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Bi-Additive *s*-Functional Inequalities and Quasi-$*$-Multipliers on Banach Algebras

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Abstract: Using the fixed point method, we prove the Hyers-Ulam stability of quasi-$*$-multipliers on Banach $*$-algebras and unital C^*-algebras, associated to bi-additive s-functional inequalities.

Keywords: quasi-multiplier on C^*-algebra; quasi-$*$-multiplier on Banach algebra; Hyers-Ulam stability; fixed point method; bi-additive s-functional inequality

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1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [6] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [7]. Fechner [8] and Gilányi [9] proved the Hyers-Ulam stability of the functional inequality (1).

Park [10,11] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [12–15]).

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra, which was introduced by Akemann and Pedersen [16] for C^*-algebras. McKennon [17] extended the definition to a general complex Banach algebra with bounded approximate identity as follows.

Definition 1. *[17] Let A be a complex Banach algebra. A $\mathbb{C}$-bilinear mapping $P : A \times A \to A$ is called a quasi-multiplier on A if P satisfies*

$$P(xy, zw) = xP(y,z)w$$

for all $x, y, z, w \in A$.

Definition 2. *Let* A *be a complex Banach* $*$*-algebra. A bi-additive mapping* $P : A \times A \to A$ *is called a quasi-*$*$*-multiplier on* A *if* P *is* $\mathbb{C}$*-linear in the first variable and satisfies*

$$\begin{aligned} P(x,z) &= P(z,x)^*, \\ P(xy,z) &= xP(y,z) \end{aligned}$$

for all $x, y, z \in A$.

It is easy to show that if P is a quasi-$*$-multiplier, then P is conjugate $\mathbb{C}$-linear in the second variable and $P(xy, zw) = xP(y,w)z^*$ for all $x, y, z, w \in A$.

We recall a fundamental result in fixed point theory.

Theorem 1. *[18,19] Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $\alpha < 1$*. Then for each given element* $x \in X$*, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [20] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [21–25]).

This paper is organized as follows: In Sections 2 and 3, we prove the Hyers-Ulam stability of the following bi-additive s-functional inequalities

$$\begin{aligned} &\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\| \\ &\leq \left\| s\left(4f\left(\frac{x+y}{2}, z-w\right)+4f\left(\frac{x-y}{2}, z+w\right)-4f(x,z)+4f(y,w)\right)\right\|, \end{aligned} \tag{2}$$

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}, z-w\right)+4f\left(\frac{x-y}{2}, z+w\right)-4f(x,z)+4f(y,w)\right\| \\ &\leq \|s\left(f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\right)\| \end{aligned} \tag{3}$$

in complex Banach spaces by using the fixed point method. Here s is a fixed nonzero complex number with $|s| < 1$. In Section 4, we prove the Hyers-Ulam stability and the superstability of quasi-$*$-multipliers on Banach $*$-algebras and unital C^*-algebras associated to the bi-additive s-functional inequalities (2) and (3).

Throughout this paper, let X be a complex normed space and Y a complex Banach space. Let A be a complex Banach $*$-algebra. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2. Bi-Additive s-Functional Inequality (2)

Park [26] solved the bi-additive s-functional inequality (2) in complex normed spaces.

Lemma 1. *([26] Lemma 2.1) If a mapping $f : X^2 \to Y$ satisfies $f(0,z) = f(x,0) = 0$ and*

$$\begin{aligned} &\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\| \\ &\leq \left\| s\left(4f\left(\frac{x+y}{2},z-w\right)+4f\left(\frac{x-y}{2},z+w\right)-4f(x,z)+4f(y,w)\right)\right\| \end{aligned} \tag{4}$$

for all $x,y,z,w \in X$, then $f : X^2 \to Y$ is bi-additive.

Using the fixed point method, we prove the Hyers-Ulam stability of the bi-additive s-functional inequality (4) in complex Banach spaces.

Theorem 2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y) \tag{5}$$

for all $x,y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and

$$\begin{aligned} &\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\| \\ &\leq \left\| s\left(4f\left(\frac{x+y}{2},z-w\right)+4f\left(\frac{x-y}{2},z+w\right)-4f(x,z)+4f(y,w)\right)\right\| \\ &+\varphi(x,y)\varphi(z,w) \end{aligned} \tag{6}$$

for all $x,y,z,w \in X$. Then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that

$$\|f(x,z)-B(x,z)\| \leq \frac{L}{4(1-L)}\varphi(x,x)\varphi(z,0) \tag{7}$$

for all $x,z \in X$.

Proof. Letting $w = 0$ and $y = x$ in (6), we get

$$\|2f(2x,z)-4f(x,z)\| \leq \varphi(x,x)\varphi(z,0) \tag{8}$$

for all $x,z \in X$. So

$$\left\|f(x,z)-2f\left(\frac{x}{2},z\right)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0) \leq \frac{L}{4}\varphi(x,x)\varphi(z,0)$$

for all $x,z \in X$.

Consider the set

$$S := \{h : X^2 \to Y,\ \ h(x,0) = h(0,z) = 0\ \ \forall x,z \in X\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x,z)-h(x,z)\| \leq \mu\varphi(x,x)\,\varphi(z,0),\ \ \forall x,z \in X\right\},$$

where, as usual, $\inf\phi = +\infty$. It is easy to show that (S,d) is complete (see [27]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := 2g\left(\frac{x}{2},z\right)$$

for all $x,z \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x,z) - h(x,z)\| \leq \varepsilon\varphi\left(x,x\right)\varphi(z,0)$$

for all $x, z \in X$. Hence

$$\begin{aligned}\|Jg(x,z) - Jh(x,z)\| &= \left\|2g\left(\frac{x}{2},z\right) - 2h\left(\frac{x}{2},z\right)\right\| \leq 2\varepsilon\varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0) \\ &\leq 2\varepsilon\frac{L}{2}\varphi\left(x,x\right)\varphi(z,0) = L\varepsilon\varphi\left(x,x\right)\varphi(z,0)\end{aligned}$$

for all $x, z \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (8) that

$$\left\|f(x,z) - 2f\left(\frac{x}{2},z\right)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{2},\frac{x}{2}\right)\varphi(z,0) \leq \frac{L}{4}\varphi(x,x)\varphi(z,0)$$

for all $x, z \in X$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1, there exists a mapping $B : X^2 \to Y$ satisfying the following:

(1) B is a fixed point of J, i.e.,

$$B\left(x,z\right) = 2B\left(\frac{x}{2},z\right) \tag{9}$$

for all $x, z \in X$. The mapping B is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that B is a unique mapping satisfying (9) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x,z) - B(x,z)\| \leq \mu\varphi\left(x,x\right)\varphi(z,0)$$

for all $x, z \in X$;

(2) $d(J^l f, B) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n},z\right) = B(x,z)$$

for all $x, z \in X$;

(3) $d(f,B) \leq \frac{1}{1-L}d(f,Jf)$, which implies

$$\|f(x,z) - B(x,z)\| \leq \frac{L}{4(1-L)}\varphi\left(x,x\right)\varphi(z,0)$$

for all $x, z \in X$. So we obtain (7).

It follows from (5) and (6) that

$$\begin{aligned}
&\|B(x+y,z+w)+B(x+y,z-w)+B(x-y,z+w)+B(x-y,z-w)-4B(x,z)\| \\
&= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n},z+w\right)+f\left(\frac{x+y}{2^n},z-w\right)+f\left(\frac{x-y}{2^n},z+w\right)\right.\right.\\
&\qquad\left.\left.+f\left(\frac{x-y}{2^n},z-w\right)-4f\left(\frac{x}{2^n},z\right)\right)\right\| \\
&\le \lim_{n\to\infty}\left\|2^n s\left(4f\left(\frac{x+y}{2^{n+1}},z-w\right)+4f\left(\frac{x-y}{2^{n+1}},z+w\right)-4f\left(\frac{x}{2^n},z\right)+4f\left(\frac{y}{2^n},w\right)\right)\right\| \\
&\qquad+\lim_{n\to\infty}2^n\varphi\left(\frac{x}{2^n},\frac{x}{2^n}\right)\varphi(z,0) \\
&\le \left\|s\left(4B\left(\frac{x+y}{2},z-w\right)+4B\left(\frac{x-y}{2},z+w\right)-4B(x,z)+4B(y,w)\right)\right\|
\end{aligned}$$

for all $x,y,z,w\in X$, since $2^n\varphi\left(\frac{x}{2^n},\frac{x}{2^n}\right)\varphi(z,0)\le\frac{2^nL^n}{2^n}\varphi(x,x)\varphi(z,0)$ tends to zero as $n\to\infty$. So

$$\begin{aligned}
&\|B(x+y,z+w)+B(x+y,z-w)+B(x-y,z+w)+B(x-y,z-w)-4B(x,z)\| \\
&\le \left\|s\left(4B\left(\frac{x+y}{2},z-w\right)+4B\left(\frac{x-y}{2},z+w\right)-4B(x,z)+4B(y,w)\right)\right\|
\end{aligned}$$

for all $x,y,z,w\in X$. By Lemma 1, the mapping $B:X^2\to Y$ is bi-additive. □

Corollary 1. *Let $r>1$ and θ be nonnegative real numbers and let $f:X^2\to Y$ be a mapping satisfying $f(x,0)=f(0,z)=0$ and*

$$\begin{aligned}
&\|f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\| \\
&\le\left\|s\left(4f\left(\frac{x+y}{2},z-w\right)+4f\left(\frac{x-y}{2},z+w\right)-4f(x,z)+4f(y,w)\right)\right\| \\
&+\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r)
\end{aligned}\tag{10}$$

for all $x,y,z,w\in X$. Then there exists a unique bi-additive mapping $B:X^2\to Y$ such that

$$\|f(x,z)-B(x,z)\|\le\frac{\theta}{2^r-2}\|x\|^r\|z\|^r$$

for all $x,z\in X$.

Proof. The proof follows from Theorem 2 by taking $\varphi(x,y)=\sqrt{\theta}(\|x\|^r+\|y\|^r)$ for all $x,y\in X$. Choosing $L=2^{1-r}$, we obtain the desired result. □

Theorem 3. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)\tag{11}$$

for all $x,y\in X$. Let $f:X^2\to Y$ be a mapping satisfying (6) and $f(x,0)=f(0,z)=0$ for all $x,z\in X$. Then there exists a unique bi-additive mapping $B:X^2\to Y$ such that

$$\|f(x,z)-B(x,z)\|\le\frac{1}{4(1-L)}\varphi(x,x)\varphi(z,0)$$

for all $x,z\in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := \frac{1}{2} g\left(2x,z\right)$$

for all $x \in X$.

It follows from (8) that

$$\left\| f(x,z) - \frac{1}{2} f(2x,z) \right\| \leq \frac{1}{4} \varphi(x,x)\varphi(z,0)$$

for all $x, z \in X$.

The rest of the proof is similar to the proof of Theorem 2. □

Corollary 2. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying (10) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that*

$$\|f(x,z) - B(x,z)\| \leq \frac{\theta}{2 - 2^r} \|x\|^r \|z\|^r$$

for all $x, z \in X$.

Proof. The proof follows from Theorem 3 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

3. Bi-Additive *s*-Functional Inequality (3)

Park [26] solved the bi-additive s-functional inequality (3) in complex normed spaces.

Lemma 2. *([26] Lemma 3.1) If a mapping $f : X^2 \to Y$ satisfies $f(0,z) = f(x,0) = 0$ and*

$$\begin{aligned} &\left\| 4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x,z) + 4f(y,w) \right\| \\ &\quad \leq \| s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right) \| \end{aligned} \tag{12}$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is bi-additive.

Using the fixed point method, we prove the Hyers-Ulam stability of the bi-additive s-functional inequality (12) in complex Banach spaces.

Theorem 4. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (5). Let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned} &\left\| 4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x,z) + 4f(y,w) \right\| \\ &\leq \| s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right) \| \\ &\quad + \varphi(x,y)\varphi(z,w) \end{aligned} \tag{13}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that

$$\|f(x,z) - B(x,z)\| \leq \frac{1}{4(1-L)} \varphi(x,0)\varphi(z,0) \tag{14}$$

for all $x, z \in X$.

Proof. Letting $y = w = 0$ in (13), we get

$$\left\| 8f\left(\frac{x}{2}, z\right) - 4f(x,z) \right\| \le \varphi(x,0)\varphi(z,0) \tag{15}$$

for all $x, z \in X$.

Consider the set

$$S := \{h : X^2 \to Y,\ \ h(x,0) = h(0,z) = 0\ \ \forall x, z \in X\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x,z) - h(x,z)\| \le \mu\varphi\left(x,0\right)\varphi(z,0),\ \ \forall x, z \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [27]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := 2g\left(\frac{x}{2}, z\right)$$

for all $x, z \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x,z) - h(x,z)\| \le \varepsilon\varphi\left(x,0\right)\varphi(z,0)$$

for all $x, z \in X$. Hence

$$\begin{aligned} \|Jg(x,z) - Jh(x,z)\| &= \left\| 2g\left(\frac{x}{2}, z\right) - 2h\left(\frac{x}{2}, z\right) \right\| \le 2\varepsilon\varphi\left(\frac{x}{2}, 0\right)\varphi(z,0) \\ &\le 2\varepsilon\frac{L}{2}\varphi\left(x,0\right)\varphi(z,0) = L\varepsilon\varphi\left(x,0\right)\varphi(z,0) \end{aligned}$$

for all $x, z \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (15) that

$$\left\| f(x,z) - 2f\left(\frac{x}{2}, z\right) \right\| \le \frac{1}{4}\varphi(x,0)\varphi(z,0)$$

for all $x, z \in X$. So $d(f, Jf) \le \frac{1}{4}$.

By Theorem 1, there exists a mapping $B : X^2 \to Y$ satisfying the following:

(1) B is a fixed point of J, i.e.,

$$B\left(x,z\right) = 2B\left(\frac{x}{2}, z\right) \tag{16}$$

for all $x, z \in X$. The mapping B is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that B is a unique mapping satisfying (16) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x,z) - B(x,z)\| \le \mu\varphi\left(x,0\right)\varphi(z,0)$$

for all $x, z \in X$;

(2) $d(J^l f, B) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}, z\right) = B(x,z)$$

for all $x, z \in X$;

(3) $d(f, B) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x,z) - P(x,z)\| \leq \frac{1}{4(1-L)} \varphi(x,0)\,\varphi(z,0)$$

for all $x, z \in X$. So we obtain (14).

The rest of the proof is similar to the proof of Theorem 2. □

Corollary 3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x,z) + 4f(y,w)\right\| \\ &\leq \|s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right)\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{17}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that

$$\|f(x,z) - B(x,z)\| \leq \frac{2^{r-2}\theta}{2^r - 2} \|x\|^r \|z\|^r$$

for all $x, z \in X$.

Proof. The proof follows from Theorem 4 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 5. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (11). Let $f : X^2 \to Y$ be a mapping satisfying (13) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that*

$$\|f(x,z) - B(x,z)\| \leq \frac{L}{4(1-L)} \varphi(x,0)\,\varphi(z,0) \tag{18}$$

for all $x, z \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 4.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x,z) := \frac{1}{2} g(2x, z)$$

for all $x \in X$.

It follows from (15) that

$$\left\|f(x,z) - \frac{1}{2} f(2x,z)\right\| \leq \frac{1}{8}\varphi(2x,0)\varphi(z,0) \leq \frac{L}{4}\varphi(x,0)\varphi(z,0)$$

for all $x, z \in X$.

The rest of the proof is similar to the proofs of Theorems 2 and 4. $\square$

Corollary 4. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying (17) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $B : X^2 \to Y$ such that*

$$\|f(x,z) - B(x,z)\| \leq \frac{\theta}{4(2-2^r)}\|x\|^r\|z\|^r$$

for all $x, z \in X$.

Proof. The proof follows from Theorem 5 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-1}$, we obtain the desired result. $\square$

4. Quasi-$*$-Multipliers in C^*-Algebras

In this section, we investigate quasi-$*$-multipliers on complex Banach $*$-algebras and unital C^*-algebras associated to the bi-additive s-functional inequalities (4) and (12).

Theorem 6. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y) \tag{19}$$

for all $x, y \in A$. Let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and

$$\begin{aligned} &\|f(\lambda(x+y), z+w) + f(\lambda(x+y), z-w) + f(\lambda(x-y), z+w) \\ &\quad + f(\lambda(x-y), z-w) - 4\lambda f(x,z)\| \\ &\leq \left\| s\left(4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x,z) + 4f(y,w)\right)\right\| + \varphi(x,y)\varphi(z,w) \end{aligned} \tag{20}$$

for all $\lambda \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y, z, w \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - B(x,z)\| \leq \frac{L}{4(1-L)}\varphi(x,x)\varphi(z,0) \tag{21}$$

for all $x, z \in A$.

Furthermore, If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x,z) = 2f(x,z)$ and

$$\|f(xy,z) - xf(y,z)\| \leq \varphi(x,y)^2\varphi(z,0), \tag{22}$$

$$\|f(x,z) - f(z,x)^*\| \leq \varphi(x,0)\varphi(z,0) \tag{23}$$

for all $x, y, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Proof. Let $\lambda = 1$ in (20). By Theorem 2, there is a unique bi-additive mapping $B : A^2 \to A$ satisfying (21) defined by

$$B(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then we can easily show that $B(x,z) = f(x,z)$ for all $x, z \in A$.

Letting $y = x$ and $w = 0$ in (20), we get

$$\|2f(2\lambda x, z) - 4\lambda f(x,z)\| \leq \varphi(x,x)\varphi(z,0)$$

for all $x, z \in A$ and all $\lambda \in \mathbb{T}^1$. So

$$\begin{aligned}\|2B(2\lambda x, z) - 4\lambda B(x,z)\| &= \lim_{n\to\infty} 2^n \left\|2f\left(2\lambda \frac{x}{2^n}, z\right) - 4\lambda f\left(\frac{x}{2^n}, z\right)\right\| \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right)\varphi(z,0) \leq \lim_{n\to\infty} \frac{2^n L^n}{2^n}\varphi(x,x)\varphi(z,0) = 0\end{aligned}$$

for all $x, z \in A$ and all $\lambda \in \mathbb{T}^1$. Hence $2B(2\lambda x, z) = 4\lambda B(x,z)$ and so $B(\lambda x, z) = \lambda B(x,z)$ for all $x, z \in A$ and all $\lambda \in \mathbb{T}^1$. By ([28] Theorem 2.1), the bi-additive mapping $B : A^2 \to A$ is $\mathbb{C}$-linear in the first variable.

It follows from (22) that

$$\begin{aligned}\|B(xy, z) - xB(y,z)\| &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{xy}{2^n \cdot 2^n}, z\right) - \frac{x}{2^n} f\left(\frac{y}{2^n}, z\right)\right\| \\ &\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)^2 \varphi(z,0) \leq \lim_{n\to\infty} \frac{4^n L^{2n}}{4^n}\varphi(x,y)^2\varphi(z,0) = 0\end{aligned}$$

for all $x, y, z \in A$. Thus

$$B(xy, z) = xB(y,z)$$

for all $x, y, z \in A$.

It follows from (23) that

$$\begin{aligned}\|B(x,z) - B(z,x)^*\| &= \lim_{n\to\infty} 2^n \left\|f\left(x, \frac{z}{2^n}\right) - f\left(\frac{z}{2^n}, x\right)^*\right\| \\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, 0\right)\varphi(z,0) \leq \lim_{n\to\infty} \frac{2^n L^n}{2^n}\varphi(x,0)\varphi(z,0) = 0\end{aligned}$$

for all $x, z \in A$. Thus

$$B(x,z) = B(z,x)^*$$

for all $x, z \in A$. Hence the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier. □

Corollary 5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\|f(\lambda(x+y), z+w) + f(\lambda(x+y), z-w) + f(\lambda(x-y), z+w) \\ &\quad + f(\lambda(x-y), z-w) - 4\lambda f(x,z)\| \\ &\leq \left\|s\left(4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x,z) + 4f(y,w)\right)\right\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)\end{aligned} \tag{24}$$

for all $\lambda \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y, z, w \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - B(x,z)\| \leq \frac{\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{25}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x,z)$ and

$$\|f(xy, z) - xf(y,z)\| \leq \theta(\|x\|^r + \|y\|^r)\|z\|^r, \tag{26}$$

$$\|f(x,z) - f(z,x)^*\| \leq \theta \|x\|^r \|z\|^r \tag{27}$$

for all $x, y, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Proof. The proof follows from Theorem 6 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 7. *Let $\varphi : A^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{28}$$

for all $x, y \in A$. Let $f : A^2 \to A$ be a mapping satisfying (20) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - B(x,z)\| \leq \frac{1}{4(1-L)}\varphi(x,x)\varphi(z,0) \tag{29}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (22), (23) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Proof. The proof is similar to the proof of Theorem 6. □

Corollary 6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (24) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that*

$$\|f(x,z) - B(x,z)\| \leq \frac{\theta}{2 - 2^r}\|x\|^r \|z\|^r \tag{30}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Proof. The proof follows from Theorem 7 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

Similarly, we can obtain the following results.

Theorem 8. *Let $\varphi : A^2 \to [0, \infty)$ be a function satisfying (19) and let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned} &\left\|4f\left(\lambda\frac{x+y}{2}, z-w\right) + 4f\left(\lambda\frac{x-y}{2}, z+w\right) - 4\lambda f(x,z) + 4\lambda f(y,w)\right\| \\ &\leq \|s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right)\| \\ &\quad + \varphi(x,y)\varphi(z,w) \end{aligned} \tag{31}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - B(x,z)\| \leq \frac{1}{4(1-L)}\varphi(x,0)\varphi(z,0) \tag{32}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (22), (23) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Corollary 7. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned} &\left\| 4f\left(\lambda\frac{x+y}{2}, z-w\right) + 4f\left(\lambda\frac{x-y}{2}, z+w\right) - 4\lambda f(x,z) + 4\lambda f(y,w) \right\| \\ &\quad \le \| s\left(f(x+y,z+w) + f(x+y,z-w) + f(x-y,z+w) + f(x-y,z-w) - 4f(x,z)\right)\| \\ &\qquad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{33}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - B(x,z)\| \le \frac{2^{r-2}\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{34}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Proof. The proof follows from Theorem 8 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{1-r}$, we obtain the desired result. $\square$

Theorem 9. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (28). Let $f : A \to A$ be a mapping satisfying (31) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that*

$$\|f(x,z) - B(x,z)\| \le \frac{L}{4(1-L)}\varphi(x,0)\varphi(z,0) \tag{35}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (22), (23) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Corollary 8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (33) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that*

$$\|f(x,z) - B(x,z)\| \le \frac{\theta}{4(2 - 2^r)}\|x\|^r\|z\|^r \tag{36}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (26), (27) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier.

Proof. The proof follows from Theorem 9 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{r-1}$, we obtain the desired result. $\square$

From now on, assume that A is a unital C^*-algebra with unit e and unitary group $U(A)$.

Theorem 10. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (19) and let $f : A^2 \to A$ be a mapping satisfying (20) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (21).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (23), $f(2x,z) = 2f(x,z)$ and

$$\|f(uy,z) - uf(y,z)\| \leq \varphi(u,y)^2\varphi(z,0), \tag{37}$$

for all $u \in U(A)$ and all $x,y,z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x,w \in A$.

Proof. By the same reasoning as in the proof of Theorem 6, there is a unique bi-additive mapping $B : A^2 \to A$ satisfying (21), which is $\mathbb{C}$-linear in the first variable, defined by

$$B(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x,z \in A$.

If $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then we can easily show that $B(x,z) = f(x,z)$ for all $x,z \in A$.

By the same reasoning as in the proof of Theorem 6, $B(uy,z) = uB(y,z)$ for all $u \in U(A)$ and all $y,z \in A$.

Since B is $\mathbb{C}$-linear in the first variable and each $x \in A$ is a finite linear combination of unitary elements (see [29]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$),

$$\begin{aligned} B(xy,z) &= B(\sum_{j=1}^{m} \lambda_j u_j y, z) = \sum_{j=1}^{m} \lambda_j B(u_j y, z) = \sum_{j=1}^{m} \lambda_j u_j B(y,z) \\ &= (\sum_{j=1}^{m} \lambda_j u_j) B(y,z) = xB(y,z) \end{aligned}$$

for all $x,y,z \in A$. So by the same reasoning as in the proof of Theorem 6, $B : A^2 \to A$ is a quasi-$*$-multiplier and satisfies

$$B(x,w) = B(xe,we) = xB(e,we) = xB(we,e)^* = x(wB(e,e))^* = xB(e,e)^*w^* = xB(e,e)w^*$$

for all $x,w \in A$. Thus $f : A^2 \to A$ is a quasi-$*$-multiplier and satisfies $f(x,w) = f(xe,we) = xf(e,e)w^*$ for all $x,w \in A$. □

Corollary 9. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (24) and $f(x,0) = f(0,z) = 0$ for all $x,z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (25).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (27), $f(2x,z) = 2f(x,z)$ and

$$\|f(uy,z) - uf(y,z)\| \leq \theta(1 + \|y\|^r)\|z\|^r \tag{38}$$

for all $u \in U(A)$ and all $x,y,z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x,w \in A$.

Proof. The proof follows from Theorem 10 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x,y \in A$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 11. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (28). Let $f : A^2 \to A$ be a mapping satisfying (20) and $f(x,0) = f(0,z) = 0$ for all $x,z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (35).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (37), (23) and $f(2x,z) = 2f(x,z)$ for all $x,z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x,w \in A$.

Proof. The proof is similar to the proofs of Theorems 7 and 10. □

Corollary 10. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (24) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (36).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (38), (27) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x, w \in A$.

Proof. The proof follows from Theorem 11 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

Similarly, we can obtain the following results.

Theorem 12. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (19) and let $f : A^2 \to A$ be a mapping satisfying (31) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (32).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (37), (23) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x, w \in A$.

Corollary 11. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (33) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (34).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (38), (27) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x, w \in A$.

Proof. The proof follows from Theorem 12 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{1-r}$, we obtain the desired result. □

Theorem 13. *Let $\varphi : A^2 \to [0,\infty)$ be a function satisfying (28). Let $f : A^2 \to A$ be a mapping satisfying (31) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (35).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (37), (23) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x, w \in A$.

Corollary 12. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (33) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $B : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (36).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (38), (27) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a quasi-$*$-multiplier satisfying $f(x,w) = xf(e,e)w^*$ for all $x, w \in A$.

Proof. The proof follows from Theorem 13 by taking $\varphi(x,y) = \sqrt{\theta}(\|x\|^r + \|y\|^r)$ for all $x, y \in A$. Choosing $L = 2^{r-1}$, we obtain the desired result. □

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