

## Article

# Common Fixed Points for Mappings under Contractive Conditions of $(\alpha, \beta, \psi)$ -Admissibility Type

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**Abstract:** In this paper, we introduce the notion of  $(\alpha, \beta, \psi)$ -contraction for a pair of mappings  $(S, T)$  defined on a set  $X$ . We use our new notion to create and prove a common fixed point theorem for two mappings defined on a metric space  $(X, d)$  under a set of conditions. Furthermore, we employ our main result to get another new result. Our results are modifications of many existing results in the literature. An example is included in order to show the authenticity of our main result.

**Keywords:**  $\alpha$ -admissible mapping; common fixed point; metric spaces

## 1. Introduction and Preliminaries

The importance of fixed point theories lies in finding and proving the uniqueness of solutions for many questions of Applied Sciences such as Physics, Chemistry, Economics, and Engineering. The pioneer mathematician in the area of fixed point theory was Banach, who established and proved the first fixed point theorem named the “Banach contraction theorem” [1]. After that, many authors formulated and established many contractive conditions to modify the Banach contraction theorem in many different directions. Khan [2] introduced the altering distance mapping to formulate a new contractive condition in fixed point theory in order to extend the Banach fixed point theorem to new forms. For some extension to the Banach contraction theorem, we ask the readers to see References [3–20]. Recently, Abodyeh et al. [21] introduced a new notion, named almost perfect function, to formulate new contractive conditions to modify and extend some fixed point theorems known in the literature.

Now, we mention the notions of altering distance function and almost perfect function:

**Definition 1** ([2]). A self-function  $\psi$  on  $\mathbb{R}^+ \cup \{0\}$  is called an altering distance function if  $\psi$  satisfies the following conditions:

1.  $\psi(s) = 0 \iff s = 0$ .
2.  $\psi$  is a nondecreasing and continuous function.

**Definition 2** ([21]). A nondecreasing self-function  $\psi$  on  $\mathbb{R}^+ \cup \{0\}$  is called an almost perfect function if  $\psi$  satisfies the following conditions:

1.  $\psi(s) = 0 \iff s = 0$ .
2. If for all sequence  $(s_n)$  in  $\mathbb{R}^+ \cup \{0\}$  with  $\psi(s_n) \rightarrow 0$  it holds  $s_n \rightarrow 0$ .

One of the most important notions in fixed point theory to derive new contractive conditions is  $\alpha$ -admissibility, which were introduced by Samet et al. [22]. Then, E. Karapiner et al. [23] generated

the concept of triangular  $\alpha$ -admissibility. In meantime, Abdeljawad [24] expanded the notion of  $\alpha$ -admissibility to a pair of functions. For some fixed point theorems on  $\alpha$ -admissibility, we direct readers to read References [25–31].

The notions of  $\alpha$ -admissibility mapping and  $\alpha$ -admissibility for a pair of mappings are introduced as follows:

**Definition 3** ([22]). Let  $S$  be a self-mapping on  $X$  and  $\alpha: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be a function. Then,  $S$  is called  $\alpha$ -admissible if for all  $v, w \in X$  with  $\alpha(v, w) \geq 1$  it holds  $\alpha(Sv, Sw) \geq 1$ .

The definition of triangular  $\alpha$ -admissibility for a single mapping is:

**Definition 4** ([23]). Let  $S$  be a self-mapping on  $X$  and  $\alpha: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$ . Then, we call  $S$  triangular  $\alpha$ -admissible if

1.  $S$  is  $\alpha$ -admissible; and
2. For all  $v, w, u \in X$  with  $\alpha(v, w) \geq 1$  and  $\alpha(w, u) \geq 1$  it holds  $\alpha(v, u) \geq 1$ .

**Definition 5** ([24]). Let  $S$  and  $T$  be two self mappings on  $X$  and  $\alpha: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be a function. Then, the pair  $(S, T)$  is called  $\alpha$ -admissible if  $z, w \in X$  and  $\alpha(z, w) \geq 1$  imply  $\alpha(Sz, Tw) \geq 1$  and  $\alpha(Tz, Sw) \geq 1$ .

In our work we need the following definitions:

**Definition 6** ([30]). Let  $d$  be a metric on a set  $X$  and  $\alpha, \beta: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be functions. Then,  $X$  is called  $\alpha, \beta$ -complete if and only if  $\{x_n\}$  is a Cauchy sequence in  $X$  and  $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$  for all  $n \in \mathbb{N}$  imply  $(x_n)$  converges to some  $x \in X$ .

**Definition 7** ([30]). Let  $d$  be a metric on a set  $X$  and  $\alpha, \eta: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be functions. A self-mapping  $T$  on  $X$  is called  $\alpha, \beta$ -continuous if  $\{x_n\}$  is a sequence in  $X$ ,  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$  for all  $n \in \mathbb{N}$  imply  $Tx_n \rightarrow Tx$  as  $n \rightarrow \infty$ .

In this paper, we introduce a new contractive condition of type  $(\alpha, \beta, \psi)$ -admissibility for a pair of mappings  $(S, T)$  defined on a set  $X$ . We utilize our new contractive condition to formulate and prove a common fixed point theorem for two self-mappings defined on a metric space  $(X, d)$  under a set of conditions. Then, we utilize our main result to obtain some fixed point results.

This paper is divided into three sections. In the first section, we collect all necessary definitions and preliminaries that cover the subject of our paper. In Section 2, we give our new definitions and our main result. In addition, we give an example to validate our main result. In Section 3, we write our conclusion.

## 2. Main Results

We begin our work with the following new definition:

**Definition 8.** Let  $S, T$  be two self-mappings on the set  $X$  and  $\alpha, \beta: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be functions. We say that  $(S, T)$  is a pair of  $(\alpha, \beta)$ -admissibility if  $z, w \in X$  and  $\alpha(z, w) \geq \beta(z, w)$  imply  $\alpha(Sz, Tw) \geq \beta(Sz, Tw)$  and  $\alpha(Tz, Sw) \geq \beta(Tz, Sw)$ .

**Example 1.** Define self-mappings  $S$  and  $T$  on a set of real numbers by  $Sv = v^2$  and

$$Tv = \begin{cases} -v^2, & \text{if } v < 0; \\ v^2, & \text{if } v \geq 0. \end{cases}$$

Additionally, define  $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  via  $\alpha(v, w) = e^{v+w}$  and  $\beta(v, w) = e^v$ . Then,  $(S, T)$  is a pair of  $(\alpha, \beta)$ -admissibility.

**Proof.** Let  $v, w \in X$  such that  $\alpha(v, w) \geq \beta(v, w)$ . Then,  $e^{v+w} \geq e^v$ . So  $v + w \geq v$  and hence  $w$  is a nonnegative real number. Therefore

$$\alpha(Sv, Tw) = \alpha(v^2, w^2) = e^{v^2+w^2} \geq e^{v^2} = \beta(Sv, Tw).$$

Consequently,  $\alpha(Sv, Tw) \geq \beta(Sv, Tw)$ .

Now, if  $v \geq 0$ , then

$$\alpha(Tv, Sw) = \alpha(v^2, w^2) = e^{v^2+w^2} \geq e^{v^2} = \beta(Tv, Sw).$$

While, if  $v < 0$ , then

$$\alpha(Tv, Sw) = \alpha(-v^2, w^2) = e^{-v^2+w^2} \geq e^{-v^2} = \beta(Tv, Sw).$$

$$\alpha(Tv, Sw) \geq \beta(Tv, Sw). \quad \square$$

**Definition 9.** Let  $\psi$  be a nondecreasing function on  $\mathbb{R}^+ \cup \{0\}$ . We call  $\psi$  a perfect function if the following conditions hold:

1.  $\psi(t) = 0 \iff t = 0$ .
2. If  $(t_n)$  is a sequence in  $\mathbb{R}^+ \cup \{0\}$  and  $\psi(t_n) \rightarrow 0$  as  $n \rightarrow +\infty$  implies  $t_n \rightarrow 0$  as  $n \rightarrow +\infty$ .
3.  $\psi(u + v) \leq \psi(u) + \psi(v)$  for all  $u, v \in \mathbb{R}^+ \cup \{0\}$ .

**Example 2.** Define the self-function  $\psi$  on  $\mathbb{R}^+ \cup \{0\}$  by

$$\psi(u) = \begin{cases} \ln(1 + u), & \text{if } u \leq 1; \\ 1, & \text{if } u > 1. \end{cases}$$

Then,  $\psi$  is a perfect function.

Our main definition in this paper is:

**Definition 10.** Let  $d$  be a metric on a set  $X$ . Let  $S, T$  be two self-mappings on  $X$ ,  $\psi$  be a perfect self-mapping on  $\mathbb{R}^+ \cup \{0\}$ ,  $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be functions. We say that the pair  $(S, T)$  is an  $(\alpha, \beta, \psi)$ -contraction if there exists  $k \in [0, 1)$  such that  $z, w \in X$  and  $\alpha(z, w) \geq \beta(z, w)$  imply

$$\psi(d(Sz, Tw)) \leq \max \left\{ k\psi(d(z, w)), k\psi(d(z, Sz)), k\psi(d(w, Tw)), k\psi(d(w, Sz)), \frac{1}{2}k\psi(d(z, Tw)) \right\} \quad (1)$$

and

$$\psi(d(Tz, Sw)) \leq \max \left\{ k\psi(d(z, w)), k\psi(d(z, Tz)), k\psi(d(w, Sw)), k\psi(w, Tz), \frac{1}{2}\psi(d(z, Sw)) \right\}. \quad (2)$$

**Example 3.** Define  $d : [0, \frac{1}{4}] \times [0, \frac{1}{4}] \rightarrow \mathbb{R}^+ \cup \{0\}$  by  $d(v, w) = |v - w|$  and  $S, T : [0, \frac{1}{4}] \rightarrow [0, \frac{1}{4}]$  by  $Su = u^2$  and  $Tu = u^4$ . Also define the self-function  $\psi$  on  $\mathbb{R}^+ \cup \{0\}$  by  $\psi(s) = \frac{s}{1+s}$  and the functions  $\alpha, \beta : [0, \frac{1}{4}] \times [0, \frac{1}{4}] \rightarrow \mathbb{R}^+ \cup \{0\}$  by  $\alpha(u, w) = e^u$  and  $\beta(u, w) = e^{u+w}$ . Then,  $(S, T)$  is an  $(\alpha, \beta, \psi)$ -contraction.

**Proof.** Given  $v, w \in [0, \frac{1}{4}]$  is such that  $\alpha(v, w) \geq \beta(v, w)$ . Then,  $e^v \geq e^{v+w}$ . Therefore, we conclude that  $w = 0$ . Since  $v \leq \frac{1}{4}$ , we have

$$\psi(d(Sv, Tw)) = \psi(d(v^2, 0)) = \psi(v^2) = \frac{v^2}{1+v^2} \leq \frac{1}{4} \frac{v}{1+v} = \frac{1}{4} \psi(d(v, w))$$

and

$$\psi(d(Tv, Sw)) = \psi(d(v^4, 0)) = \psi(v^4) = \frac{v^4}{1+v^4} \leq \frac{1}{4} \frac{v}{1+v} = \frac{1}{4} \psi(d(v, w)).$$

So the pair  $(S, T)$  is an  $(\alpha, \beta, \psi)$ -contraction.  $\square$

The main result of this paper is:

**Theorem 1.** On the set  $X$ , let  $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be two functions and  $S, T : X \rightarrow X$  be two mappings. Assume there exists a metric  $d$  on  $X$  such that the following hypotheses hold:

1.  $(X, d)$  is an  $\alpha, \beta$ -complete metric space.
2.  $S$  and  $T$  are  $\alpha, \beta$ -continuous.
3.  $(S, T)$  is an  $(\alpha, \beta, \psi)$ -contraction.
4.  $(S, T)$  is a pair of  $(\alpha, \beta)$ -admissibility.
5. If  $v, w, z \in X$  satisfy the condition  $\alpha(v, w) \geq \beta(v, w)$  and  $\alpha(w, z) \geq \beta(w, z)$ , then  $\alpha(v, z) \geq \beta(v, z)$ .
6. There exists  $x_0 \in X$  such that  $\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0)$  and  $\alpha(TSx_0, Sx_0) \geq \beta(TSx_0, Sx_0)$ .

Then, both mappings  $S$  and  $T$  have a common fixed point.

**Proof.** In view of hypothesis (6), we start with  $x_0 \in X$  in such a way that  $\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0)$  and  $\alpha(TSx_0, Sx_0) \geq \beta(TSx_0, Sx_0)$ . Now, let  $x_1 = Sx_0$  and  $x_2 = Tx_1$ . Then,  $\alpha(x_0, x_1) \geq \beta(x_0, x_1)$  and  $\alpha(x_1, x_0) \geq \beta(x_1, x_0)$ . In view of hypothesis (4), we have

$$\alpha(x_1, x_2) = \alpha(Sx_0, Tx_1) \geq \beta(Sx_0, Tx_1) = \beta(x_1, x_2)$$

and

$$\alpha(x_2, x_1) = \alpha(Tx_1, Sx_0) \geq \beta(Tx_1, Sx_0) = \beta(x_1, x_2).$$

Again, we put  $x_3 = Sx_2$ . Then, hypothesis (4) implies that

$$\alpha(x_2, x_3) = \alpha(Tx_1, Sx_2) \geq \beta(Tx_1, Sx_2) = \beta(x_2, x_3)$$

and

$$\alpha(x_3, x_2) = \alpha(Sx_2, Tx_1) \geq \beta(Sx_2, Tx_1) = \beta(x_3, x_2).$$

Putting  $x_4 = Tx_3$  and referring to hypothesis (4), we conclude

$$\alpha(x_3, x_4) = \alpha(Sx_2, Tx_3) \geq \beta(Sx_2, Tx_3) = \beta(x_3, x_4)$$

and

$$\alpha(x_4, x_3) = \alpha(Tx_3, Sx_2) \geq \beta(Tx_3, Sx_2) = \beta(x_4, x_3).$$

Continuing in the same manner, we construct a sequence  $(x_n)$  in  $X$  with  $x_{2n+1} = Sx_{2n}$  and  $x_{2n+2} = Tx_{2n+1}$  such that

$$\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1}) \quad \forall \quad n \in \mathbb{N}$$

and

$$\alpha(x_{n+1}, x_n) \geq \beta(x_{n+1}, x_n) \quad \forall \quad n \in \mathbb{N}.$$

From hypothesis (5), we see that

$$\alpha(x_n, x_m) \geq \beta(x_n, x_m) \quad \forall n, m \in \mathbb{N}.$$

If there exists  $t \in \mathbb{N}$  such that  $x_{2t} = x_{2t+1}$ , then  $x_{2t} = Sx_{2t+1}$  and hence  $S$  has a fixed point. From contractive condition (1), we have

$$\begin{aligned} & \psi(d(x_{2t+1}, d(x_{2t+2})) \\ &= \psi(d(Sx_{2t}, Tx_{2t+1})) \\ &\leq \max \left\{ k\psi(d(x_{2t}, x_{2t+1})), k\psi(d(x_{2t}, Sx_{2t+1})), \right. \\ & \quad \left. k\psi(d(x_{2t+1}, Tx_{2t+1})), k\psi(d(x_{2t+1}, Sx_{2t})), \frac{1}{2}k\psi(d(x_{2t}, Tx_{2t+1})) \right\} \\ &\leq \max \left\{ k\psi(d(x_{2t+1}, x_{2t+2})), \frac{1}{2}k\psi(d(x_{2t}, x_{2t+2})) \right\} \\ &\leq \max \left\{ k\psi(d(x_{2t+1}, x_{2t+2})), \frac{1}{2}k\psi(d(x_{2t+1}, x_{2t+2})) \right\} \\ &\leq k\psi(d(x_{2t+1}, x_{2t+2})). \end{aligned}$$

The last inequality is correct only if  $\psi(d(x_{2t+1}, x_{2t+2})) = 0$ . The properties of  $\psi$  and  $d$  imply that  $x_{2t+1} = x_{2t+2}$ . Hence,  $x_{2t} = Sx_{2t} = Tx_{2t}$ . Thus,  $S$  and  $T$  have a common fixed point of  $S$  and  $T$ .

If there is a natural number  $t$  with  $x_{2t+1} = x_{2t+2}$ , then  $x_{2t+1} = Tx_{2t+1}$  and hence  $T$  has a fixed point. From contractive condition (2), we have

$$\begin{aligned} & \psi(d(x_{2t+2}, d(x_{2t+3})) \\ &= \psi(d(Tx_{2t+1}, Sx_{2t+2})) \\ &\leq \max \left\{ k\psi(d(x_{2t+1}, x_{2t+2})), k\psi(d(x_{2t+1}, Tx_{2t+1})), k\psi(d(x_{2t+2}, Sx_{2t+2})), \right. \\ & \quad \left. k\psi(d(x_{2t+2}, Tx_{2t+1})), \frac{1}{2}k\psi(d(x_{2t+1}, Sx_{2t+2})) \right\} \\ &\leq \max \left\{ k\psi(d(x_{2t+2}, x_{2t+3})), \frac{1}{2}k\psi(d(x_{2t+1}, x_{2t+3})) \right\} \\ &\leq \max \left\{ k\psi(d(x_{2t+2}, x_{2t+3})), \frac{1}{2}k\psi(d(x_{2t+2}, x_{2t+3})) \right\} \\ &\leq k\psi(d(x_{2t+2}, x_{2t+3})). \end{aligned}$$

The last inequality holds only if  $\psi(d(x_{2t+2}, x_{2t+3})) = 0$ . The properties of  $\psi$  and  $d$  imply that  $x_{2t+2} = x_{2t+3}$ . Hence  $x_{2t+1} = Sx_{2t+1} = Tx_{2t+1}$ . Thus, we conclude that  $x_{2t+1}$  is a common fixed point of  $S$  and  $T$ .

Now, assume that  $x_n \neq x_{n+1} \quad \forall n \in \mathbb{N}$ .  
For  $n \in \mathbb{N} \cup \{0\}$ , we get

$$\begin{aligned} & \psi(d(x_{2n+1}, x_{2n+2})) \\ &= \psi(d(Sx_{2n}, Tx_{2n+1})) \\ &\leq \max \left\{ k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n}, Sx_{2n})), k\psi(d(x_{2n+1}, Tx_{2n+1})), \right. \\ & \quad \left. k\psi(d(x_{2n+1}, Sx_{2n})), \frac{1}{2}k\psi(d(x_{2n}, Tx_{2n+1})) \right\} \\ &= \max \left\{ k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2})), \frac{1}{2}k\psi(d(x_{2n}, x_{2n+2})) \right\} \\ &= \max \left\{ k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2})) \right\}. \end{aligned}$$

Thus, if

$$\max\{k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2}))\} = k\psi(d(x_{2n+1}, x_{2n+2})),$$

then  $\psi(d(x_{2n+1}, x_{2n+2})) \leq k\psi(d(x_{2n+1}, x_{2n+2}))$ . Since  $k < 1$ , condition (1) on  $\psi$  implies that  $x_{2n+1} = x_{2n+2}$ , a contradiction. Therefore,

$$\max\{k\psi(d(x_{2n}, x_{2n+1})), k\psi(d(x_{2n+1}, x_{2n+2}))\} = k\psi(d(x_{2n}, x_{2n+1})).$$

Hence,

$$\psi(d(x_{2n+1}, x_{2n+2})) \leq k\psi(d(x_{2n}, x_{2n+1})). \quad (3)$$

Using arguments similar to the above, we may show that

$$\psi(d(x_{2n}, x_{2n+1})) \leq k\psi(d(x_{2n-1}, x_{2n})). \quad (4)$$

Combining Equations (3) and (4) together, we reach

$$\psi(d(x_n, x_{n+1})) \leq k\psi(d(x_{n-1}, x_n)). \quad (5)$$

By recurring Equation (5)  $n$ -times, we deduce

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq k\psi(d(x_{n-1}, x_n)) \\ &\leq k^2\psi(d(x_{n-2}, x_{n-1})) \\ &\vdots \\ &\leq k^n\psi(d(x_0, x_1)). \end{aligned} \quad (6)$$

On allowing  $n \rightarrow +\infty$  in Equation (6), we get

$$\lim_{n \rightarrow +\infty} \psi(d(x_n, x_{n+1})) = 0. \quad (7)$$

Condition (2) on the function  $\psi$  implies that

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0. \quad (8)$$

We intend to prove that  $(x_n)$  is a Cauchy sequence in  $X$ , take  $n, m \in \mathbb{N}$  with  $m > n$ . We divide the proof into four cases:

**Case 1:**  $n$  is an odd integer and  $m$  is an even integer.

Therefore, there exist  $s \in \mathbb{N}$  and an odd integer  $h$  such that  $n = 2s + 1$  and  $m = n + h = 2s + 1 + h$ . Since  $\alpha(x_n, x_m) \geq \beta(x_n, x_m)$ , we have

$$\begin{aligned}
 & \psi(d(x_n, x_m)) \\
 &= \psi(d(x_{2s+1}, x_{2s+1+h})) \\
 &= \psi(d(Sx_{2s}, Tx_{2s+h})) \\
 &\leq \max \left\{ k\psi(d(x_{2s}, x_{2s+h})), k\psi(d(x_{2s}, Sx_{2s})), k\psi(d(x_{2s+h}, Tx_{2s+h})), \right. \\
 &\quad \left. k\psi(d(x_{2s+h}, Sx_{2s})), \frac{1}{2}k\psi(d(x_{2s}, Tx_{2s+h})) \right\} \\
 &\leq \max \left\{ k\psi\left(\sum_{j=2s}^{2s+h-1} d(x_j, x_{j+1})\right), k\psi(d(x_{2s}, x_{2s+1})), k\psi(d(x_{2s+h}, x_{2s+h+1})), \right. \\
 &\quad \left. k\psi(d(x_{2s+h}, x_{2s+1})), \frac{1}{2}k\psi(d(x_{2s}, x_{2s+h+1})) \right\} \\
 &\leq \max \left\{ k \sum_{j=2s}^{2s+h-1} \psi(d(x_j, x_{j+1})), k\psi(d(x_{2s}, x_{2s+1})), k\psi(d(x_{2s+h}, x_{2s+h+1})), \right. \\
 &\quad \left. k\psi\left(\sum_{j=2s+1}^{2s+h-1} d(x_j, x_{j+1})\right), \frac{1}{2}k\psi\left(\sum_{j=2s}^{2s+h} d(x_j, x_{j+1})\right) \right\} \\
 &\leq \max \left\{ k \sum_{j=2s}^{2s+h-1} \psi(d(x_j, x_{j+1})), k\psi(d(x_{2s}, x_{2s+1})), k\psi(d(x_{2s+h}, x_{2s+h+1})), \right. \\
 &\quad \left. k \sum_{j=2s+1}^{2s+h-1} \psi(d(x_j, x_{j+1})), \frac{1}{2}k \sum_{j=2s}^{2s+h} \psi(d(x_j, x_{j+1})) \right\} \\
 &\leq \max \left\{ k \sum_{j=2s}^{+\infty} \psi(d(x_j, x_{j+1})), k\psi(d(x_{2s}, x_{2s+1})), k\psi(d(x_{2s+h}, x_{2s+h+1})) \right\} \\
 &\leq \max \left\{ \frac{k^{2s+1}}{1-k} \psi(d(x_0, x_1)), k\psi(d(x_{2s}, Sx_{2s})), k\psi(d(x_{2s+h}, Sx_{2s+h})) \right\}.
 \end{aligned}$$

By permitting  $n, m \rightarrow +\infty$  in above inequalities and considering Equation (7), we have

$$\lim_{n \rightarrow +\infty} \psi(d(x_n, x_m)) = 0.$$

The properties of  $\psi$  imply that

$$\lim_{n \rightarrow +\infty} d(x_n, x_m) = 0. \quad (9)$$

**Case 2:**  $n$  and  $m$  are both even integers.

Applying the triangular inequality of the metric  $d$ , we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m).$$

Letting  $n \rightarrow +\infty$  and in view of Equations (8) and (9), we get  $\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = 0$ .

**Case 3:**  $n$  is an even integer and  $m$  is an odd integer.

Applying the triangular inequality of the metric  $d$ , we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m-1}) + d(x_{m-1}, x_m).$$

On permitting  $n \rightarrow +\infty$  and considering Equations (8) and (9), we get  $\lim_{n,m \rightarrow +\infty} d(x_n, x_m) = 0$ .

**Case 4:**  $n$  and  $m$  are both odd integers. Applying the triangular inequality of the metric  $d$ , we have

$$d(x_n, x_m) \leq d(x_n, x_{m-1}) + d(x_{m-1}, x_m).$$

On permitting  $n \rightarrow +\infty$  and in view of Equations (8) and (9), we get  $\lim_{n,m \rightarrow +\infty} d(x_n, x_m) = 0$ . Combining all cases with each other, we conclude that

$$\lim_{n,m \rightarrow +\infty} d(x_n, x_m) = 0.$$

Thus, we conclude that  $(x_n)$  is a Cauchy sequence in  $X$ . The  $\alpha, \beta$ -completeness of the metric space  $(X, d)$  ensures that there is  $x \in X$  such that  $x_n \rightarrow x$ . Using the  $\alpha, \beta$ -continuity of the mappings  $S$  and  $T$ , we deduce that

$x_{2n+1} = Sx_{2n} \rightarrow Sx$  and  $x_{2n+2} = Tx_{2n+1} \rightarrow Tx$ . By uniqueness of limit, we obtain  $Sx = Tx = x$ . Thus,  $x$  is a fixed point of  $S$ .  $\square$

**Corollary 1.** Let  $d$  be a metric on the set  $X$ , let  $\alpha, \beta: X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  be functions and  $S, T$  be self-mappings on  $X$ . Assume following hypotheses:

1.  $(X, d)$  is an  $\alpha, \beta$ -complete metric space.
2.  $S$  and  $T$  are  $\alpha, \beta$ -continuous.
3.  $(S, T)$  is a pair of  $(\alpha, \beta)$ -admissibility.
4. There exist positive numbers  $a_1, a_2, a_3, a_4$  and  $a_5$  with  $a_1 + a_2 + a_3 + a_4 + 2a_5 < 1$  and a perfect function  $\psi$  such that if  $z, w \in X$  are so that  $\alpha(z, w) \geq \beta(z, w)$ , then

$$\psi(d(Sz, Tw)) \leq a_1\psi(d(z, w)) + a_2\psi(d(z, Sz)) + a_3\psi(d(w, Tw)) + a_4\psi(d(w, Sz)) + a_5\psi(d(z, Tw))$$

and

$$\psi(d(Tz, Sw)) \leq a_1\psi(d(z, w)) + a_2\psi(d(z, Tz)) + a_3\psi(d(w, Sw)) + a_4\psi(d(w, Tz)) + a_5\psi(d(z, Sw)).$$

5. If  $v, w, z$  are in  $X$ , with  $\alpha(v, w) \geq \beta(v, w)$  and  $\alpha(w, z) \geq \beta(w, z)$ , then  $\alpha(v, z) \geq \beta(v, z)$ .
6. There exists  $x_0 \in X$  such that

$$\alpha(Sx_0, TSx_0) \geq \beta(Sx_0, TSx_0) \text{ and } \alpha(TSx_0, Sx_0) \geq \beta(TSx_0, Sx_0).$$

Then  $S$  and  $T$  have a common fixed point.

**Example 4.** Define  $d: [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$  by

$$d(z, w) = \begin{cases} \max\{z, w\}, & \text{if } z \neq w; \\ 0, & \text{if } z = w. \end{cases}$$

Let  $S, T$  be two self-mappings on  $[0, +\infty)$  defined by  $Sz = \frac{1}{2}\sin^2(z)$  and  $Tz = \frac{1}{4}\sin^2(z)$ . In addition, define the function  $\psi: [0, +\infty) \rightarrow [0, +\infty)$  by  $\psi(s) = \frac{s}{1+s}$ .

Furthermore, we define the functions  $\alpha, \beta: X \times X \rightarrow [0, +\infty)$  by

$$\alpha(s, t) = \begin{cases} e^{s+t}, & \text{if } s, t \in [0, 1]; \\ 0, & \text{if } s > 1 \text{ or } t > 1, \end{cases}$$

and



$$\beta(s, t) = \begin{cases} e^s, & \text{if } s, t \in [0, 1]; \\ 1, & \text{if } s > 1 \text{ or } t > 1. \end{cases}$$

Then:

1.  $\psi$  is a perfect function.
2. There exists  $x_0 \in X$  such that

$$\alpha(Sx_0, S^2x_0) \geq \beta(Sx_0, S^2x_0) \text{ and } \alpha(S^2x_0, Sx_0) \geq \beta(S^2x_0, Sx_0).$$

3.  $(S, T)$  is a pair of  $(\alpha, \beta)$ -admissibility.
4.  $S$  and  $T$  are  $\alpha, \beta$ -continuous.
5.  $(X, d)$  is an  $\alpha, \beta$ -complete metric space.
6.  $(S, T)$  is an  $(\alpha, \beta, \psi)$ -contraction.

**Proof.** It is an easy matter to see Equations (1)–(3). To prove Equation (4), let  $(x_n)$  be any sequence in  $[0, +\infty)$  such that  $x_n \rightarrow x \in [0, +\infty)$  and  $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$  for all  $n \in \mathbb{N}$ . Thus,  $x_n \in [0, 1]$  for all  $n \in \mathbb{N}$ . If  $x_n = x$  for all but finitely many, we conclude that  $Sx_n \rightarrow Sx$  as  $n \rightarrow +\infty$ . If  $x_n \neq x$  for all but finitely many, we notice that  $x = 0$ . Hence,  $x_n \rightarrow 0$  in  $([0, 1], |\cdot|)$ . Therefore,  $\max\{\frac{1}{2}\sin^2 x_n, 0\} \rightarrow 0 = Sx$  in  $([0, +\infty), d)$ ; that is,  $S$  is  $\alpha, \beta$ -continuous.

To prove (5), let  $(x_n)$  be a Cauchy sequence in  $([0, +\infty), d)$  such that  $\alpha(x_n, x_{n+1}) \geq \beta(x_n, x_{n+1})$ . Then,  $x_n \in [0, 1]$  for all  $n \in \mathbb{N}$ . If there exists  $x \in [0, 1]$  such that  $x_n = x$  for all but finitely many, then  $x_n \rightarrow x$  as  $n \rightarrow +\infty$ . Now, suppose the elements of  $(x_n)$  are distinct for all but finitely many. Given  $\varepsilon > 0$ , since  $(x_n)$  is a Cauchy sequence in  $([0, +\infty), d)$ , then there exists  $n_0 \in \mathbb{N}$  such that  $\max\{x_n, x_m\} < \varepsilon$  for all  $m > n \geq n_0$ . Therefore,  $\max\{x_n, 0\} < \varepsilon$  for all  $n \geq n_0$ . So,  $x_n \rightarrow 0$  in  $([0, +\infty), d)$ . Thus,  $([0, +\infty), d)$  is an  $\alpha, \beta$ -complete metric space.

To prove (6), let  $z, w \in X$  be such that  $\alpha(z, w) \geq \beta(z, w)$ . Then,  $z, w \in [0, 1]$ . So

$$\begin{aligned} \psi(d(Sz, Tw)) &= \psi\left(d\left(\frac{1}{2}\sin^2 z, \frac{1}{4}\sin^2 w\right)\right) \\ &= \psi\left(\max\left\{\frac{1}{2}\sin^2 z, \frac{1}{4}\sin^2 w\right\}\right) \\ &= \frac{\max\left\{\frac{1}{2}\sin^2 z, \frac{1}{4}\sin^2 w\right\}}{1 + \max\left\{\frac{1}{2}\sin^2 z, \frac{1}{4}\sin^2 w\right\}} \\ &= \frac{\max\left\{\sin^2 z, \frac{1}{2}\sin^2 w\right\}}{2 + \max\left\{\sin^2 z, \frac{1}{2}\sin^2 w\right\}} \\ &\leq \frac{4}{5} \left( \frac{\max\left\{\sin^2 z, \sin^2 w\right\}}{1 + \max\left\{\sin^2 z, \sin^2 w\right\}} \right) \\ &\leq \frac{4}{5} \left( \frac{\max\{z, w\}}{1 + \max\{z, w\}} \right) \\ &= \frac{4}{5} \psi(d(z, w)) \\ &\leq \max\left\{\frac{4}{5}\psi(d(z, w)), \frac{4}{5}\psi(d(z, Sz)), \frac{4}{5}\psi(d(w, Tw)), \frac{4}{5}\psi(d(w, Sz)), \frac{4}{10}\psi(d(z, Tw))\right\}. \end{aligned}$$

Similarly, we can show that

$$\psi(d(Tz, Sw)) \leq \max \left\{ \frac{4}{5}\psi(d(z, w)), \frac{4}{5}\psi(d(z, Tz)), \frac{4}{5}\psi(d(w, Sw)), \frac{4}{5}d(w, Tz), \frac{4}{10}d(z, Sw) \right\}.$$

Hence,  $S$  and  $T$  satisfy Definition 2.3 for  $k = \frac{4}{5}$ . Therefore,  $S$  and  $T$  satisfy all the conditions of Theorem 1. Therefore,  $S$  and  $T$  have a common fixed point.  $\square$

### Remark 1

1. By taking  $S = T$  in Theorem 1 and Corollary 1, we can formulate and get some fixed point results.
2. By Defining the self-function  $\psi$  on  $[0, +\infty)$  via  $\psi(t) = t$ , and the two functions  $\alpha, \beta: X \times X \rightarrow [0, +\infty)$  via  $\alpha(s, t) = \beta(s, t) = 1$  in Theorem 1 and Corollary 1, we may formulate and get some common fixed point results.

### 3. Conclusions

New notions of  $(\alpha, \beta)$ -admissibility and  $(\alpha, \beta)$ -contraction for a pair of self-mappings on a set  $X$  are given. According to these notions, we introduced and proved our main result. Additionally, we gave an example to validate our main result.

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