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# Ekeland's Variational Principle and Minimization Takahashi's Theorem in Generalized Metric Spaces

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**Abstract:** We consider a distance function on generalized metric spaces and we get a generalization of Ekeland Variational Principle (EVP). Next, we prove that EVP is equivalent to Caristi–Kirk fixed point theorem and minimization Takahashi's theorem.

**Keywords:** Ekeland variational principle; minimization theorem; fixed point; $\gamma$-function; quasi-G-metric

## 1. Introduction

After presentation of Ekeland Variational Principle (EVP) in 1972, it becomes clear that this principle is equivalent to Caristi fixed point theorem [1–9], Drop theorem [10,11], Flower Petal theorem [10,11] and Takahashi's nonconvex minimization theorem. Many scholars have studied EVP on complete convex space and on locally convex space. EVP is proved for the investigation of best proximity points for Cyclic contractions, Reich type cyclic contractions, Kannan type cyclic contractions, Ciric type cyclic contractions, Hardy and Rogers type cyclic contraction, Chatterjee type cyclic, contractions, and Zamfirescu type cyclic contractions (for more details, see [12]).

## 2. EVP

**Theorem 1. (Ekeland Theorem)** *Assume that $(Z, d)$ is a complete metric space (see [13]). Let $f : Z \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, semicontinuous and bounded below function. Then, there exists $y \in Z$ such that*

$$f(y) \leq f(x); \quad d(x, y) \leq 1 \text{ and } f(z) > f(y) - \epsilon d(y, z) \text{ for all } y \neq z.$$

Now, we study generalized metric spaces and their properties, for more details and application, we refer to [7,14–23].

**Definition 1.** *Assume that $Z$ is a nonempty set and mapping*

$$G : Z \times Z \times Z \longrightarrow [0, \infty)$$

*is satisfied in the following conditions (see [24]):*

*(i)* $G(u, v, w) = 0$ *if* $u = v = w$,
*(ii)* $G(u, u, v) > 0$ *for all* $u, v \in X$, *where* $u \neq v$,
*(iii)* $G(u, u, w) \leq G(u, v, w)$ *for all* $u, v, w \in Z$ *with* $v \neq w$,
*(iv)* $G(u, v, w) = G(p\{u, v, w\})$ *such that* $p$ *is a permutation of* $u, v, w$,
*(v)* $G(u, v, w) \leq G(u, \alpha, \alpha) + G(\alpha, v, w)$ *for all* $u, v, w, \alpha$ *in* $Z$.

*Then, $G$ is said to be G-metric and pair $(Z, G)$ is said to be G-metric space.*

**Definition 2.** *Let $(Z, G)$ be a G-metric space (see [24]). A sequence $\{u_n\}$ in $Z$ is said to be*

*(a) G-Cauchy sequence if, for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $m, n, l \in \mathbb{N}$ and $m, n, l \geq n_0$ then $G(u_n, u_m, u_l) < \epsilon$,*
*(b) G-convergent to $u \in Z$ if for all epsilon $> 0$, there exists natural number $n_0$ such that for all $m, n \geq n_0$, then $G(u_n, u_m, u) < \epsilon$.*

**Proposition 1.** *Assuming that $(Z, G)$ is a G-metric space, then the following statements are equivalent (see [24]):*

*(a) $\{u_n\}$ is a G-Cauchy sequence,*
*(b) for each $\epsilon > 0$, there exists natural number $n_0$ such that for all $m, n \geq n_0$, then $G(u_n, u_m, u_m) < \epsilon$.*

**Definition 3.** *A function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is sub-additive when $\xi(u + v) \leq \xi(u) + \xi(v)$ and $\xi(\alpha u) = \alpha\xi(u)$, for every $\alpha > 0$.*

**Definition 4.** *Let $Z$ be a nonempty set. A function*

$$G : Z \times Z \times Z \longrightarrow [0, \infty)$$

*is said to be quasi-G-metric (shortly q-G-m) if the following conditions hold,*

1. *$G(u, v, w) = 0$ if $u = v = w$,*
2. *$G(u, u, v) > 0$ for all $u, v \in Z$ , $u \neq v$,*
3. *$G(u, u, w) \leq G(u, v, w)$ for all $u, v, w \in Z$, $w \neq v$,*
4. *$G(u, v, w) \leq G(u, \alpha, \alpha) + G(\alpha, v, w)$ for all $u, v, w, \alpha \in X$.*

*$(Z, G)$ is said to be q-G-m space when $Z$ is a nonempty set and $G$ is a q-G-m. The concept of Cauchy sequence, convergence and complete space are defined as G-metric space.*

Throughout the paper, unless otherwise specified, $Z$ is a nonempty set and $(Z, G)$ is a q-G-m space. $\psi : (-\infty, \infty) \longrightarrow (0, \infty)$ is a nondecreasing function. A function $h : Z \longrightarrow (-\infty, \infty)$ is said to be lower semicontinuous from above (shortly *Lsca*) at $u_0$, when, for each sequence $\{u_n\}$ in $Z$ such that $u_n \longrightarrow u_0$ and $h(u_1) \geq h(u_2) \geq \cdots \geq h(u_n) \geq \cdots$, we have $h(u_0) \leq \lim_{n\to\infty} h(u_n)$. The function $h$ is said to be *Lsca* on $Z$, when $h$ is *Lsca* at every point of $Z$, $h$ is proper when $h \not\equiv \infty$.

**Definition 5.** *Assume that $Z$ is an order space with order $\preceq$ on $Z$.*

*(1) $\preceq$ will be quasi-order on $Z$ if it is reflexive and transitive;*
*(2) The sequence $\{u_n\}_{n\in\mathbb{N}}$ on $Z$ will be decreasing if for all $n \in \mathbb{N}$, $u_{n+1} \preceq u_n$;*
*(3) Let $P(u) = \{v \in Z : v \preceq u\}$ and if $\{u_n\}_{n\in\mathbb{N}} \subseteq P(u)$ be decreasing and convergent to $\tilde{u} \in Z$, then $\tilde{u} \in P(u)$. Then, the quasi-order $\preceq$ is said to be lower closed if each $u \in Z$.*

**Definition 6.** *Assume that $(Z, G)$ is a q-G-m space with quasi-order $\preceq$. The set $P(u) = \{v \in Z : v \preceq u\}$ $u \in Z$, would be said to be $\preceq$ $-$complete if every decreasing sequence in $P(u)$ be convergent in it.*

**Definition 7.** *Let $(Z, G)$ be a q-G-m space. A function $\gamma : Z \times Z \times Z \longrightarrow [0, \infty)$ is said to be $\gamma$-function when*

*(1) $\gamma(u, v, w) \leq \gamma(u, \alpha, \alpha) + \gamma(\alpha, v, w)$ for all $u, v, w, \alpha \in Z$,*
*(2) if $u \in Z$, $\{v_n\}_{n\in\mathbb{N}}$ be a sequence in $Z$ which is convergent to $v$ in $Z$ and $\gamma(u, v_n, v_n) \leq M$, then $\gamma(u, v, v) \leq M$,*
*(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that $\gamma(u, \alpha, \alpha) \leq \delta$ and $\gamma(\alpha, v, w) \leq \delta$ imply that $G(u, v, w) \leq \epsilon$.*

*We show the class of all the $\gamma$-function by $\Gamma$.*

**Remark 1.** *Assume that $(Z,G)$ is a q-G-m space and $\gamma \in \Gamma$ . If $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing and sub-additive function such that $\xi(0) = 0$, then $\xi \circ \gamma \in \Gamma$.*

**Example 1.** *Assume that*

$$G : Z^3 \longrightarrow [0, \infty),$$
$$G(u,v,w) = \frac{1}{2}(\mid w-u \mid + \mid u-v \mid).$$

*Then, G is a q-G-m but is not G-metric.*

**Proof.** q-G-m is obvious. We show that $G(u,v,w) \neq G\{p(u,v,w)\}$ ($p$ is a permutation of $u,v,w$). Since

$$G(3,5,2) = \frac{1}{2}(\mid 2-3 \mid + \mid 3-5 \mid) = \frac{3}{2},$$
$$G(2,3,5) = \frac{1}{2}(\mid 3-2 \mid + \mid 5-2 \mid) = 2,$$

then $G$ is not a $G$-metric. □

**Example 2.** *Let $G(u,v,w)$ be the same as in the previous example. Then, $\gamma = G$ is a $\gamma$-function.*

**Proof.** (a) and (b) are obvious. Let $\epsilon > 0$ be given, put $\delta = \frac{\epsilon}{2}$ if $\gamma(u,\alpha,\alpha) = \frac{1}{2}(\mid w-\alpha \mid + \mid \alpha - v \mid) < \frac{\epsilon}{2}$ then

$$\begin{aligned} G(u,v,w) &= \frac{1}{2}(\mid w-v \mid + \mid u-v \mid) \\ &\leq \frac{1}{2}(\mid w-\alpha \mid + \mid \alpha-u \mid + \mid u-\alpha \mid + \mid \alpha-v \mid) \\ &< \epsilon. \end{aligned}$$

□

**Example 3.** *Assume that $(Z, \|.\|)$ is a normed space. Then, the function $\gamma : Z \times Z \times Z \to [0,\infty)$ defined by $\gamma(u,v,w) = \|v\|$ for each u,v,w $\in Z$, is a $\gamma$-function. However, it is not a q-G-m on Z.*

**Example 4.** *Let $Z = \mathbb{R}$. Define a function $G : Z \times Z \times Z \to [0,\infty)$, by*

$$G(u,v,w) = \begin{cases} 0, & \text{if } u = v = w, \\ |u|+1, & \text{otherwise.} \end{cases}$$

*Then, G is q-G-m but is not a $\gamma$-function.*

**Lemma 1.** *Assume that $(Z,G)$ is a G-metric space and $\gamma \in \Gamma$ . Let $\{x_n\}$ and $\{y_n\}$ be two sequences in Z, $\{\rho_n\}$ and $\{\varphi_n\}$ be in $[0,\infty]$, which are convergent to zero (see [20]). Let $x,y,z,\alpha \in Z$, then*

*(1) If $\gamma(y,x_n,x_n) \leq \rho_n$ and $\gamma(x_n,y,z) \leq \varphi_n$ for all $n \in \mathbb{N}$, then $G(y,y,z) < \epsilon$ and hence $z = y$,*
*(2) If $\gamma(y_n,x_n,x_n) \leq \rho_n$ and $\gamma(x_n,x_m,z) \leq \varphi_n$ for every $m > n$, then $G(y_n,y_m,z)$ convergent to zero and hence $y_n \to z$,*
*(3) If $\gamma(x_n,x_m,x_l) \leq \rho_n$ for all $m,n,l \in \mathbb{N}$ with $n \leq m \leq l$, then $\{x_n\}$ is a $G-$cauchy sequence.*
*(4) If $\gamma(x_n,\alpha,\alpha) \leq \rho_n$ for all $n \in \mathbb{N}$ then $\{u_n\}$ is a G-cauchy sequence.*

**Lemma 2.** *Let $\gamma \in \Gamma$ . If sequence $\{u_n\}$ be in $Z$ that*

$$\limsup_{n \longrightarrow \infty} \{\gamma(u_n, u_m, u_l), n \leq m \leq l\} = 0.$$

*Then, $\{u_n\}$ will be a G-Cauchy sequence in Z.*

**Proof.** Assume $\rho_n = \sup\{\gamma(u_n, u_m, u_l)\}$, then $\lim_{n\to\infty} \rho_n = 0$. By Lemma 1 (3), $\{u_n\}$ is a $G$-Cauchy sequence. □

**Lemma 3.** *Let $h : Z \longrightarrow [-\infty, \infty]$ be a function and $\gamma \in \Gamma$. Let $P(u)$ be defined by*

$$P(u) = \{v \in Z;\ \ v \neq u,\ \ \gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v))\}\cdot$$

*If $P(u)$ be nonempty, then, for every $v \in P(u)$,*

$$P(v) \subseteq P(u) \ \text{ and } \ h(v) \leq h(u).$$

**Proof.** Let $v \in P(u)$. Thus, $v \neq u$ and $\gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v))$. Since $\gamma(u, v, v) \geq 0$ and $\psi$ is a nondecreasing and positive function, then $h(u) \geq h(v)$. If $P(v) = \emptyset$, then $P(v) \subseteq P(u)$. Therefore, $w \neq v$ and $\gamma(v, w, w) \leq \psi(h(v))(h(v) - h(w))$ as above $h(v) \geq h(w)$. Since $\Gamma \in \Gamma$, then

$$\gamma(u, w, w) \leq \gamma(u, v, v) + \gamma(v, w, w) \leq \psi(h(u))(h(u) - h(w)).$$

We claim that $w \neq u$. Assume that $w = u$ so $\gamma(u, w, w) = 0$. On the other hand,

$$\begin{aligned}
&\gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v)) \leq \psi(h(u))(h(u) - h(w)) = 0 \\
&\Longrightarrow \gamma(u, v, v) = 0.
\end{aligned}$$

Then, $\gamma(u, v, v) = 0$. For every $\epsilon > 0$, we have $\gamma(u, w, w) = 0 < \epsilon$ and $\gamma(w, v, v) = 0 < \epsilon$. Then, by definition $\gamma$-function, we have $G(w, v, v) < \epsilon$, so $G(w, v, v) = 0$ and $w = v$, which is a contradiction; therefore, $w \in P(u)$ and $P(v) \subseteq P(u)$. □

**Proposition 2.** *Assume that $(Z, G)$ is a complete q-G- m space, $h : Z \longrightarrow [-\infty, \infty]$ is a proper and bounded below function and $\gamma \in \Gamma$. Let*

$$P(u) = \{v \in Z; v \neq u, \gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v))\}\cdot$$

*Let $\{u_n\}$ be a sequence in $Z$ such that $S(u_n)$ is nonempty and for all $n \in \mathbb{N}$, $u_{n+1} \in S(u_n)$. Then, there exists $u_0 \in Z$ such that $u_n \longrightarrow u_0$ and $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$. In addition, if, for every $n \in \mathbb{N}$, $h(u_{n+1}) \leq \inf_{w \in S(u_n)} h(w) + \frac{1}{n}$, then $\bigcap_{n=1}^{\infty} S(u_n)$ is a singleton set.*

**Proof.** At first, we prove that $\{u_n\}$ is a Cauchy sequence by Lemma 3, $h(u_n) \geq h(u_{n+1})$ for all $n \in \mathbb{N}$. Therefore, $\{h(u_n)\}$ is non-increasing. On the other hand, if $h$ is bounded below, then $\lim_{n\to\infty} h(u_n) = r$, and $h(u_n) \geq r$, for all $n \in \mathbb{N}$. We claim that

$$\limsup_{n \to \infty} \{\gamma(u_n, u_m, u_m) : m > n\} = 0\cdot$$

We have

$$\begin{aligned}\gamma(u_n, u_m, u_m) &\leq \gamma(u_n, u_{n+1}, u_{n+1}) + \gamma(u_{n+1}, u_m, u_m)\\ &\leq \gamma(u_n, u_{n+1}, u_{n+1}) + \gamma(u_{n+1}, u_{n+2}, u_{n+2}) + \cdots\\ &+ \gamma(u_{m-1}, u_m, u_m).\end{aligned}$$

Thus,

$$\gamma(u_n, u_m, u_m) \leq \sum_{j=n}^{m-1} \gamma(u_n, u_m, u_m) \leq \psi(h(u))(h(u_n) - r)$$

for all $m, n \in \mathbb{N}$ with $m > n$.

Put $\rho_n = \psi(h(u))(h(u_n) - r)$; then, $\sup\{\gamma(u_n, u_m, u_m) : m > n\} \leq \rho_n$, for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} h(u_n) = r$, we have that

$$\lim_{n\longrightarrow\infty} \sup\{\gamma(u_n, u_m, u_m) : m > n\} = 0$$

and $\lim_{n\to\infty} \rho_n = 0$. By Lemma 2, $\{u_n\}$ is a $G$-Cauchy sequence. Then, there exists $u_0 \in Z$ such that $u_n \to u_0$. Now, we show that $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$. Since $h$ is *Lsca*, then $h(u_0) \leq \lim_{n\to\infty} h(u_n) = r \leq h(u_k)$.

Letting $n \in \mathbb{N}$, we have

$$\gamma(u_n, u_m, u_m) \leq \sum_{j=n}^{m-1} \gamma(u_j, u_{j+1}, u_{j+1}) \leq \psi(h(u_n))(h(u_n) - h(u_0))$$

for all $m \in \mathbb{N}$ and $m > n$. Since $\gamma(u, v, \cdot)$ and $\gamma(u, \cdot, v) : Z \longrightarrow (0, \infty)$ is semicontinuous from below, then

$$\gamma(u_n, u_0, u_0) \leq \psi(h(u_n)(h(u_n) - h(u_0))$$

for all $n \in \mathbb{N}$. In addition, $u_0 \neq u$ for all $n \in \mathbb{N}$. Otherwise, there exists $j \in \mathbb{N}$ such that $u_0 = u_j$. Since

$$\begin{aligned}\gamma(u_j, u_{j+1}, u_{j+1}) &\leq \psi(h(u_j))(h(u_j) - h(u_{j+1}))\\ &\leq \psi(h(u_j))(h(u_j) - h(u_0)) = 0,\end{aligned}$$

then we have $\gamma(u_j, u_{j+1}, u_{j+1}) = 0$ and in the same way

$$\gamma(u_{j+1}, u_{j+2}, u_{j+2}) = 0.$$

Now assume that $\epsilon > 0$, $\gamma(u_j, u_{j+1}, u_{j+1}) = 0 < \delta$ and $\gamma(u_{j+1}, u_{j+2}, u_{j+2}) = 0 < \delta$. Therefore, by Definition 7 (3), we get $G(u_j, u_{j+2}, u_{j+2}) < \epsilon$. Then, $u_j = u_{j+2}$ that is a contradiction since $u_j \neq u_{j+2}$. Since $u_{j+1} \in P(u_j)$, then $P(u_{j+1}) \subseteq P(u_j)$ and $u_{j+2} \in P(u_{j+1})$. Thus, $u_{j+2} \in P(u_j)$. Suppose that $u_{j+2} \neq u_j$ for all $n \in \mathbb{N}$. We have $u_0 \in \bigcap_{n=1}^{\infty} P(u_n)$, then $\bigcap_{n=1}^{\infty} P(u_n) \neq \emptyset$. Let $h(u_{n+1}) \leq \inf_{w\in P(u_n)} h(w) + \frac{1}{n}$, for all $u_0 \neq u_n$. We show that $\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$. Assume that $z \in \bigcap_{n=1}^{\infty} S(u_n)$, then

$$\begin{aligned}\gamma(u_n, z, z) &\leq \psi(h(u_n))(h(u_n) - h(z))\\ &\leq \psi(h(u_1))(h(u_n) - \inf_{w\in P(u_n)} h(w))\\ &\leq \psi(h(u_1))(h(u_n) - h(u_{n+1}) + \frac{1}{n}).\end{aligned}$$

Let

$$\psi_n = \psi(h(u_1))(h(u_n) - h(u_{n+1}) + \frac{1}{n})$$

for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \psi_n = 0$, so $\lim_{n\to\infty} \gamma(u_n, z, z) = 0$. On the other hand, $\{u_m\}$ is a $G$-Cauchy sequence. Then, $\lim_{n\to\infty} \gamma(u_m, u_m, u_n) = 0$ and $u_n \to \infty$, by uniqueness $z = u_0$. Then, $\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$. □

**Theorem 2.** **(Generalized Ekeland's variational principle)** *Assume that $(Z, G)$ is a complete q-G-m space and $h : Z \longrightarrow (-\infty, \infty]$ is a proper, bounded below and Lsca function. Suppose that $\gamma$ is a $\gamma$-function on $Z \times Z \times Z$, then there exists $u \in Z$ such that*

$$\gamma(y, u, u) > \psi(h(u))(h(u) - h(y))$$

*for all $u \in Z$ with $y \neq u$.*

**Proof.** Suppose it is not true. Then, for every $u \in Z$, there exists $v \in Z$, $v \neq u$ such that $\gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v))$. Then $P(u) \neq \emptyset$. We define the sequence $\{u_n\}$ as follows. Put $u_1 = \alpha$, we choose $u_2 \in S(u_1)$ such that $h(u_2) \leq \inf_{u \in S(u_1)} h(u) + 1$. In the same way suppose that $u_n \in Z$ is given. We choose $u_{n+1} \in S(u_n)$ such that $h(u_{n+1}) \leq \inf_{u \in P(u_n)} h(u) + \frac{1}{n}\cdot$. By Proposition 2, there exists $u_0 \in Z$ such that

$$\bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}.$$

By Lemma 3, we have $P(u_0) \subseteq \bigcap_{n=1}^{\infty} P(u_n) = \{u_0\}$ and then $P(u_0) = \{u_0\}$. This is a contradiction. Therefore, there exists $y \in Z$ such that

$$\gamma(y, u, u) > \psi(h(y))(h(y) - h(u)).$$

Now, we present two generalizations of Ekeland-type variational principles in the q-G-m spaces and complete q-G-m spaces. □

**Theorem 3.** *Assume that $(Z, G)$ is a q-G-m space, $\gamma : Z \times Z \times Z \longrightarrow \mathbb{R}_+$ is a $\gamma$-function on $Z$. Suppose that $\psi : (-\infty, \infty) \longrightarrow (0, \infty)$ is a nondecreasing function and the function $h : Z \longrightarrow \mathbb{R} \cup \{+\infty\}$ is proper and bounded below. We define the quasi-order $\preceq$ as follows:*

$$v \preceq u \;\; iff \;\; u = v \;\; or \;\; \gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v)). \tag{1}$$

*Suppose that there exists $\hat{u} \in Z$ such that $\inf_{u \in Z} h(u) < h(\hat{u})$ and $P(\hat{u}) = \{v \in Z : v \preceq \hat{u}\}$, is $\preceq$ $-$complete. Then, there exists $\bar{u} \in Z$ such that*
*(a) $\gamma(\hat{u}, \bar{u}, \bar{u}) \leq \psi(h(\hat{u}))(h(\hat{u}) - h(\bar{u}))$,*
*(b) $\gamma(\bar{u}, u, u) > \psi(h(\bar{u}))(h(\bar{u}) - h(u))$.*

**Proof.** The reflexive property is obvious. We show that $\preceq$ is transitive. Let $w \preceq v$ and $v \preceq u$, then

$$w = v \;\; or \;\; \gamma(v, w, w) \leq \psi(h(v))(h(v) - h(w)) \tag{2}$$

and

$$v = u \;\; or \;\; \gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v)). \tag{3}$$

If $w = v$ or $u = v$, the transitive is established. Let $u \neq v \neq w$, since $\gamma(u, v, w) \geq 0$ and $\psi(u) > 0$; then, by Equations (2) and (3), we have $h(v) \geq h(w)$ and $h(u) \geq h(v)$. Thus, $h(w) \leq h(u)$. Since $\psi$ is nondecreasing, then $\psi(h(v)) \leq \psi(h(u))$ by Definition 7 (1), (2) and (3), we get

$$\begin{aligned}
\gamma(u, w, w) &\leq \gamma(u, v, v) + \gamma(v, w, w) \\
&\leq \psi(h(u))(h(u) - h(v)) + \psi(h(v))(h(v) - h(w)) \\
&\leq \psi(h(u))(h(u) - h(v)) + \psi(h(u))(h(v) - h(w)) \\
&= \psi(h(u))(h(u) - h(w)).
\end{aligned}$$

Thus, $w \preceq u$, which means $\preceq$ is quasi-order on $Z$. We define the sequence $\{u_n\}$ in $P(\hat{u})$ as follows. Let

$$\begin{aligned}
P(u_n) &= \{v \in P(\hat{u}) : v = u_n \ \ or \ \ \gamma(u_n, v, v) \leq \psi(h(u_n))(h(u_n) - h(v))\} \\
&= \{v \in P(\hat{u}) : v \preceq u_n\}.
\end{aligned}$$

Putting $\hat{u} = u_0$, we choose $u_2 \in p(u_1)$ such that $h(u_2) \leq \inf\limits_{u \in P(u_1)} h(u) + \frac{1}{2}$. Let $u_{n-1}$ be specified. Then, we choose $u_n \in P(u_{n-1})$ such that

$$h(u_n) \leq \inf_{u \in P(u_{n-1})} h(u) + \frac{1}{n}. \tag{4}$$

Then, we have that $u_n \preceq u_{n-1}$ and $\{u_n\}$ is decreasing. Moreover,

$$\gamma(u_{n-1}, u_n, u_n) \leq \psi(h(u_{n-1}))(h(u_{n-1}) - h(u_n)),$$

and then, $h(u_n) \leq h(u_{n-1})$ for all $n \in \mathbb{N}$. This means that $\{h(u_n)\}$ is decreasing. On the other hand, $h$ is bounded below, Thus, $\{h(u_n)\}$is convergent. Letting $\lim\limits_{n \to \infty} h(u_n) = r$, we prove that $\{u_n\}$ is a Cauchy sequence in $P(\hat{u})$. Assuming that $n < m$, then

$$\begin{aligned}
\gamma(u_n, u_m, u_m) &\leq \gamma(u_n, u_{n+1}, u_{n+1}) + \gamma(u_{n+1}, u_m, u_m) \\
&\leq \gamma(u_n, u_{n+1}, u_{n+1}) + \gamma(u_{n+1}, u_{n+2}, u_{n+2}) + \cdots \\
&+ \gamma(u_{m-1}, u_m, u_m) \\
&\leq \psi(h(u_n))(h(u_n) - h(u_{n+1})) + \psi(h(u_{n+1}))(h(u_{n+1}) \\
&- h(u_{n+2})) + \cdots + \psi(h(u_{m-1}))(h(u_{m-1}) - h(u_m)) \\
&\leq \psi(h(u_n))(h(u_n) - h(u_{n+1})) + \psi(h(u_n))(h(u_{n+1}) \\
&- h(u_{n+2})) + \cdots + \psi(h(u_n))(h(u_{m-1}) - h(u_m)) \\
&\leq \psi(h(u_n))(h(u_n) - h(u_m)) \\
&\leq \psi(h(u_n))(h(u_n) - r).
\end{aligned}$$

Put $\rho_n = \psi(h(u_n))(h(u_n) - r)$, so $\lim\limits_{n \longrightarrow \infty} \rho_n = 0$ and by Lemma 1 (3), $\{u_n\}$ is a Cauchy sequence in $P(\hat{u})$. Since $P(\hat{u})$, is $\preceq$ $-$complete, then $\{u_n\}$ is convergent to $\bar{u} \in P(\hat{u})$. Since $\preceq$ is transitive, $P(u_n) \subset P(u_{n-1})$ for all $n \in \mathbb{N}$. Therefore, $(a)$ holds. We show that $\{\bar{u}\} = P(\bar{u})$. On the other hand, assuming that $u \in p(\bar{u})$ and $u \neq \bar{u}$, we have:

$$\gamma(\bar{u}, u, u) \leq \psi(h(\bar{u}))(h(\bar{u}) - h(u)).$$

Since $\gamma$ and $\psi$ are nonnegative, then $h(u) \leq h(\bar{u})$. $\bar{u} \in P(u_{n-1})$, implies that $u \preceq \bar{u}$ and $\bar{u} \preceq u_{n-1}$. Then, $u \preceq u_{n-1}$ (transitive $\preceq$) for all $n \in \mathbb{N}$. In addition, we have

$$h(\bar{u}) \leq h(u_n) \leq h(u_n) + \frac{1}{n}.$$

On the other hand $\lim_{n\longrightarrow\infty} h(u_n) = r$, thus $h(\bar{u}) \leq r \leq h(u) \leq h(\bar{u})$. Therefore, $h(\bar{u}) = h(u) = r$. Since $u \preceq u_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \gamma(u_n, u, u) &\leq \psi(h(u_n))(h(u_n) - h(u)) \\ &= \psi(h(u_n))(h(u_n) - r) \\ &= \rho_n. \end{aligned} \tag{5}$$

In addition, we have

$$\begin{aligned} \gamma(u_n, \bar{u}, \bar{u}) &\leq \psi(h(u_n))(h(u_n) - h(\bar{u})) \\ &= \psi(h(u_n))(h(u_n) - r) \\ &= \rho_n. \end{aligned} \tag{6}$$

Thus, $\lim_{n\longrightarrow\infty} \rho_n = 0$. By Equations (5) and (6) and Lemma (1), (1) we have $u = \bar{u}$. Thus, $\{\bar{u}\} = P\{\bar{u}\}$. Hence,

$$\gamma(\bar{u}, u, u) > \psi(h(\bar{u}))(h(\bar{u}) - h(u)\cdot$$

for all $u \in Z, \ \ u \neq \bar{u}$. □

**Theorem 4.** *Let $(Z, G)$ be a complete q-G-m space and*

$$\gamma : Z \times Z \times Z \longrightarrow \mathbb{R}_+$$

*be a $\gamma$-function on $Z$, $\psi : (-\infty, \infty] \longrightarrow (0, \infty)$ be a nondecreasing function and function $h : Z \longrightarrow \mathbb{R} \cup \{\infty\}$ be Lsca, proper and bounded below. Suppose that there is $\hat{u} \in Z$ such that $\inf_{u\in Z} h(u) < h(\hat{u})$. Then, there is $\bar{u} \in Z$ such that*
*(a) $\gamma(\hat{u}, \bar{u}, \bar{u}) \leq \psi(h(\hat{u}))(h(\hat{u}) - h(\bar{u}))$,*
*(b) $\gamma(\bar{u}, u, u) > \psi(h(\bar{u}))(h(\bar{u}) - h(u))$ for all $u \in Z$ , $u \neq \bar{u}$.*

**Proof.** We define order $\preceq$ as follows:

$$v \leq u \quad iff \quad u = v \quad or \quad \gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v)). \tag{7}$$

We prove that $\preceq$ is quasi-order. We show that $\preceq$ is lower closed. Assume that $\{u_n\}$ is a sequence in $Z$, such that $\{u_n\}$ convergent to $u \in Z$ and $u_{n+1} \preceq u_n$, thus

$$\gamma(u_n, u_{n+1}, u_{n+1}) \leq \psi(h(u_n))(h(u_n) - h(u_{n+1})). \tag{8}$$

Since $\gamma \geq 0$ and $\psi \geq 0$, then $h(u_{n+1}) \leq h(u_n)$. Therefore, $\{h(u_n)\}$ is a nondecreasing sequence. Since $h$ is bounded below, then there is $\lim_{n\longrightarrow\infty} h(u_n)$. Let $\lim_{n\longrightarrow\infty} h(u_n) = r$ for all $n \in \mathbb{N}$. In addition, if $h$ be *Lsca*, then $h(u) \leq \lim_{n\longrightarrow\infty} h(u_n)$ and we have $h(u) \leq r \leq h(u_n)$. Let $n \in \mathbb{N}$ be fixed. Using Theorem 3, we have

$$\begin{aligned} \gamma(u_n, u_m, u_m) &\leq \psi(h(u_n))(h(u_n) - h(u_m)) \\ &\leq \psi(h(u_n))(h(u_n) - h(u)). \end{aligned}$$

Therefore, $h(u) \leq h(u_n)$ for all $n \in \mathbb{N}$. Put $M = \psi(h(u_n))(h(u_n) - h(u))$. By Definition 7, (2), we have $\gamma(u_n, u_m, u_m) \leq M$ for all $n \in \mathbb{N}$. Then,

$$\gamma(u_n, u, u) \leq M = \psi(h(u_n))(h(u_n) - h(u)) \cdot \quad for \ \ all \ \ n \in \mathbb{N}.$$

Thus, $u \preceq u_n$, which means $\preceq$ is lower closed and $P(w) = \{v \in Z : v \preceq w\}$ for all $w \in Z$, is lower closed. Let

$$\begin{aligned} P(u_n) &= \{v \in Z : v = u_n \ \ or \ \ \gamma(u_n, v, v) \leq \psi(h(u_n))(h(u_n) - h(v))\} \\ &= \{v \in Z : v \preceq u\}. \end{aligned}$$

Then, $P(u_n)$ is lower closed for every $n \in \mathbb{N}$. Therefore, $P(u_n)$ is $\preceq$ −complete space. The rest of proof obtained of Theorem 3. □

**Remark 2.** *Theorem 3 and Theorem 4 are generalizations of Theorem* 2.1 *in [5] , Theorem* 1.1 *in [25], Theorem* 1 *in [13], Theorem* 3 *in [26] , Theorem* 2.1 *in [27] and Theorem* 3 *in [28].*

**Corollary 1.** *Let $\psi$, $Z$, $\gamma$ and $h$ be the same as Theorem 4. Let $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be non-decreasing and sub-additive such that $\xi(0) = 0$. Assume that there is $\hat{u} \in Z$ such that $\inf_{u \in Z} h(u) < h(\hat{u})$, then there exists $\bar{u} \in Z$ such that*
*(a) $\xi(\gamma(\hat{u}, \bar{u}, \bar{u})) \leq \psi(h(\hat{u}))(h(\hat{u}) - h(\bar{u}))$,*
*(b) $\xi(\gamma(\bar{u}, u, u)) > \psi(h(\bar{u}))(h(\bar{u}) - h(u)), \ \ u \neq \bar{u}$.*

**Proof.** By Remark 1, we conclude that $\eta \circ \gamma \in \Gamma$. The rest of proof is obtained from Theorem 4. □

## 3. EVP Results

**Theorem 5.** *Let $(Z, G)$ be a complete q-G-m space and $\gamma \in \Gamma$. Let $\psi : (-\infty, \infty] \longrightarrow (0, \infty)$ be nondecreasing and the function $h$ be a proper, bounded below and Lsca, then the following statements are equivalent to Theorem 4.*
***(a) (Caristi–Kirk type fixed point theorem)** Assume that $T : Z \longrightarrow 2^Z$ is a multi-valued map with nonempty value. If*

$$\gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v)) \tag{9}$$

*is satisfied for all $v \in T(u)$, and $T$ has a fixed point, i.e., there is $\bar{u} \in Z$ such that $\{\bar{u}\} = T(\bar{u})$. If*

$$\gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v)) \tag{10}$$

*for some $v \in T(u)$, then there exists $\bar{u} \in Z$, such that $\bar{u} \in T(\bar{u})$.*
***(b) (Takahashi's minimization theorem)** Let $\inf_{w \in Z} h(w) < h(\hat{u})$ for every $\hat{u} \in Z$, and let there be $u \in Z$ such that*

$$\gamma(\hat{u}, u, u) \leq \psi(h(\hat{u}))(h(\hat{u}) - h(u)) \quad ; \quad u \neq \hat{u}. \tag{11}$$

*Then, there exists $\bar{u} \in Z$ such that $h(\bar{u}) = \inf_{v \in Z} h(v)$.*
***(c) (Equilibrium version of Ekeland-type variational principle)** Assume that $H : Z \times Z \longrightarrow \mathbb{R} \cup \{\infty\}$ is a function with the following properties:*

*$(E_1)$ $H(u, w) \leq H(u, v) + H(v, w)$ for every $u, v, w \in Z$,*
*$(E_2)$ $H(u, \cdot) : Z \longrightarrow \mathbb{R} \cup \{\infty\}$, for every $u \in Z$ is proper and Lsca,*
*$(E_3)$ There exists $u \in Z$ that $\inf_{u \in Z} H(\hat{u}, u) > -\infty$.*

*Then, there is $u \in Z$ such that*

*(i) $\psi(h(\hat{u}))H(\hat{u}, \bar{u}) + \gamma(\hat{u}, \bar{u}, \bar{u}) \leq 0$,*
*(ii) $\psi(h(\bar{u}))H(\bar{u}, u) + \gamma(\bar{u}, u, u) > 0$ for each $u \in Z$ and $u \neq \bar{u}$.*

**Proof.** We show that Theorem 4 implies (a). □

By Theorem 4, there is $\bar{u} \in Z$ such that $u \neq \bar{u}$

$$\gamma(\bar{u}, u, u) > \psi(h(\bar{u}))(h(\bar{u}) - h(u)) \ \ for \ all u \in Z, \ \ u \neq \bar{u}. \tag{12}$$

We show that $\{\bar{u}\} = T(\bar{u})$. Suppose that it is not true. By Equation (9), we have $\gamma(\bar{u}, v, v) \leq \psi(h(\bar{u}))(h(\bar{u}) - h(u))$ and $\gamma(\bar{u}, v, v) > \psi(h(\bar{u}))(h(\bar{u}) - h(u))\cdot$, for every $v \in \psi(\bar{u})$ with $y \neq \bar{u}$. This is a contradiction. Thus, $\{\bar{u}\} = T(\bar{u})$.

$(a) \Rightarrow (b)$:

Let $T : Z \longrightarrow 2^Z$ and

$$T(u) = \{v \in Z; \gamma(u, v, v) \leq \psi(h(u))(h(u) - h(v))\}$$

for all $u \in Z$ and for every $T$ in Equation (9).

By $(a)$, there is $\bar{u} \in Z$ such that $\{\bar{u}\} = T(\bar{u})$. On the other hand, according to assumptions, for every $\hat{u} \in Z$ with $\inf_{w \in Z} h(w) < h(\hat{u})$, there is $u \in Z$ such that $\gamma(\hat{u}, u, u) \leq \psi(h(\hat{u}))(h(\hat{u}) - h(u)), \ \ u \neq \hat{u}$. Therefore, $u \in T(\hat{u})$ and $T(\hat{u}) \backslash \{\hat{u}\} \neq \emptyset$, thus $f(\bar{u}) = \inf_{u \in Z} h(u)$.

$(b) \Rightarrow (c)$:

Suppose $h : Z \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined by $h(u) = H(\hat{u}, u)$, $\hat{u}$ is the same as in $(E_3)$. By $(E_3)$, we have $\inf_{u \in Z} h(u) > -\infty$, and then $h$ is bounded below. Let $(ii)$ not be true. Then, for each $u \in Z$, $v \in Z$ exists such that

$$\psi(h(u))H(u, v) + \gamma(u, v, v) \leq 0. \tag{13}$$

By $(E_1)$, we have $H(\hat{u}, v) \leq H(\hat{u}, u) + H(u, v)$, then $H(\hat{u}, v) - H(\hat{u}, u) \leq H(u, v)$. Therefore, by Equation (13), we have

$$\psi(h(u))(H(\hat{u}, v) - H(\hat{u}, u)) + \gamma(u, v, v) \leq \psi(h(u))H(u, v) + \gamma(u, v, v) \leq 0. \tag{14}$$

Thus, for all $u \in Z$, there is $v \in Z$, such that

$$\psi(h(u))(h(v) - h(u)) + \gamma(u, v, v) \leq 0\cdot, \quad v \neq u.$$

On the other hand, $\gamma(u, v, v) \leq \psi(h(u))(h(v) - h(u))$. Now, by $(b)$,

$$h(\bar{u}) = \inf_{v \in Z} h(v) \leq h(w)\cdot$$

Replace $u$ by $\bar{u}$, then

$$\psi(h(\bar{u}))(H(\hat{u}, v) - H(\hat{u}, v)) + \gamma(\bar{u}, v, v) \leq 0, \quad v \neq u.$$

Thus,

$$\begin{aligned} & \psi(h(\bar{u}))(h(v) - h(\bar{u})) + \gamma(\bar{u}, v, v) \leq 0 \\ or \qquad & \gamma(\bar{u}, v, v) \leq \psi(h(\bar{u}))(h(\bar{u}) - h(v)). \end{aligned} \tag{15}$$

Since $v \neq u$, by Lemma 1, (1), we get $\gamma(\bar{u}, \bar{u}, \bar{u}) \neq 0$ and $\gamma(\bar{u}, u, u) \neq 0$. Thus, we have $\gamma(\bar{u}, v, v) > 0$ and by Equation (15),

$$0 < \psi(h(\bar{u}))(h(\bar{u}) - h(v)) \Longrightarrow h(v) < h(\bar{u}),$$

which is a contradiction.

$(c) \Rightarrow$Theorem 4:

Assume that $H : Z \times Z \longrightarrow \mathbb{R} \cup \{\infty\}$ is a function such that $H(u, v) = h(v) - h(u), \ \ u, v \in Z$. Then, by hypothesis, $H$ is satisfied in condition $(c)$. By $(i)$, we have

$$\psi(h(\hat{u}))H(\hat{u}, \bar{u}) + \gamma(\hat{u}, \bar{u}, \bar{u}) \leq 0.$$

Then,

$$\psi(h(\hat{u}))(h(\bar{u}) - h(\hat{u})) + \gamma(\hat{u}, \bar{u}, \bar{u}) \leq 0$$

and so

$$\gamma(\hat{u}, \bar{u}, \bar{u}) \leq \psi(h(\hat{u}))(h(\hat{u}) - h(\bar{u})).$$

In addition, by $(ii)$, we have

$$\psi(h(\bar{u}))H(\bar{u}, u) + \gamma(\bar{u}, u, u) > 0$$

for each $u \in Z$ and $u \neq \bar{u}$. Then,

$$\psi(h(\bar{u}))(h(u) - h(\bar{u})) + \gamma(\bar{u}, u, u) > 0.$$

**Corollary 2.** ***(Equilibrium version of Ekeland-type variational principle.)*** *Assume that $(Z, G)$ is a complete q-G-m space and $\gamma \in \Gamma$. Let $H : Z \times Z \longrightarrow \mathbb{R}$ be a function such that:*

*($E_1$) $H(u, w) \leq H(u, v) + H(v, w)$ for all $u, v, w \in Z$,*
*($E_2$) $H(u, \cdot) : Z \longrightarrow \mathbb{R}$, for every $u \in Z$ be a Lsca and bounded below, then for every $\epsilon > 0$ and for each $\hat{u} \in Z$, there is $\bar{u} \in Z$ such that*

*(i) $H(\hat{u}, \bar{u}) + \gamma(\hat{u}, \bar{u}, \bar{u}) \leq 0$,*
*(ii) $H(\bar{u}, u) + \epsilon\Gamma(\bar{u}, u, u) > 0$ for every $u \in Z$, $u \neq \bar{u}$.*

**Proof.** Put $h : Z \longrightarrow \mathbb{R} \cup \{+\infty\}$, that $h(\hat{u}) = H(u, \hat{u})$ and $\hat{u} \in Z$. Then, function $h$ is proper, *Lsca* and bounded by the below. For every $\epsilon > 0$ and $\hat{u} \in Z$, we define $\psi(h(\hat{u})) = \frac{1}{\epsilon}$. Then, by Theorem 5, we get $(c)$, $(i)$ and $(ii)$. □

**Corollary 3.** (**Nonconvex equilibrium theorem**). *Assume that the function $H : Z \times Z \longrightarrow (-\infty, \infty)$ is a proper and lsca and bounded below in the first argument, and $\psi : (-\infty, \infty) \to (0, \infty)$ is a nondecreasing function. Let, for each $u \in Z$ with $\{x \in Z : H(u, x) < 0\} \neq \emptyset$, there exist $v = v(u) \in Z$ with $v \neq u$ such that*

$$\gamma(u, v, v) \leq \psi\big(H(u, t)\big)\big(H(u, t) - H(v, t)\big) \tag{16}$$

*holds for all $z \in Z$. Then, there exists $y \in Z$ such that $H(y, v) \geq 0$ for all $v \in Z$.*

**Proof.** By Theorem 2 for each $w \in Z$, there exists $y(w) \in Z$ such that $\gamma(y(w), u, u) > \psi\big(H(y(w), w)\big)\big(H(y(w), w) - H(u, w)\big)$ for each $u \in Z$ with $u \neq y(w)$. We show that there exists $y \in Z$ such that$H(y, v) \geq 0$ for all $v \in Z$. On the contrary, for each $u \in Z$, there exists $v \in Z$ such that $H(u, v) < 0$. Then, for each $u \in Z$, $\{x \in Z : H(u, x) < 0\} \neq \emptyset$. According to the assumption, there exists $v = v(y(w)), v \neq y(w)$ such that $\gamma(y(w), v, v) \leq \psi\big(H(y(w), w)\big)\big(H(y(w), w) - H(v, w)\big)$, which is a contradiction. □

**Example 5.** *Assume that $Z = [0, 1]$ and $G(u, v, w) = \max\{|u - v|, |u - w|, |v - w|\}$. Then, $(Z, G)$ is a complete q-G-m space. Let a,b be positive real numbers with $a \geq b$. Assume that $H : Z \times Z \longrightarrow R$ with $H(u, v) = au - bv$. Therefore, function $x \longrightarrow H(u, v)$ is proper, lower semicontinuous and bounded below, and $H(1, v) \geq 0$ for every $v \in Z$. In addition, $H(u, v) \geq 0$ for every $u \in [\frac{b}{a}, 1]$ and for each $v \in Z$. In fact, for each $u \in [0, \frac{b}{a}]$, $H(u, v) = au - bv < 0$ when $v \in [\frac{a}{b}u, 1]$. Then, set $\{x \in Z : H(u, x) < 0\} \neq \emptyset$ for each $u \in [0, \frac{b}{a}]$. Supposing $u, v \in Z$, $u \geq v$, we have $u - v = \frac{1}{a}\{(au - bx) - (av - bx)\}$, for each $x \in z$. Let $\psi : [0, \infty) \longrightarrow [0, \infty)$ with $\psi(t) = \frac{1}{a}$ be define. Therefore, $G(u, v, v) \leq \psi\big(H(u, x)\big)\big(H(u, x) - H(v, x)\big)$, for each $u \geq v$, and $u, v, x \in Z$. By Corollary 3, there exists $y \in Z$ such that $H(y, v) \geq 0$ for every $v \in Z$.*

**Definition 8.** *Let $M$ be a nonempty subset of metric space (see [29]). Assume that $H : M \times M \longrightarrow \mathbb{R}$ is a real valued function and $\gamma \in \Gamma$ . Let $\epsilon > 0$, if there exists $\bar{u} \in Z$, such that*

$$H(\bar{u}, v) + \epsilon\gamma(\bar{u}, v, v) \geq 0 \tag{17}$$

*for every $v \in M$, then $\bar{u}$ is a $\sigma$-solution of EP if the inequality in Equation (17) is strict for all $u \neq v$. Then, $\bar{u}$ is said to be strictly a $\sigma$-solution of EP.*

**Theorem 6.** *Assume that $M$ is a nonempty compact of complete metric space $Z$ and $\gamma \in \Gamma$ Assume that $H : M \times M \longrightarrow \mathbb{R}$ is a real valued function such that:*

$(E_1)$ *$H(u, w) \leq H(u, v) + H(v, w)$ for each $u, v, w \in M$,*
$(E_2)$ *For all $u \in M$, $H(u, \cdot) : M \longrightarrow \mathbb{R}$ is an Lsca and bounded below function,*
$(E_3)$ *For all $v \in M$, $H(\cdot, v) : M \longrightarrow \mathbb{R}$ is a lower semicontinuous function.*

*Then, $\bar{u} \in M$ is a solution of EP.*

**Proof.** By using Corollary 2, for all $n \in \mathbb{N}$, there is $u_n \in M$ such that

$$H(u_n, v) + \frac{1}{n}\gamma(u_n, v, v) \geq 0$$

for every $v \in M$.

On the other hand, for $n \in \mathbb{N}$, $u_n \in M$ is a $\sigma$-solution of $EP$ for $\sigma = \frac{1}{n}$. $M$ is compact. Therefore, there is subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \longrightarrow \bar{u}$. Since $H(\cdot, v)$ is upper semicontinuous, then

$$F(\bar{u}, v) \geq \limsup_{n \longrightarrow \infty}(H(u_n, v) + \frac{1}{n_k}\gamma(u_{n_k}, v, v)) \geq 0$$

$\bar{u}$ is a solution of $EP$. □

**Definition 9.** *Assume that $(Z, G)$ is a complete q-G-m space and $\gamma$ is a $\gamma$-function on $Z$. We said that $u_0 \in Z$ is satisfied in condition $\Xi$ if and only if every sequence $\{u_n\}$ in $Z$, $H(u_0, u_n) \leq \frac{1}{n}$ for each $n \in \mathbb{N}$ and $H(u_n, u) + \frac{1}{n}\gamma(u_n, u, u) \geq 0$ for all $u \in Z$ and $n \in N$ has a convergent subsequence.*

**Theorem 7.** *Assume that $(Z, G)$ be a complete q-G-m space. Suppose $\gamma \in \Gamma$ and $H : Z \times Z \longrightarrow \mathbb{R}$ satisfied in conditions $(E_1)$ and $(E_2)$ of Corollary 2. In addition, $H$ will be upper semicontinuous in first argument, if $u_0 \in Z$ be satisfied in condition $\Xi$. Then, there is a solution $\bar{u} \in Z$ of EP.*

**Proof.** Put $\sigma = \frac{1}{n}$ in Corollary 2, then for each $n \in \mathbb{N}$ and for every $u_0 \in X$, there is $u_n \in Z$ such that

$$H(u_0, u_n) + \frac{1}{n}\gamma(u_0, u_n, u_n) \leq 0 \tag{18}$$

and

$$H(u_n, u) + \frac{1}{n}\gamma(u_n, u, u) > 0, \tag{19}$$

for all $u \in Z$. Since $\gamma(u_0, u_n, u_n) \geq 0$, by Equation (18), we have, $H(u_0, u_n) \leq 0$ for all $n \in \mathbb{N}$. □

By using condition $\Xi$, there is subsequence $\{u_n\}$ that is convergent to point $\bar{u} \in Z$. On the other hand, by using semicontinuous $h(\cdot, u)$ and Equation (19), we find that $\bar{u}$ is a solution of $EP$.

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