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Credibility Measure for Intuitionistic Fuzzy Variables

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Received: 7 March 2018; Accepted: 26 March 2018; Published: 2 April 2018



Abstract: Credibility measures in vague environments are to quantify the approximate chance of occurrence of fuzzy events. This paper presents a novel definition about credibility for intuitionistic fuzzy variables. We axiomatize this credibility measure and to clarify, give some examples. Based on the notion of these concepts, we provide expected values, entropy, and general formulae for the central moments and discuss them through examples.

Keywords: credibility measure; intuitionistic fuzzy variables; expected value; entropy

1. Introduction

Fuzzy set theory, proposed by Zadeh [1] and intuitionistic fuzzy set proposed by Atanassov [2,3], have influenced human judgment a great deal and decreased uncertainties in available information. Entropy, as an important tool of measuring the degree of uncertainty, and also the main core of information theory, for the first time was introduced by Shannon [4], and Zadeh [5] was the first who defined entropy for fuzzy sets by introducing weighted Shannon entropy. However, the synthesis of entropy and fuzzy set theory was first defined by Du Luca and Termini [6] using the Shannon function. He replaced the membership degrees of elements with the variable in the Shannon function. Later, this definition was extended by Szmidt and Kacprzyk [7] by introducing the intuitionistic fuzzy sets. The fuzzy and intuitionistic fuzzy entropy are now often employed in various scientific studies. For example, Huang et al. [8] used fuzzy two dimensional entropy to develop a novel approach for the automatic recognition of red Fuji apples in natural scenes, Yari et al. [9,10] employed it in option pricing and portfolio optimization, Song et al. [11] used fuzzy logics in psychology while studying children's emotions, and Farnoosh et al. [12] proposed a method for image processing based on intuitionistic fuzzy entropy. Additionally, many researchers have recently conceptualized fuzzy entropy from different aspects. Some of them can be found in [13–16].

In 2002, using credibility, a new formula was presented by Liu and Liu [17] for expected values of fuzzy variables. By these notations, a new environment has been created in the fuzzy area, both in pure and applied branches. Decision-making, portfolio optimization, pricing models, and supply chain problems are some of the areas which have used these conceptions. Now, in this paper, we define a new concept of credibility measure for intuitionistic fuzzy sets to be used in all of the mentioned areas.

After Liu and Liu [17], new concepts and properties of fuzzy credibility functions were proposed by some researchers. For example, a sufficient and necessary condition was given by Li and Liu [18] for credibility measures and, in 2008, an entropy measure was defined by Li and Liu [19] for discrete and continuous fuzzy variables, based on the credibility distributions. Therefore, in the rest of the paper, several additional concepts of fuzzy credibility functions are presented as a basis of developing the credibility measure.

In this paper, following the introduction in Section 1 and introducing some concepts and knowledge about credibility and fuzzy entropy measures in Section 2, based on the measure defined by Liu and Liu [17], we present a novel definition about credibility for intuitionistic fuzzy variables in Section 3. Section 3 also presents central moments and entropy formulation. All of these definitions are followed by their corresponding examples. We finally discuss and conclude in Section 4.

2. Preliminaries

Suppose that A is a fuzzy subset of the universe of discourse, U. Then the possibility and necessity measures are defined as (Zadeh, [20]):

$$\text{Pos}\{X \text{ is } A\} \triangleq \text{Sup}_{u \in A} \pi_X\{u\} \in [0, 1],$$

$$\text{Nec}\{X \text{ is } A\} \triangleq 1 - \text{Sup}_{u \in A^c} \pi_X\{u\},$$

where $\pi_X\{u\}$ is the possibility distribution function of Π_X , a possibility distribution associated with the variable X taking values in U.

For a fuzzy variable, ξ , with membership function μ , the credibility inversion theorem or, in other words, the credibility of $\xi \in \beta \subset \mathbb{R}$ employed by Liu and Liu [17] is:

$$\text{Cr}\{\xi \in \beta\} = \frac{1}{2}(\text{Pos}\{\xi \in \beta\} + \text{Nec}\{\xi \in \beta\}) = \frac{1}{2}(\text{Sup}_{x \in \beta} \mu(x) + 1 - \text{Sup}_{x \in \beta^c} \mu(x)) \quad (1)$$

Later, this formula was extended by Mandal et al. [21] to its general form as:

$$\text{Cr}\{\xi \in \beta\} = \rho \text{Pos}\{\xi \in \beta\} + (1 - \rho) \text{Nec}\{\xi \in \beta\}, \quad 0 \leq \rho \leq 1 \quad (2)$$

Liu and Liu [17] also defined the expected value of a fuzzy variable by the credibility function as:

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\} dr, \quad (3)$$

Later, Li and Liu [19] formulated a definition of entropy based on the notion of credibility for continuous distributions:

$$H(\xi) = \int_{-\infty}^{+\infty} S(\text{Cr}\{\xi = x\}) dx, \quad (4)$$

where the integral reduces to sigma for discrete distributions and $S(t) = -t \text{Lnt} - (1 - t) \text{Ln}(1 - t)$, $0 \leq t \leq 1$.

Further, De Luca and Termini [6] for the first time debated the fuzzy entropy measure (formulated in the following manner) which was later extended for intuitionistic fuzzy entropy measures by Szmidt and Kacprzyk [7].

Let H be a real-valued function: $F(X) \rightarrow [0, 1]$. H is an entropy measure of fuzzy set, if it satisfies the four axiomatic requirements:

$$\text{FS1} : H(\tilde{A}) = 0 \text{ iff } \tilde{A} \text{ is a crisp set, i.e., } \mu_{\tilde{A}}(x_i) = 0 \text{ or } 1 \forall x_i \in X.$$

$$\text{FS2} : H(\tilde{A}) = 1 \text{ iff } \mu_{\tilde{A}}(x_i) = 0.5 \forall x_i \in X.$$

$$\text{FS3} : H(\tilde{A}) \leq H(\tilde{B}) \text{ if } \tilde{A} \text{ is less fuzzy than } \tilde{B} \forall x_i \in X.$$

$$\text{FS4} : H(\tilde{A}) = H(\tilde{A}^c), \text{ where } \tilde{A}^c \text{ is the complement of } \tilde{A}.$$

Here, $H(\tilde{A}) = \sum_{i=1}^n S(\mu_A(x_i))$ or $H(\tilde{A}) = \int_{-\infty}^{+\infty} S(\mu_A(x)) dx$, for discrete and continuous distributions, respectively.

3. Credibility Measures in Intuitionistic Fuzzy Environment

Definition 1. *Determinacy of an intuitionistic fuzzy set*

Let A be an intuitionistic fuzzy subset of the universe of discourse, U ; and $f: A \rightarrow B$ be a function that changes the intuitionistic fuzzy elements $u \in A$ to fuzzy elements $v \in B$. Then, the determinacy measure is defined as follows:

$$det\{X \text{ is } B\} \triangleq \text{Sup}_{v \in B} \pi_X\{u\} \in [0, 1],$$

where $v \in B$ is a fuzzy number with γ and $1-\gamma$ as the degrees of membership and non-membership, respectively; γ is the degree of non-membership of the corresponding value of $u \in A$, and $\pi_X\{u\}$ is the possibility distribution function of Π_X .

3.1. Axioms of a Possibility-Determinacy Space

The quadruplet $(\Theta, P(\Theta), \text{Pos}, \text{Det})$ is called a possibility-determinacy space of an intuitionistic fuzzy variable if:

$$\left\{ \begin{array}{l} \text{i) Pos}\{\Theta\} = 1, \\ \text{ii) Pos}\{\emptyset\} = 0, \\ \text{iii) } 0 \leq \text{Pos}^+\{A\} = \text{Pos}\{A\} + \text{Det}\{A\} \leq 1, \\ \quad \text{and } 0 \leq \text{Pos}^-\{A\} = \text{Pos}\{A\} \leq 1, \\ \text{iv) Pos}^+\{U_i A_i\} = \sup_i \{\text{Pos}\{A_i\} + \text{Det}\{A_i\}\}, \\ \quad \text{and Pos}^-\{U_i A_i\} = \sup_i \{\text{Pos}\{A_i\}\}, \end{array} \right. \quad \text{for } A \text{ in Pos}\{\Theta\} \quad (5)$$

where Θ is a nonempty set, $P(\Theta)$ the power set of Θ , Pos a distribution of possibility from 2^U to $[0,1]$ and Det , the determinacy (Definition 1). It is easy to check that the above axioms tend to the possibility space axioms when $\text{Det}\{A\}=0$; that is, when we have a fuzzy variable.

Possibility and necessity in the intuitionistic fuzzy environment will be denoted by duals $(\text{Pos}^+, \text{Pos}^-)$ and $(\text{Nec}^+, \text{Nec}^-)$. These expressions represent the maximum and the minimum of the possibility and necessity, respectively.

3.2. Necessity Measure of an Intuitionistic Fuzzy Set

Following the concepts of the triangular fuzzy numbers by Dubois and Prade [22], let A be an intuitionistic fuzzy variable on a possibility-determinacy space $(\Theta, P(\Theta), \text{Pos}, \text{Det})$. Then, the necessity measure of A is defined as follows:

$$\text{Nec}^+\{A\} = 1 - \text{Pos}^+\{A^c\}, \quad \text{Nec}^-\{A\} = 1 - \text{Pos}^-\{A^c\}, \quad \varphi = \text{Sup}_{x \in \mathbb{R}} \mu\{x\}. \quad (6)$$

Example 1. Let ξ be a triangular intuitionistic fuzzy number with the following membership and non-membership functions, μ and γ :

$$\mu(x) = \begin{cases} \frac{x-a}{b-a} \varphi, & a \leq x < b, \\ \varphi, & x = b, \\ \frac{c-x}{c-b} \varphi, & b < x \leq c, \\ 0 & \text{otherwise,} \end{cases} \quad \gamma(x) = \begin{cases} \frac{b-x+\omega(x-a)}{b-a}, & a \leq x < b, \\ \omega, & x = b, \\ \frac{x-b+\omega(c-x)}{c-b}, & b < x \leq c, \\ 0 & \text{otherwise,} \end{cases} \quad 0 \leq \omega + \varphi \leq 1$$

Then, we have Pos and Det given as:

$$\text{Pos}^- \{X \geq x_0\} = \sup_{x \geq x_0} \mu(x) = \begin{cases} \varphi, & a < x_0 \leq b, \\ \frac{c - x_0}{c - b} \varphi, & b < x_0 \leq c, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{Pos}^+ \{X \geq x_0\} = \sup_{x \geq x_0} \mu(x) = \begin{cases} 1, & a \leq x_0 \leq c, \\ 0 & \text{otherwise,} \end{cases}$$

3.3. Credibility Measure in Intuitionistic Fuzzy Environment

Based on the Pos (Pos^+ and Pos^-) and Nec (Nec^+ and Nec^-) measures, the credibility measure in intuitionistic fuzzy environment is given as:

$$\begin{aligned} \text{Cr}^- \{A\} &= \rho \text{Pos}^- (A) + (1 - \rho) \text{Nec}^- (A), & 0 \leq \rho \leq 1 \\ \Leftrightarrow \text{Cr}^- \{\xi \in B\} &= \rho \text{Sup}_{x \in B} \mu\{x\} + (1 - \rho)(1 - \text{Sup}_{x \in B^c} \mu\{x\}). \end{aligned} \tag{7}$$

$$\begin{aligned} \text{detCr}^- \{A\} &= \rho(\text{Pos}^+ \{A\} - \text{Pos}^- \{A\}) + (1 - \rho)(\text{Nec}^- \{A\} - \text{Nec}^+ \{A\}) \\ \Leftrightarrow \text{detCr}^- \{\xi \in B\} &= \rho \text{Sup}_{x \in B} \gamma\{x\} + (1 - \rho)(1 - \text{Sup}_{x \in B^c} \gamma\{x\}). \end{aligned} \tag{8}$$

Here, the $\text{Cr}^- \{A\}$ and $\text{detCr}^- \{A\}$ are, respectively, the fixed and the determinacy of the credibility measure, and ξ is an intuitionistic fuzzy variable with membership function μ and non-membership function γ .

We can see that Cr satisfies the following conditions:

- (i) $\text{Cr}^- \{\emptyset\} = \text{detCr}^- \{\emptyset\} = 0,$
- (ii) $\text{Cr}^- \{\mathbb{R}\} = 1,$
- (iii) for $A \subset B, \text{Cr}^- \{A\} \leq \text{Cr}^- \{B\}$ & $\text{detCr}^- \{A\} \leq \text{detCr}^- \{B\},$ for any $A, B \in 2^{\mathbb{R}}.$

Thus, similar to the credibility measure defined in Liu and Liu [17] and Equation (3.2) in Mandal et al. [10], Cr^- is an intuitionistic fuzzy measure on $(\mathbb{R}, 2^{\mathbb{R}}).$

Example 2. Following Example 1, the credibility for a standard triangular intuitionistic fuzzy number ξ is as follows:

$$\begin{aligned} \text{Cr}^- \{\xi \geq x_0\} &= \begin{cases} \rho \varphi + (1 - \rho) \left(1 - \frac{x_0 - a}{b - a} \varphi\right), & a \leq x_0 < b, \\ \rho \left(\frac{c - x_0}{c - b}\right) \varphi + (1 - \rho)(1 - \varphi), & b < x_0 \leq c, \\ 0, & \text{otherwise.} \end{cases} \\ \text{detCr}^- \{\xi \geq x_0\} &= \begin{cases} \rho(1 - \varphi) + (1 - \rho) \left(1 - \frac{x_0 - a}{b - a} \varphi\right), & a \leq x_0 < b, \\ \rho \left(1 - \frac{c - x_0}{c - b}\right) \varphi + (1 - \rho)(1 - \varphi), & b < x_0 \leq c, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 1. Let ξ be an intuitionistic fuzzy variable taking values in \mathbb{R} . If there exist an interval, $B,$ such that $\text{Cr}^- \{\xi \in B\} = \varphi$ or $\text{detCr}^- \{\xi \in B\} = \varphi + \omega,$ then for every interval, α (s.t. $\alpha \cap B = \emptyset,$ we have $\text{Cr}^- \{\xi \in \alpha\} = 0,$ where φ and ω are, respectively, the supremum values of membership and non-membership functions.

Proof of Lemma 1. From Equations (7) and (8), the maximum value for Cr^- and $detCr^-$ occur when $Sup_{x \in B^c} \mu(x) = 0$ and $Sup_{x \in B^c} \gamma(x) = 0$, respectively. Then, since $\alpha \subset B^c$, we have $Sup_{x \in \alpha} \mu(x) = Sup_{x \in \alpha} \gamma(x) = 0$. Therefore, $Cr^- \{ \xi \in \alpha \} = detCr^- \{ \xi \in \alpha \} = 0$. \square

Credibility is a value between 0 to φ for possibility function and increases to $\varphi + \omega$ when determinacy is involved. In this lemma, it is shown when we have an interval containing the highest credibility value, the existence of any other disjoint interval containing positive credibility is impossible and, therefore, we can ignore the intervals having no positive possibility and determinacy values in the following definitions, especially, for entropy.

To check this lemma for discrete fuzzy variables, see Li and Liu [19].

Definition 2. For the expected value and central moments of an intuitionistic fuzzy variable based on Cr^- and $detCr^-$, the general form for the n th moments of real-valued continuous intuitionistic fuzzy variable about a value c are introduced as $E^- [(\xi - c)^n]$ and $detE^- [(\xi - c)^n]$, where:

$$\begin{cases} IE^- [\xi] = \int_0^{+\infty} Cr^- \{ \xi \geq r \} dr - \int_{-\infty}^0 Cr^- \{ \xi \leq r \} dr, \\ detE^- = \int_0^{+\infty} Cr^+ \{ \xi \geq r \} dr - \int_{-\infty}^0 Cr^+ \{ \xi \leq r \} dr. \end{cases} \tag{9}$$

Here E^- is the fixed value which is similar to the expected value in fuzzy variables, whereas $detE^-$ measures the determinacy of an expected value. The expected value does not exist and is not defined if the right-hand side of Equation (9) is $\infty - \infty$.

For the central moments such as variance, skewness and kurtosis, similar to Liu and Liu [17], and based on the defined credibility measures in Section 3.3, for each intuitionistic fuzzy variable ξ with finite expected value, we have:

$$CM^- [\xi, n] = E^- [(\xi - E^-(\xi))^n], \quad detCM^- [\xi, n] = detE^- [(\xi - detE^-(\xi))^n],$$

where $CM^- [\xi, n]$ and $detCM^- [\xi, n]$ for $n = 2, 3$, and 4 , respectively, represent the variance, skewness, and kurtosis.

Note: In this new Definition 2, the expected value for membership degrees is isolated from the expected value for non-membership degrees wherein both are calculated from the credibility functions. It means that we have a dual $[E^- [\xi], detE^- [\xi]]$ which denotes the expected values, separately. A linear combination of the elements of this dual can be used as the score function, if one wants to compare some intuitionistic fuzzy variables.

Definition 3. Entropy for intuitionistic fuzzy variables

Similar to entropy measure by credibility functions, entropy is formulized for intuitionistic fuzzy variables. In this definition, we have again two measures, fixed and the determinacy entropies:

$$H(\xi) = \int_{-\infty}^{+\infty} S(Cr^- \{ \xi = x_i \}) + S(detCr^- \{ \xi = x_i \}) dx, \quad x \subset \beta$$

where according to Lemma 1, β is the smallest interval containing the positive possibilities.

Example 3. Let ξ be a triangular fuzzy variable with the membership and non-membership functions introduced in Section 3.1. Then, the entropy is:

$$H(\xi) = \left(\frac{c-a}{2}\right)(2\varphi + \omega).$$

If ξ be a trapezoidal fuzzy variable (a,b,c,d) , then:

$$H(\xi) = \left(\left(\frac{d-a}{2}\right) + \left(\ln 2 - \frac{1}{2}\right)(c-b)\right)(2\varphi + \omega).$$

4. Discussion and Conclusions

In this paper, we defined the notion of credibility in intuitionistic fuzzy environment as an extension of credibility for fuzzy values which was not described before and, thus, by this conception, we have created a new environment as we have had for fuzzy in [17]. Based on these conceptions, we presented novel definitions of expected value, entropy and a general formula for central moments of intuitionistic fuzzy variables. In each step, all the definitions and axioms in the paper are provided by illustrative examples.

Acknowledgments: Authors sincerely thank reviewers for their valuable comments and suggestions which have led to the present form of the paper. Mohamadtaghi Rahimi thanks University of Northern British Columbia for awarding the post-doctoral fellowship and acknowledges the partial funding support from the NSERC Discovery Development Grant of Prof. Pranesh Kumar.

Author Contributions: All the authors contributed equally in this work. They read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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