression (11) becomes:

$$V(\alpha) = \frac{1}{2} \int_0^1 [u(w + \alpha(k\mu + a_1(\gamma))) + u(w + \alpha(k\mu + a_2(\gamma)))] f(\gamma) d\gamma.$$

By deriving, one obtains:

$$V'(\alpha) = \frac{1}{2} \int_0^1 [(k\mu + a_1(\gamma))u'(w + \alpha(k\mu + a_1(\gamma))) + (k\mu + a_2(\gamma))u'(w + \alpha(k\mu + a_2(\gamma)))]f(\gamma)d\gamma,$$

which can be written

$$V'(\alpha) = E_f[(k\mu + A)u'(w + \alpha(k\mu + A))]. \quad (12)$$

Let $\alpha(k)$ be the solution of the maximization problem $\max_{\alpha} V(\alpha)$, with $V(\alpha)$ being written under the form (12). Then, the first order condition $V'(\alpha(k)) = 0$ will be written:

$$E_f[(k\mu + A)u'(w + \alpha(k)(k\mu + A))] = 0. \quad (13)$$

As in [5] (Section 5.2), we assume that $\alpha(0) = 0$.

Everywhere in this paper, we will keep the notations and hypotheses from above.

4. The Effect of Prudence on the Optimal Allocation

The main result of this section is a formula for the approximate calculation of the solution $\alpha(k)$ of Equation (13). In the formula will appear the indicators of absolute risk aversion and prudence, marking how these influence the optimal investment level $\alpha(k)$ in the risky asset.

We will consider the second-order Taylor approximation of $\alpha(k)$ around $k = 0$:

$$\alpha(k) \approx \alpha(0) + k\alpha'(0) + \frac{1}{2}k^2\alpha''(0) = k\alpha'(0) + \frac{1}{2}k^2\alpha''(0). \quad (14)$$

For the approximate calculation of $\alpha(k)$, we will determine the approximate values of $\alpha'(k)$ and $\alpha''(k)$. Note that the calculation of the approximate values of $\alpha'(0)$ and $\alpha''(0)$ follows an analogous line to the one used in [10] in the analysis of the probabilistic model. In the proof of the approximate calculation formulas of $\alpha'(0)$ and $\alpha''(0)$, we will use the properties of the possibilistic expected utility from Section 2.1. Before this, we will recall the Arrow–Pratt index $r_u(w)$ and prudence index $P_u(w)$ associated with the utility function u :

$$r_u(w) = -\frac{u''(w)}{u'(w)}; P_u(w) = -\frac{u'''(w)}{u''(w)}. \quad (15)$$

Proposition 3. $\alpha'(0) \approx \frac{\mu}{E_f(A^2)} \frac{1}{r_u(w)}$.

Proposition 4. $\alpha''(0) \approx \frac{P_u(w)}{(r_u(w))^2} \frac{E_f(A^3)}{(E_f(A^2))^3} \mu^2$.

We recall from Section 3 that $A = B - E_f(B)$. The following result gives us an approximate expression of $\alpha(k)$:

Theorem 1. $\alpha(k) \approx \frac{1}{r_u(w)} \frac{E_f(B)}{\text{Var}_f(B)} + \frac{1}{2} \frac{P_u(w)}{(r_u(w))^2} \frac{E_f[(B - E_f(B))^3]}{(\text{Var}_f(B))^3} (E_f(B))^2$.

Remark 2. The previous theorem gives us an approximate solution of the maximization problem $\max_{\alpha} V(\alpha)$ with respect to the indices of absolute risk aversion and prudence $r_u(w)$, $P_u(w)$, and the first three possibilistic moments $E_f(B)$, $\text{Var}_f(B)$, and $E_f[(B - E_f(B))^3]$.

This result can be seen as a possibilistic version of the formula (A.6) of [10], which gives us the optimal allocation of investment in the context of a probabilistic portfolio choice model.

Example 1. We consider the triangular fuzzy number $B = (b, \alpha, \beta)$ defined by:

$$B(t) = \begin{cases} 1 - \frac{b-x}{\alpha} & \text{if } b - \alpha \leq x \leq b, \\ 1 - \frac{x-b}{\beta} & \text{if } b \leq x \leq b + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

The level sets of B are $[B]^\gamma = [b_1(\gamma), b_2(\gamma)]$, where $b_1(\gamma) = b - (1 - \gamma)\alpha$ and $b_2(\gamma) = b + (1 - \gamma)\beta$, for $\gamma \in [0, 1]$. We assume that the weighting function f has the form $f(\gamma) = 2\gamma$, for $\gamma \in [0, 1]$. Then, by [25], Lemma 2.1:

$$\begin{aligned} E_f(B) &= b + \frac{\beta - \alpha}{6}; \text{Var}_f(B) = \frac{\alpha^2 + \beta^2 + \alpha\beta}{18}, \\ E_f[(B - E_f(B))^2] &= \int_0^1 \gamma[(b_1(\gamma) - E_f(B))^2 + (b_2(\gamma) - E_f(B))^2] d\gamma \\ &= \frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}. \end{aligned}$$

By replacing these indicators in the formula of Theorem 1, we obtain

$$\alpha(k) \approx \frac{1}{r_u(w)} \frac{b + \frac{\beta - \alpha}{6}}{\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}} + \frac{1}{2} \frac{P_u(w)}{(r_u(w))^2} \frac{\frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}}{(\frac{\alpha^2 + \beta^2 + \alpha\beta}{18})^3} (b + \frac{\beta - \alpha}{6})^2.$$

Assume that the utility function u is HARA-type (see [5], Section 3.6):

$$u(w) = \zeta(\eta + \frac{w}{\gamma})^{1-\gamma}, \text{ for } \eta + \frac{w}{\gamma} > 0.$$

Then, according to [5] (Section 3.6):

$$\begin{aligned} r_u(w) &= (\eta + \frac{w}{\gamma})^{-1}; P_u(w) = \frac{\gamma + 1}{\gamma} (\eta + \frac{w}{\gamma})^{-1}, \\ \frac{1}{r_u(w)} &= \eta + \frac{w}{\gamma} \text{ and } \frac{P_u(w)}{(r_u(w))^2} = \frac{\frac{\gamma+1}{\gamma} (\eta + \frac{w}{\gamma})^{-1}}{(\eta + \frac{w}{\gamma})^{-2}} = \frac{\gamma + 1}{\gamma} (\eta + \frac{w}{\gamma}). \end{aligned}$$

Replacing in the approximation calculation formula of $\alpha(k)$, it follows:

$$\alpha(k) \approx (\eta + \frac{w}{\gamma}) \frac{b + \frac{\beta - \alpha}{6}}{\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}} + \frac{1}{2} \frac{\gamma + 1}{\gamma} (\eta + \frac{w}{\gamma}) \frac{\frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}}{(\frac{\alpha^2 + \beta^2 + \alpha\beta}{18})^3} (b + \frac{\beta - \alpha}{6})^2.$$

If $B = (b, \alpha)$ is a symmetric triangular fuzzy number ($\alpha = \beta$), then the approximate solution $\alpha(k)$ gets a very simple form:

$$\alpha(k) \approx 18 \frac{b}{\alpha^2} (\eta + \frac{w}{\gamma}).$$

Following [5] (Section 3.6), we consider the CRRA-type utility function:

$$u(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ \ln(w) & \text{if } \gamma = 1. \end{cases}$$

For $\gamma \neq 1$, we have $r_u(w) = \frac{\gamma}{w}$ and $P_u(w) = \frac{\gamma+1}{w}$. A simple calculation leads to the following form of the solution:

$$\alpha(k) \approx \frac{w}{\gamma} \frac{b + \frac{\beta - \alpha}{6}}{\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}} + \frac{1}{2} \frac{w(\gamma + 1)}{\gamma^2} \frac{\frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}}{\left(\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}\right)^3} \left(b + \frac{\beta - \alpha}{6}\right)^2.$$

5. Models with Background Risk

In the two standard portfolio choice problems (8) and (10), a single risk parameter appears: in (8) the risk is represented by the random variable X , and in (10) by the fuzzy number B . In both cases, we will call it investment risk. More complex situations may exist in which other risk parameters may appear in addition to the investment risk. This supplementary risk is called background risk (see [4,5]).

For simplicity, in this paper we will study investment models with a single background risk parameter. In the interpretation from [4], this background risk is associated with labor income. Therefore, the considered portfolio choice problems will have two types of risk: investment risk and background risk. Each can be random variables or fuzzy numbers, according to the following table.

The models corresponding to the four cases in Table 1 are obtained by adding in (7) and (9) the background risk as a random variable of a fuzzy number. For each problem we will have an approximate solution expressed in terms of indicators, Arrow–Pratt index, and prudence.

Table 1. Models with background risk.

	Investment Risk	Background Risk
1	probabilistic	probabilistic
2	possibilistic	possibilistic
3	possibilistic	probabilistic
4	probabilistic	possibilistic

Case 1. Besides the return of the risky asset X_0 , we will have a probabilistic background risk represented by a random variable Z . Starting from the standard model (7), the following optimization problem is obtained by adding the background risk Z :

$$\max_{\alpha} M[u(w + \alpha(X_0 - r) + Z)]. \quad (16)$$

Case 2. Besides the return of the possibilistic risky-asset B_0 , a possibilistic background risk represented by a fuzzy number C appears. In the standard model (8) the fuzzy number C is added and the following optimization problem is obtained:

$$\max_{\alpha} E_f[u(w + \alpha(B_0 - r) + C)]. \quad (17)$$

Case 3. Besides the investment risk B_0 a probabilistic background risk represented by a random variable Z appears. The optimization problem is obtained adding the random variable Z in (9):

$$\max_{\alpha} E_f[u(w + \alpha(B_0 - r) + Z)]. \quad (18)$$

Case 4. Besides the investment risk X_0 of (7) the possibilistic background risk represented by a fuzzy number C appears:

$$\max_{\alpha} E_f[u(w + \alpha(X_0 - r) + C)]. \quad (19)$$

Problem (17) is formulated in terms of a bidimensional possibilistic expected utility (see [16], p. 60), and (18) and (19) use the mixed expected utility defined in Section 2.

By denoting with $X = X_0$ and $B = B_0 - r$ the probabilistic excess return and the possibilistic excess return, respectively, the optimization problems (16)–(19) become

$$\max_{\alpha} M[u(w + \alpha X + Z)], \quad (20)$$

$$\max_{\alpha} E_f[u(w + \alpha B + C)], \quad (21)$$

$$\max_{\alpha} E_f[u(w + \alpha B + Z)], \quad (22)$$

$$\max_{\alpha} E_f[u(w + \alpha X + C)]. \quad (23)$$

In the following section we will study model 3 in detail, proving an approximate calculation formula of the solution of the optimization problem (18). The proof of the approximate solutions of the other three optimization problems is done similarly.

6. Approximate Solutions of Portfolio Choice Model with Background Risk

In this section we will prove the approximate calculation formulas for the solutions of the optimization problems (20)–(23). These formulas will emphasize how risk aversion and the agent's prudence influence the optimal proportions invested in the risky asset in the case of the four portfolio choice models with background risk. We will study in detail only the mixed model (22), in which, besides this possibilistic risk, a probabilistic background risk may appear, modeled by a random variable Z . This mixed model comes from the possibilistic standard model by adding Z in the composition of the total utility function. More precisely, the total utility function $W(\alpha)$ will be:

$$W(\alpha) = E_f[u(w + \alpha(k\mu + A) + Z)], \quad (24)$$

where the other components of the model have the same meaning as in Section 3.

Assume that the level sets of A are $[A]^\alpha = [a_1(\gamma), a_2(\gamma)]$, $\gamma \in [0, 1]$. By definition (5) of the mixed expected utility, formula (24) can be written as:

$$W(\alpha) = \frac{1}{2} \int_0^1 [M(u(w + \alpha(k\mu + a_1(\gamma)) + Z)) + M(u(w + \alpha(k\mu + a_2(\gamma)) + Z))] f(\gamma) d\gamma.$$

One computes the first derivative of $W(\alpha)$:

$$\begin{aligned} W'(\alpha) &= \frac{1}{2} \int_0^1 (k\mu + a_1(\gamma)) M(u'(w + \alpha(k\mu + a_1(\gamma)) + Z)) f(\gamma) d\gamma + \\ &+ \frac{1}{2} \int_0^1 (k\mu + a_2(\gamma)) M(u'(w + \alpha(k\mu + a_2(\gamma)) + Z)) f(\gamma) d\gamma. \end{aligned}$$

$W'(\alpha)$ can be written as:

$$W'(\alpha) = E_f[(k\mu + A)u'(w + \alpha(k\mu + A) + Z)]. \quad (25)$$

By deriving one more time, we obtain:

$$W''(\alpha) = E_f[(k\mu + A)^2 u''(w + \alpha(k\mu + A) + Z)].$$

Since $u'' \leq 0$, it follows that $W''(\alpha) \leq 0$, and thus W is concave. Then, the solution $\beta(k)$ of the optimization problem $\max_{\alpha} W(\alpha)$ will be given by $W'(\beta(k)) = 0$. By (25),

$$E_f[(k\mu + A)u'(w + \beta(k)(k\mu + A) + Z)] = 0. \quad (26)$$

In this case we will also make the natural hypothesis $\beta(0) = 0$.

To compute an approximate value of $\beta(k)$ we will write the second-order Taylor approximation of $\beta(k)$ around $k = 0$:

$$\beta(k) \approx \beta(0) + k\beta'(0) + \frac{1}{2}k^2\beta''(0) = k\beta'(0) + \frac{1}{2}k^2\beta''(0). \quad (27)$$

We propose to find some approximate values of $\beta'(0)$ and $\beta''(0)$.

Proposition 5. $\beta'(0) \approx \frac{\mu}{E_f(A^2)} \left(\frac{1}{r_u(w)} - M(Z) \right)$.

Proposition 6. $\beta''(0) \approx \frac{P_u(w)(\beta'(0))^2}{\text{Var}_f(B)} \frac{E_f[(B - E_f(B))^3]}{1 - M(Z)P_u(w)}$.

Theorem 2.

$$\beta(k) \approx \frac{E_f(B)}{\text{Var}_f(B)} \left[\frac{1}{r_u(w)} - M(Z) \right] + \frac{1}{2}P_u(w) \left[\frac{1}{r_u(w)} - M(Z) \right]^2 \frac{E_f^2(B)E_f[(B - E_f(B))^3]}{\text{Var}_f^3(B)[1 - M(Z)P_u(w)]}.$$

Remark 3. In the approximate expression of $\beta(k)$ from the previous theorem appear the Arrow index and the prudence index of the utility function u , the possibilistic indicators $E_f(B)$, $\text{Var}_f(B)$, and the possibilistic expected value $M(Z)$.

Example 2. We consider that the investment risk is represented by a fuzzy number $B = (b, \alpha, \beta)$ and the background risk by the random variable Z with the normal distribution $N(m, \sigma^2)$. We will consider a HARA-type utility:

$$u(w) = \zeta \left(\eta + \frac{w}{\gamma} \right)^{1-\gamma}, \text{ for } \eta + \frac{w}{\gamma} > 0.$$

Using the computations from Example 1 and taking into account that $M(Z) = m$, one reaches the following form of the approximate solution:

$$\begin{aligned} \beta(k) \approx & \left(\eta + \frac{w}{\gamma} - m \right) \frac{b + \frac{\beta - \alpha}{6}}{\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}} + \\ & + \frac{\gamma + 1}{2\gamma} \frac{\left(\eta + \frac{w}{\gamma} - m \right)^2}{\eta + \frac{w}{\gamma}} \frac{\frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}}{\left(\frac{\alpha^2 + \beta^2 + \alpha\beta}{18} \right)^3} \frac{1}{1 - m \frac{\gamma + 1}{\gamma} \left(\eta + \frac{w}{\gamma} \right)}. \end{aligned}$$

Let us assume that the utility function u is of CRRA-type: $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ if $\gamma \neq 1$ and $u(w) = \ln(w)$, if $\gamma = 1$.

For $\gamma \neq 1$ we have $r_u(w) = \frac{\gamma}{w}$, $P_u(w) = \frac{\gamma + 1}{w}$, from where it follows:

$$\begin{aligned} \beta(k) \approx & \left(\frac{w}{\gamma} - m \right) \frac{b + \frac{\beta - \alpha}{6}}{\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}} + \\ & + \frac{(\gamma + 1) \left(\frac{w}{\gamma} - m \right)^2}{2w} \frac{\frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}}{\left(\frac{\alpha^2 + \beta^2 + \alpha\beta}{18} \right)^3} \frac{1}{1 - \frac{mw(\gamma + 1)}{\gamma^2}}. \end{aligned}$$

For $\gamma = 1$:

$$\beta(k) \approx (w - m) \frac{b + \frac{\beta - \alpha}{6}}{\frac{\alpha^2 + \beta^2 + \alpha\beta}{18}} +$$

$$+ \frac{(w-m)^2}{w} \frac{19(\beta^3-\alpha^3)}{1080} + \frac{\alpha\beta(\beta-\alpha)}{72} \frac{1}{(\frac{\alpha^2+\beta^2+\alpha\beta}{18})^3} \frac{1}{1-2mv}.$$

We will state without proof some results on approximate solutions of the other three models with background risk. For the optimization problems (20) and (23), we will assume that $X = k\mu + Y$, with $\mu > 0$ and $E(Y) = 0$ (according to the model of [5], Section 5.2), and for (21), we will take $B = k\mu + A$, with $B = k\mu + A$, with $\mu > 0$ and $E_f(A) = 0$.

Theorem 3. An approximate solution $\beta_1(k)$ for the optimization problem (20) is

$$\beta_1(k) \approx \frac{M(X)}{\text{Var}(X)} \left[\frac{1}{r_u(w)} - M(Z) \right] + \frac{1}{2} P_u(w) \left[\frac{1}{r_u(w)} - M(Z) \right]^2 \frac{M^2(Z) M[(X - M(X))^3]}{\text{Var}^3(X) [1 - M(Z) P_u(w)]}.$$

Example 3. The formula from Theorem 3 may take different forms, depending on the distributions of the random variables X and Z . If X is the normal distribution $N(m, \sigma)$ then $M(X) = m$, $\text{Var}(X) = \sigma^2$ and $M[(X - M(X))^3] = 0$, thus

$$\beta_1(k) \approx \frac{m}{\sigma^2} \left[\frac{1}{r_u(w)} - M(Z) \right].$$

Assume that the utility function u is of HARA-type:

$$u(w) = \zeta \left(\eta + \frac{w}{\gamma} \right)^{1-\gamma} \text{ for } \eta + \frac{w}{\gamma} > 0,$$

and Z is the distribution $N(0, 1)$, we obtain:

$$\beta_1(k) \approx \frac{m}{\sigma^2} \frac{1}{r_u(w)} = \frac{m}{\sigma^2} \left(\eta + \frac{w}{\gamma} \right).$$

The form of $\beta_1(k)$ from the previous section extends the approximate calculation formula of the solution of the probabilistic model (8) (see [6,7]). Its proof follows some steps similar to the ones in the formula of $\beta(k)$ from Theorem 2, but uses the probabilistic techniques from [6,7].

Theorem 4. An approximate solution $\beta_2(k)$ of the optimization problem (21) is

$$\beta_2(k) \approx \frac{E_f(B)}{\text{Var}_f(B)} \left[\frac{1}{r_u(w)} - E_f(C) \right] + \frac{1}{2} P_u(w) \left[\frac{1}{r_u(w)} - E_f(C) \right]^2 \frac{E_f^2(B) E_f[(B - E_f(B))^3]}{\text{Var}_f^3(B) [1 - E_f(C) P_u(w)]}.$$

Example 4. We assume that:

- B is a triangular fuzzy number $B = (b, \alpha, \beta)$ and C is a symmetric triangular fuzzy number $C = (c, \delta)$,
- the utility function u is of HARA-type: $u(w) = \zeta \left(\eta + \frac{w}{\gamma} \right)^{-1}$ for $\eta + \frac{w}{\gamma} > 0$,
- the weighting function f has the form $f(t) = 2t$ for $t \in [0, 1]$.

By taking into account the calculations from Examples 1, 2, and the fact that $E_f(C) = c$, the approximate solution $\beta_2(k)$ becomes:

$$\beta_2(k) \approx \left(\eta + \frac{w}{\gamma} - c \right) \frac{b + \frac{\beta-\alpha}{6}}{\frac{\alpha^2+\beta^2+\alpha\beta}{18}} +$$

$$+ \frac{\gamma+1}{2\gamma} \frac{(\eta + \frac{w}{\gamma} - c)^2}{\eta + \frac{w}{\gamma}} \frac{\frac{19(\beta^3 - \alpha^3)}{1080} + \frac{\alpha\beta(\beta - \alpha)}{72}}{(\frac{\alpha^2 + \beta^2 + \alpha\beta}{18})^3} \frac{1}{1 - c \frac{\gamma+1}{\gamma} (\eta + \frac{w}{\gamma})}.$$

Theorem 5. An approximate solution $\beta_3(k)$ of the optimization problem (23) is:

$$\begin{aligned} \beta_2(k) \approx & \frac{M(X)}{\text{Var}(X)} \left[\frac{1}{r_u(w)} - E_f(C) \right] + \\ & + \frac{1}{2} P_u(w) \left[\frac{1}{r_u(w)} - E_f(C) \right]^2 \frac{M^2(X) M[(X - M(X))^3]}{\text{Var}^3(X) [1 - E_f(C) P_u(w)]}. \end{aligned}$$

Example 5. We consider the following hypotheses:

- X has the normal distribution $N(m, \sigma)$ and C is the triangular fuzzy numbers $C = (c, \delta, \epsilon)$,
- the utility function u is of HARA-type: $u(w) = \zeta(\eta + \frac{w}{\gamma})^{-1}$ for $\eta + \frac{w}{\gamma} > 0$,
- the weighting function f has the form: $f(t) = 2$ for $t \in [0, 1]$.

Then, $M(X) = m$, $\text{Var}(X) = \sigma^2$, $M[(X - M(X))^3] = 0$, and $E_f(c) = c + \frac{\epsilon - \delta}{6}$.

It follows the following form of $\beta_3(k)$:

$$\beta_3(k) \approx \frac{m}{\sigma^2} \left[\frac{1}{r_u(w)} - E_f(C) \right] = \frac{m}{\sigma^2} \left[\eta + \frac{w}{\gamma} - c - \frac{\epsilon - \delta}{6} \right].$$

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Appendix A

Proof of Corollary 2. We take $u(x, z) = xz$, and applying (5), we have

$$\begin{aligned} E_f(AZ) &= \frac{1}{2} \int_0^1 [M(a_1(\gamma)Z) + M(a_2(\gamma)Z)] f(\gamma) d\gamma \\ &= \frac{1}{2} \int_0^1 [a_1(\gamma)M(Z) + a_2(\gamma)M(Z)] f(\gamma) d\gamma = M(Z) E_f(A). \end{aligned}$$

Taking $u(x, z) = x^2z$, we obtain

$$\begin{aligned} E_f(A^2Z) &= \frac{1}{2} \int_0^1 [M(a_1^2(\gamma)Z) + M(a_2^2(\gamma)Z)] f(\gamma) d\gamma \\ &= \frac{1}{2} \int_0^1 [a_1^2(\gamma)M(Z) + a_2^2(\gamma)M(Z)] f(\gamma) d\gamma = M(Z) E_f(A^2). \end{aligned}$$

□

Proof of Proposition 3. We consider the Taylor approximation:

$$u'(w + \alpha(k\mu + x)) \approx u'(w) + \alpha(k\mu + x)u''(w).$$

Then, by (11) and Proposition 1

$$\begin{aligned} V'(\alpha) &\approx E_f[(k\mu + A)(u'(w) + u''(w)\alpha(k\mu + A))] \\ &= u'(w)(k\mu + E_f(A)) + \alpha u''(w) E_f[(k\mu + A)^2]. \end{aligned}$$

The equation $V'(\alpha(k)) = 0$, becomes

$$u'(w)(k\mu + E_f(A)) + \alpha(k)u''(w)E_f[(k\mu + A)]^2 \approx 0.$$

We derive it with respect to k :

$$u'(w)\mu + u''(w)(\alpha'(k)E_f[(k\mu + A)^2] + 2\alpha(k)\mu E_f(k\mu + A)) \approx 0.$$

In this equality we make $k = 0$. Taking into account that $\alpha(0) = 0$, it follows

$$u'(w)\mu + u''(w)\alpha'(0)E_f(A^2) \approx 0,$$

from where we determine $\alpha'(0)$:

$$\alpha'(0) \approx -\frac{\mu}{E_f(A^2)} \frac{u'(w)}{u''(w)} = \frac{\mu}{E_f(A^2)} \frac{1}{r_u(w)}.$$

□

Proof of Proposition 4. To determine the approximate value of $\alpha''(0)$ we start with the following Taylor approximation:

$$u'(w + \alpha(k\mu + x)) \approx u'(w) + \alpha(k\mu + x)u''(w) + \frac{\alpha^2}{2}(k\mu + x)^2u'''(w),$$

from which it follows:

$$(k\mu + x)u'(w + \alpha(k\mu + x)) \approx u'(w)(k\mu + x) + u''(w)\alpha(k\mu + x)^2 + \frac{u'''(w)}{2}\alpha^2(k\mu + x).$$

Then, by (11) and the linearity of the operator $E_f(\cdot)$

$$\begin{aligned} V'(\alpha) &= E_f[(k\mu + A)u'(w + \alpha(k\mu + A))] \\ &\approx u'(w)E_f(k\mu + A) + u''(w)\alpha E_f[(k\mu + A)^2] + \frac{u'''(w)}{2}\alpha^2 E_f[(k\mu + A)^3]. \end{aligned}$$

Using this approximation for $\alpha = \alpha(k)$, the equation $V'(\alpha(k)) = 0$, becomes

$$u'(w)(k\mu + E_f(A)) + u''(w)\alpha(k)E_f[(k\mu + A)^2] + \frac{u'''(w)}{2}(\alpha(k))^2 E_f[(k\mu + A)^3] \approx 0.$$

Deriving with respect to k one obtains:

$$\begin{aligned} &\mu u'(w) + u''(w)[\alpha'(k)E_f((k\mu + A)^2) + 2\mu\alpha(k)E_f(k\mu + A)] + \\ &+ \frac{u'''(w)}{2}[2\alpha(k)\alpha'(k)E_f((k\mu + A)^3) + 3(\alpha(k))^2\mu E_f((k\mu + A)^2)] \approx 0. \end{aligned}$$

We derive one more time with respect to k :

$$\begin{aligned} &u''(w)[\alpha''(k)E_f((k\mu + A)^2) + 2\mu\alpha'(k)E_f(k\mu + A) + 2\mu\alpha'(k)E_f(k\mu + A) + \\ &2\mu^2\alpha(k)] + \frac{u'''(w)}{2}[2(\alpha'(k))^2 E_f((k\mu + A)^3) + 2\alpha(k)\alpha''(k)E_f((k\mu + A)^3) + \\ &+ 6\alpha(k)\alpha'(k)E_f((k\mu + A)^2) + 6\mu\alpha(k)\alpha'(k)E_f((k\mu + A)^2) + 6\mu^2(\alpha(k)^2)E_f(k\mu + A)] \approx 0. \end{aligned}$$

In the previous relation, we take $k = 0$.

$$u''(w)[\alpha''(0)E_f(A^2) + 2\mu\alpha'(0)E_f(A) + 2\mu\alpha'(0)E_f(A) + 2\mu^2\alpha(0)] + \\ + \frac{u'''(w)}{2}[2(\alpha'(0))^2E_f(A^3) + 2\alpha(0)\alpha''(0)E_f(A^3) + 6\alpha(0)\alpha'(0)E_f(A^2) + \\ 6\mu\alpha(0)E_f(A^2) + 6\mu^2(\alpha(0))^2E_f(A)] \approx 0.$$

Taking into account that $\alpha(0) = 0$ and $E_f(A) = 0$, one obtains

$$u''(w)\alpha''(0)E_f(A^2) + u'''(w)(\alpha'(0))^2E_f(A^3) \approx 0,$$

from where we get $\alpha''(0)$:

$$\alpha''(0) \approx -\frac{u'''(w)}{u''(w)} \frac{E_f(A^3)}{E_f(A^2)} (\alpha'(0))^2.$$

By replacing $\alpha'(0)$ with the expression from Proposition 3 and taking into account (15), it follows:

$$\alpha''(0) = \frac{P_u(w)}{(r_u(w))^2} \frac{E_f(A^3)}{(E_f(A^2))^3} \mu^2.$$

□

Proof of Theorem 1. By replacing in (14) the approximate values of $\alpha'(0)$ and $\alpha''(0)$ given by Propositions 3 and 4 and taking into account that $E_f(B) = k\mu$, one obtains:

$$\alpha(k) \approx k\alpha'(0) + \frac{1}{2}k^2\alpha''(0) \\ = \frac{k\mu}{E_f(A^2)} \frac{1}{r_u(w)} + \frac{1}{2}(k\mu)^2 \frac{P_u(w)}{(r_u(w))^2} \frac{E_f(A^3)}{(E_f(A^2))^3} \\ = \frac{E_f(B)}{E_f(A^2)} \frac{1}{r_u(w)} + \frac{1}{2}(E_f(B))^2 \frac{P_u(w)}{(r_u(w))^2} \frac{E_f(A^3)}{(E_f(A^2))^3}.$$

However, $E_f(A^2) = E_f[(B - E_f(B))^2] = \text{Var}_f(B)$. Then,

$$\alpha(k) \approx \frac{1}{r_u(w)} \frac{E_f(B)}{\text{Var}_f(B)} + \frac{1}{2} \frac{P_u(w)}{(r_u(w))^2} \frac{E_f[(B - E_f(B))^3]}{(\text{Var}_f(B))^3} (E_f(B))^2.$$

□

Proof of Proposition 5. We consider the Taylor approximation:

$$u'(w + \alpha(k\mu + x) + z) \approx u'(w) + (\alpha(k\mu + x) + z)u''(w).$$

Then,

$$(k\mu + x)u'(w + \alpha(k\mu + x) + z) \approx u'(w)(k\mu + x) + u''(w)\alpha(k\mu + x)^2 + u''(w)z(k\mu + x).$$

From this relation, from (25) and the linearity of mixed expected utility, it follows:

$$W'(\alpha) \approx u'(w)(k\mu + E_f(A)) + u''(w)\alpha E_f[(k\mu + A)^2] + u''(w)E_f[(k\mu + A)Z].$$

Then, the equation $W'(\beta(k)) = 0$, will be written

$$u'(w)(k\mu + E_f(A)) + u''(w)\beta(k)E_f[(k\mu + A)^2] + u''(w)E_f[(k\mu + A)Z] \approx 0.$$

By deriving with respect to k one obtains:

$$u'(w)\mu + u''(w)(\beta'(k)E_f[(k\mu + A)^2] + 2\beta(k)\mu E_f(k\mu + A)) + u''(w)\mu M(Z) \approx 0.$$

For $k = 0$, it follows

$$u'(w)\mu + u''(w)\mu M(Z) + u''(w)\beta'(0)E_f(A^2) \approx 0,$$

from where $\beta'(0)$ is obtained:

$$\beta'(0) \approx -\frac{(u'(w) + u''(w)M(Z))\mu}{u''(w)E_f(A^2)} = \frac{\mu}{E_f(A^2)}\left(\frac{1}{r_u(w)} - M(Z)\right).$$

□

Proof of Proposition 6. We consider the Taylor approximation

$$u'(w + \alpha(k\mu + x) + z) \approx u'(w) + u''(w)[\alpha(k\mu + x) + z] + \frac{1}{2}u'''(w)[\alpha(k\mu + x) + z]^2,$$

from where it follows

$$\begin{aligned} (k\mu + x)u'(w + \alpha(k\mu + x) + z) &\approx u'(w)(k\mu + x) + u''(w)(k\mu + x)[\alpha(k\mu + x) + z] \\ &\quad + \frac{1}{2}u'''(w)(k\mu + x)[\alpha(k\mu + x) + z]^2. \end{aligned}$$

By (25), the previous relation and the linearity of mixed expected utility, we will have

$$\begin{aligned} W'(\alpha) &\approx u'(w)(k\mu + E_f(A)) + u''(w)E_f[(k\mu + A)(\alpha(k\mu + A) + Z)] + \\ &\quad + \frac{1}{2}u'''(w)E_f[(k\mu + A)(\alpha(k\mu + A) + Z)^2]. \end{aligned}$$

Then, from $W'(\beta(k)) = 0$, we will deduce:

$$\begin{aligned} u'(w)(k\mu + E_f(A)) + u''(w)E_f[(k\mu + A)(\beta(k)(k\mu + A) + Z)] + \\ + \frac{1}{2}u'''(w)E_f[(k\mu + A)(\beta(k)(k\mu + A) + Z)^2] \approx 0. \end{aligned}$$

If we denote

$$g(k) = E_f[(k\mu + A)(\beta(k)(k\mu + A) + Z)], \quad (\text{A1})$$

$$h(k) = E_f[(k\mu + A)(\beta(k)(k\mu + A) + Z)^2], \quad (\text{A2})$$

then the previous relation can be written

$$u'(w)(k\mu + E_f(A)) + u''(w)g(k) + \frac{1}{2}u'''(w)h(k) \approx 0.$$

Deriving twice with respect to k , we obtain:

$$u''(w)g''(k) + \frac{1}{2}u'''(w)h''(k) \approx 0. \quad (\text{A3})$$

We set $k = 0$ in (A2):

$$u''(w)g''(0) + \frac{1}{2}u'''(w)h''(0) \approx 0. \quad (\text{A4})$$

The computation of $g''(0)$. We notice that

$$g(k) = \beta(k)E_f[(k\mu + A)^2] + E_f[(k\mu + A)Z].$$

By denoting $g_1(k) = \beta(k)E_f[(k\mu + A)^2]$ and $g_2(k) = E_f[(k\mu + A)Z]$, we will have $g(k) = g_1(k) + g_2(k)$. One easily sees that $g_2''(k) = 0$, thus $g''(k) = g_1''(k) + g_2''(k) = g_1''(k)$. We derive $g_1(k)$:

$$\begin{aligned} g_1'(k) &= \beta'(k)E_f[(k\mu + A)^2] + 2\mu\beta(k)E_f(k\mu + A) \\ &= \beta'(k)E_f[(k\mu + A)^2] + 2\mu^2k\beta(k), \end{aligned}$$

Since $E_f(k\mu + A) = k\mu + E_f(A) = k\mu$. We derive one more time

$$g_1''(k) = \beta''(k)E_f[(k\mu + A)^2] + 2\mu\beta'(k)E_f(k\mu + A) + 2\mu^2[\beta(k) + k\beta'(k)].$$

Setting $k = 0$ in the previous relation and taking into account that $\beta(0) = E_f(A) = 0$, it follows

$$g''(0) = \beta''(0)E_f(A^2). \quad (\text{A5})$$

The computation of $h''(0)$. We write $h(k)$ as

$$h(k) = \beta^2(k)E_f[(k\mu + A)^3] + 2\beta(k)E_f[(k\mu + A)^2Z] + E_f[(k\mu + A)Z^2].$$

We denote

$$\begin{aligned} h_1(k) &= \beta^2(k)E_f[(k\mu + A)^3], \\ h_2(k) &= \beta(k)E_f[(k\mu + A)^2Z], \\ h_3(k) &= E_f[(k\mu + A)Z^2]. \end{aligned}$$

Then, $h(k) = h_1(k) + h_2(k) + h_3(k)$. One notices that $h_3''(0) = 0$, thus

$$h''(0) = h_1''(0) + 2h_2''(0). \quad (\text{A6})$$

We first compute $h_2''(0)$. One can easily notice that

$$h_2''(k) = \beta''(k)E_f[(k\mu + A)^2Z] + 2\beta'(k)\frac{d}{dk}E_f[(k\mu + A)^2Z] + \beta(k)\frac{d^2}{dk^2}E_f[(k\mu + A)^2Z].$$

Taking into account that

$$\frac{d}{dk}E_f[(k\mu + A)^2Z] = 2\mu E_f[(k\mu + A)Z],$$

and $\beta(0) = 0$,

We deduce

$$h_2''(0) = \beta''(0)E_f(A^2Z) + 4\mu\beta'(0)E_f(AZ). \quad (\text{A7})$$

We will compute $h_1''(0)$. We derive twice $h_1(k)$:

$$\begin{aligned} h_1''(k) &= \frac{d^2}{dk^2}(\beta^2(k))E_f[(k\mu + A)^3] + 2\frac{d}{dk}(\beta^2(k))\frac{d}{dk}E_f[(k\mu + A)^3] + \\ &\quad + \beta^2(k)\frac{d^2}{dk^2}E_f[(k\mu + A)^3]. \end{aligned}$$

We compute the following derivatives from the last sum:

$$\begin{aligned}\frac{d}{dk}(\beta^2(k)) &= 2\beta(k)\beta'(k), \\ \frac{d^2}{dk^2}(\beta^2(k)) &= 2[\beta''(k)\beta(k) + (\beta'(k))^2], \\ \frac{d}{dk}E_f[(k\mu + A)^3] &= 3\mu E_f[(k\mu + A)^2].\end{aligned}$$

Then, taking into account $\beta(0) = 0$:

$$\begin{aligned}h_1''(0) &= 2[\beta''(0)\beta(0) + (\beta'(0))^2]E_f(A^3) + 2\beta(0)\beta'(0)3\mu E_f[(k\mu + A)^2] + \beta^2(0) = \frac{d}{dk}E_f[(k\mu + A)^3] \\ &= 2(\beta'(0))^2E_f(A^3).\end{aligned}\quad (\text{A8})$$

By (A6)–(A8):

$$h''(0) = h_1''(0) + 2h_2''(0) = 2(\beta'(0))^2E_f(A^3) + 2(\beta''(0)E_f(A^2Z) + 4\mu\beta'(0)E_f(AZ)). \quad (\text{A9})$$

Replacing in (A3) the values of $g''(0)$ and $h''(0)$ given by (A5) and (A9):

$$u''(w)\beta''(0)E_f(A^2) + \frac{1}{2}u'''(w)[2(\beta'(0))^2E_f(A^3) + 2(\beta''(0)E_f(A^2Z) + 4\mu\beta'(0)E_f(AZ))] \approx 0,$$

from where

$$\begin{aligned}\beta''(0)[u''(w)E_f(A^2) + u'''(w)E_f(A^2Z)] &\approx \\ &\approx -\beta'(0)u'''(w)[\beta'(0)E_f(A^3) + 4\mu E_f(AZ)].\end{aligned}$$

The approximate value of $\beta''(0)$ follows:

$$\beta''(0) \approx -u'''(w)\beta'(0)\frac{\beta'(0)E_f(A^3) + 4\mu E_f(AZ)}{u''(w)E_f(A^2) + u'''(w)E_f(A^2Z)}.$$

According to Corollary 2, the expression above which approximates $\beta''(0)$ can be written:

$$\begin{aligned}\beta''(0) &\approx -u'''(w)\beta'(0)\frac{\beta'(0)E_f(A^3) + 4\mu M(Z)E_f(A)}{u''(w)E_f(A^2) + M(Z)E_f(A^2)} \\ &= -\frac{u'''(w)\beta'(0)}{E_f(A^2)}\frac{\beta'(0)E_f(A^3)}{u''(w) + M(Z)u'''(w)},\end{aligned}$$

Since $E_f(A) = 0$. If we replace A with $B - E_f(B)$, one obtains

$$\begin{aligned}\beta''(0) &\approx -\frac{(\beta'(0))^2u'''(w)}{\text{Var}_f(B)}\frac{E_f[(B - E_f(B))^3]}{u''(w) + M(Z)u'''(w)} \\ &= \frac{P_u(w)(\beta'(0))^2}{\text{Var}_f(B)}\frac{E_f[(B - E_f(B))^3]}{1 - M(Z)P_u(w)}.\end{aligned}$$

□

Proof of Theorem 2. The approximation formula of $\beta'(0)$ from Proposition 5 can be written:

$$\beta'(0) \approx \frac{\mu}{\text{Var}_f(B)}\left[\frac{1}{r_u(w)} - M(Z)\right]. \quad (\text{A10})$$

According to (27), (A9), and Proposition 6,

$$\begin{aligned}\beta(k) &\approx k\beta'(0) + \frac{1}{2}k^2\beta''(0) \\ &= \frac{\mu k}{\text{Var}_f(B)} \left[\frac{1}{r_u(w)} - M(Z) \right] + \\ &\quad + \frac{1}{2}P_u(w) \frac{(k\mu)^2 E_f[(B - E_f(B))^3]}{\text{Var}_f^3(B)[1 - M(Z)P_u(w)]} \left[\frac{1}{r_u(w)} - M(Z) \right]^2.\end{aligned}$$

Since $\mu k = E_f(B)$, it follows

$$\begin{aligned}\beta(k) &\approx \frac{E_f(B)}{\text{Var}_f(B)} \left[\frac{1}{r_u(w)} - M(Z) \right] + \\ &\quad \frac{1}{2}P_u(w) \left[\frac{1}{r_u(w)} - M(Z) \right]^2 \frac{E_f^2(B) E_f[(B - E_f(B))^3]}{\text{Var}_f^3(B)[1 - M(Z)P_u(w)]}.\end{aligned}$$

□

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