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Article

New Numerical Method for Solving Tenth Order Boundary Value Problems

Ali Akgül [1,*], Esra Karatas Akgül [2], Dumitru Baleanu [3,4] and Mustafa Inc [5]

1 Department of Mathematics, Art and Science Faculty, Siirt University, TR-56100 Siirt, Turkey
2 Department of Mathematics, Faculty of Education, Siirt University, TR-56100 Siirt, Turkey; esrakaratas@siirt.edu.tr
3 Department of Mathematics and Computer Sciences, Art and Science Faculty, Çankaya University, TR-06300 Ankara, Turkey; dumitru@cankaya.edu.tr
4 Department of Mathematics, Institute of Space Sciences, 077125 Bucharest, Romania
5 Department of Mathematics, Science Faculty, Firat University, 23100 Elazig, Turkey; minc@firat.edu.tr
* Correspondence: aliakgul00727@gmail.com

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Abstract: In this paper, we implement reproducing kernel Hilbert space method to tenth order boundary value problems. These problems are important for mathematicians. Different techniques were applied to get approximate solutions of such problems. We obtain some useful reproducing kernel functions to get approximate solutions. We obtain very efficient results by this method. We show our numerical results by tables.

Keywords: reproducing kernel Hilbert space method; tenth-order boundary value problems; reproducing kernel functions

AMS: 35C10; 46E22; 30E25

1. Introduction

A numerical approximation of tenth-order boundary value problems has been given in [1]. Usmani [2] has solved fourth order boundary value problems by using the quartic spline method. Twizell and Boutayeb [3] have improved the numerical approximations for higher order eigenvalue value problems. The approximation of second order boundary value problems has been presented by Alberg and Ito [4]. Siraj-ul-Islam et al. [5] has used a non-polynomial spline method to approximate the sixth-order boundary value problems. Papamichael and Worsey [6] have applied the cubic spline algorithm to solve linear fourth-order boundary value problems. Siddiqi and Twizell [7,8] have enhanced numerical approximations of tenth and twelfth-order boundary value problems.

We consider the following problem in the reproducing kernel Hilbert space:

$$\begin{aligned} q^{(x)}(\tau) + a_1(\tau)q^{(ix)}(\tau) + a_2(\tau)q^{(viii)}(\tau) \\ +a_3(\tau)q^{(vii)}(\tau) + a_4(\tau)q^{(vi)}(\tau) + a_5(\tau)q^{(v)}(\tau) \\ +a_6(\tau)q^{(iv)}(\tau) + a_7(\tau)q'''(\tau) + a_8(\tau)q''(\tau) \\ +a_9(\tau)q'(\tau) + a_{10}(\tau)q(\tau) = H(\tau), \quad \tau \in [a,b], \end{aligned}$$

with boundary conditions:

$$q(a) = \alpha_0, \quad q(b) = \beta_0, \quad q''(a) = \alpha_1, \quad q''(b) = \beta_1,$$
$$q^{(iv)}(a) = \alpha_2, \quad q^{(iv)}(b) = \beta_2, \quad q^{(vi)}(a) = \alpha_3,$$
$$q^{(vi)}(b) = \beta_3, \quad q^{(viii)}(a) = \alpha_4, \quad q^{(viii)}(b) = \beta_4,$$

where $\alpha_j, \beta_j, j = 0,1,2,3,4$ are arbitrary fixed real constants, and $a_j(\tau), j = 1,2,...,10$ and $H(\tau)$ are continuous functions given on $[a,b]$.

The notion of reproducing kernel has been presented by Zaremba [9]. Aronszajn has given a systematic reproducing kernel theory containing the Bergman kernel function [10]. For more details, see [11–25].

This paper is constructed as: Section 2 shows the reproducing kernel functions. The exact and approximate solutions of the problem are acquired in Section 3. Numerical results are given in Section 4. Conclusions are presented in the last section.

2. Reproducing Kernel Spaces and Their Reproducing Kernel Functions

We give some important reproducing kernel functions in this section. We define $V_2^{11}[0,1]$ by:

$$V_2^{11}[0,1] = \left\{ \begin{array}{l} h, h', h'', h''', h^{(4)}, h^{(5)}, h^{(6)}, h^{(7)}, h^{(8)}, h^{(9)}, h^{(10)} \\ \text{are absolutely continuous functions}, \quad h^{(11)} \in L^2[0,1], \\ h(0) = h''(0) = h^{(4)}(0) = h^{(6)}(0) = h^{(8)}(0) = 0 \\ h(1) = h''(1) = h^{(4)}(1) = h^{(6)}(1) = h^{(8)}(1) = 0. \end{array} \right\}$$

We define the inner product as:

$$\langle h, P_y \rangle_{V_2^{11}[0,1]} = \sum_{i=0}^{10} h^i(0) P_y^i(0) + + \int_0^1 h^{(11)}(\tau) P_y^{(11)}(\tau) d(\tau).$$

We can obtain:

$$\begin{aligned} \langle h, P_y \rangle_{V_2^{11}[0,1]} &= h(0)P_y(0) + h'(0)P_y'(0) + h''(0)P_y''(0) \\ &+ h'''(0)P_y'''(0) + h^{(4)}(0)P_y^{(4)}(0) \\ &+ h^{(5)}(0)P_y^{(5)}(0) + h^{(6)}(0)P_y^{(6)}(0) \\ &+ h^{(7)}(0)P_y^{(7)}(0) + h^{(8)}(0)P_y^{(8)}(0) \\ &+ h^{(9)}(0)P_y^{(9)}(0) + h^{(10)}(0)P_y^{(10)}(0) \\ &+ \int_0^1 h^{(11)}(\tau) P_y^{(11)}(\tau) d(\tau). \end{aligned}$$

$P_y(\tau)$ is the reproducing kernel function of $V_2^{11}[0,1]$. Therefore, we have

$$P_y(0) = P_y''(0) = P_y^{(4)}(0) = P_y^{(6)}(0) = P_y^{(8)}(0) = 0,$$
$$P_y(1) = P_y''(1) = P_y^{(4)}(1) = P_y^{(6)}(1) = P_y^{(8)}(1) = 0.$$

Thus, we get

$$\begin{aligned}
\langle h, P_y \rangle_{V_2^{11}[0,1]} &= h'(0)P_y'(0) + h'''(0)P_y'''(0) \\
&+ h^{(5)}(0)P_y^{(5)}(0) + h^{(7)}(0)P_y^{(7)}(0) \\
&+ h^{(9)}(0)P_y^{(9)}(0) + h^{(10)}(0)P_y^{(10)}(0) \\
&+ \int_0^1 h^{(11)}(\tau)P_y^{(11)}(\tau)d(\tau).
\end{aligned}$$

We use integration by parts and we get

$$\begin{aligned}
\langle h, P_y \rangle_{V_2^{11}[0,1]} &= h'(0)P_y'(0) + h'''(0)P_y'''(0) \\
&+ h^{(5)}(0)P_y^{(5)}(0) + h^{(7)}(0)P_y^{(7)}(0) \\
&+ h^{(9)}(0)P_y^{(9)}(0) + h^{(10)}(0)P_y^{(10)}(0) \\
&+ h^{(10)}(1)P_y^{(11)}(1) - h^{(10)}(0)P_y^{(11)}(0) \\
&- h^{(9)}(1)P_y^{(12)}(1) + h^{(9)}(0)P_y^{(12)}(0) \\
&+ h^{(8)}(1)P_y^{(13)}(1) - h^{(8)}(0)P_y^{(13)}(0) \\
&- h^{(7)}(1)P_y^{(14)}(1) + h^{(4)}(1)P_y^{(17)}(1) \\
&+ h''(1)P_y^{(19)}(1) + h^{(7)}(0)P_y^{(14)}(0) \\
&+ h^{(6)}(1)P_y^{(15)}(1) + h(1)P_y^{(21)}(1) \\
&- h^{(6)}(0)P_y^{(15)}(0) - h^{(5)}(1)P_y^{(16)}(1) \\
&+ h^{(5)}(0)P_y^{(16)}(0) - h^{(4)}(0)P_y^{(17)}(0) \\
&- h'''(1)P_y^{(18)}(1) + h'''(0)P_y^{(18)}(0) \\
&- h''(0)P_y^{(19)}(0) - h'(1)P_y^{(20)}(1) \\
&+ h'(0)P_y^{(20)}(0) - h(0)P_y^{(21)}(0) \\
&- \int_0^1 h(\tau)P_y^{(22)}(\tau)d(\tau).
\end{aligned}$$

Therefore, we achieve

$$\begin{aligned}
\langle h, P_y \rangle_{V_2^{11}[0,1]} &= h'(0)P_y'(0) + h'''(0)P_y'''(0) \\
&+ h^{(5)}(0)P_y^{(5)}(0) + h^{(7)}(0)P_y^{(7)}(0) \\
&+ h^{(9)}(0)P_y^{(9)}(0) + h^{(10)}(0)P_y^{(10)}(0) \\
&+ h^{(10)}(1)P_y^{(11)}(1) - h^{(10)}(0)P_y^{(11)}(0) \\
&- h^{(9)}(1)P_y^{(12)}(1) + h^{(9)}(0)P_y^{(12)}(0) \\
&- h^{(7)}(1)P_y^{(14)}(1) + h^{(7)}(0)P_y^{(14)}(0) \\
&- h^{(5)}(1)P_y^{(16)}(1) + h^{(5)}(0)P_y^{(16)}(0) \\
&- h'''(1)P_y^{(18)}(1) + h'''(0)P_y^{(18)}(0) \\
&- h'(1)P_y^{(20)}(1) + h'(0)P_y^{(20)}(0) \\
&- \int_0^1 h(\tau)P_y^{(22)}(\tau)d(\tau).
\end{aligned}$$

We choose

$$\begin{gathered}P'_y(0)+P_y^{(20)}(0)=0,\quad P'''_y(0)+P_y^{(18)}(0)=0,\\ P_y^{(5)}(0)+P_y^{(16)}(0)=0,\quad P_y^{(7)}(0)+P_y^{(14)}(0)=0,\\ P_y^{(9)}(0)+P_y^{(12)}(0)=0,\quad P_y^{(10)}(0)-P_y^{(11)}(0)=0,\\ P_y^{(11)}(1)=0,\quad P_y^{(12)}(1)=0,\quad P_y^{(14)}(1)=0,\\ P_y^{(16)}(1)=0,\quad P_y^{(18)}(1)=0,\quad P_y^{(20)}(1)=0.\end{gathered}$$

Thus, we acquire

$$\langle h,P_y\rangle_{V_2^{11}[0,1]}=-\int_0^1 h(\tau)P_y^{(22)}(\tau)d(\tau)=h(y).$$

The reproducing kernel function P_y is obtained as

$$P_y(\tau)=\begin{cases}\sum\limits_{i=1}^{22}c_i\tau^{i-1} & , \quad \tau\le y,\\ \sum\limits_{i=1}^{22}d_i\tau^{i-1} & , \quad \tau> y,\end{cases}$$

by

$$P_y^{(22)}(\tau)=-\delta(\tau-y).$$

Definition 1. *$V_2^1[0,1]$ is given as:*

$$V_2^1[0,1]=\{m\in AC[0,1]:\ m'\in L^2[0,1]\},$$

where AC shows the space of absolutely continuous functions.

$$\langle m,n\rangle_{V_2^1}=\int_0^1\big(m(\tau)n(\tau)+m'(\tau)n'(\tau)\big)\,\mathrm{d}\tau,\quad m,n\in V_2^1[0,1] \tag{1}$$

and

$$\|m\|_{V_2^1}=\sqrt{\langle m,m\rangle_{V_2^1}},\quad m\in V_2^1[0,1], \tag{2}$$

are the inner product and the norm in $V_2^1[0,1]$. $T_\tau(\varsigma)$ of $V_2^1[0,1]$ is obtained as [26]:

$$T_\tau(\varsigma)=\frac{1}{2\sinh(1)}\left[\cosh(\tau+\varsigma-1)+\cosh(|\tau-\varsigma|-1)\right]. \tag{3}$$

3. Solutions in $V_2^{11}[0,1]$

The solution of the problem is considered in the $V_2^{11}[0,1]$. We define

$$Y:\ V_2^{11}[0,1]\to V_2^1[0,1]$$

as

$$
\begin{aligned}
Yh(\tau) = h^{(x)}(\tau) + a_1(\tau)h^{(ix)}(\tau) + a_2(\tau)h^{(viii)}(\tau) + a_3(\tau)h^{(vii)}(\tau) \\
+a_4(\tau)h^{(vi)}(\tau) + a_5(\tau)h^{(v)}(\tau) + a_6(\tau)h^{(iv)}(\tau) + a_7(\tau)h'''(\tau) \\
+a_8(\tau)h''(\tau) + a_9(\tau)h'(\tau) + a_{10}(\tau)h(\tau).
\end{aligned}
$$

The model problem changes to the following problem:

$$Yh = M(\tau), \quad \tau \in [0,1], \tag{4}$$

with the following boundary conditions:

$$
\begin{gathered}
h(a) = 0, \quad h(b) = 0, \quad h''(a) = 0, \qquad h''(b) = 0, \\
h^{(iv)}(a) = 0, \quad h^{(iv)}(b) = 0, \quad h^{(vi)}(a) = 0, \\
h^{(vi)}(b) = 0, \quad h^{(viii)}(a) = 0, \quad h^{(viii)}(b) = 0.
\end{gathered}
$$

Theorem 1. *Y is a bounded linear operator.*

Proof. We need to show $\|Yh\|^2_{V_2^1} \leq M\,\|h\|^2_{V_2^{11}}$, where $M > 0$. We acquire

$$\|Yh\|^2_{V_2^1} = \langle Yh, Yh\rangle_{V_2^1} = \int_0^1 \left(Yh(\tau)^2 + Yh'(\tau)^2\right) \mathrm{d}\tau$$

by Equations (1) and (2). We obtain

$$h(\tau) = \langle h(\cdot), P_\tau(\cdot)\rangle_{V_2^{11}}$$

and

$$Yh(\tau) = \langle h(\cdot), YP_\tau(\cdot)\rangle_{V_2^{11}},$$

by reproducing property. Thus, we reach

$$|Yh(\tau)| \leq \|h\|_{V_2^{11}} \|YP_\tau\|_{V_2^{11}} = M_1\,\|h\|_{V_2^{11}},$$

where $M_1 > 0$. Therefore, we find

$$\int_0^1 [(Yh)(\tau)]^2 \,\mathrm{d}\tau \leq M_1^2\,\|h\|^2_{V_2^{11}}.$$

Since

$$(Yh)'(\tau) = \langle h(\cdot), (YP_\eta)'(\cdot)\rangle_{V_2^{11}},$$

then

$$\left|(Yh)'(\eta)\right| \leq \|h\|_{V_2^{11}} \left\|(YP_\tau)'\right\|_{V_2^{11}} = M_2\,\|f\|_{V_2^{11}},$$

where $M_2 > 0$. Therefore, we reach

$$\left[(Yh)'(\tau)\right]^2 \leq M_2^2\,\|h\|^2_{V_2^{11}}$$

and

$$\int_0^1 \left[(Yh)'(\tau)\right]^2 \mathrm{d}\tau \leq M_2^2\,\|h\|^2_{V_2^{11}}.$$

Finally, we obtain

$$\begin{aligned} \|Yh\|^2_{V_2^1} &\leq \int_0^1 \left([(Yh)(\tau)]^2 + [(Yh)'(\tau)]^2 \right) \mathrm{d}\tau \\ &\leq \left(M_1^2 + M_2^2 \right) \|h\|^2_{V_2^{11}} = M \|h\|^2_{V_2^{11}}, \end{aligned}$$

where $M = M_1^2 + M_2^2 > 0$. □

The Main Results

Let $\varphi_i(\tau) = T_{\tau_i}(\tau)$ and $\psi_i(\tau) = Y^* \varphi_i(\tau)$, Y^* is an adjoint operator of Y. The orthonormal system $\{\overline{\Psi}_i(\tau)\}_{i=1}^{\infty}$ of $V_2^{11}[0,1]$ can be obtained as:

$$\overline{\psi}_i(\tau) = \sum_{k=1}^{i} \beta_{ik} \psi_k(\tau), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots). \tag{5}$$

Theorem 2. *Let $\{\tau_i\}_{i=1}^{\infty}$ be dense in $[0,1]$ and $\psi_i(\tau) = Y_\varsigma T_\tau(\varsigma)|_{\varsigma=\tau_i}$. Then, the sequence $\{\psi_i(\tau)\}_{i=1}^{\infty}$ is a complete system in $V_2^{11}[0,1]$.*

Proof. We get

$$\begin{aligned} \psi_i(\tau) &= (Y^* \varphi_i)(\tau) = \langle (Y^* \varphi_i)(\varsigma), T_\tau(\varsigma) \rangle \\ &= \langle \varphi_i)(\varsigma), Y\varsigma T_\tau(\varsigma) \rangle = Y_\varsigma T_\tau(\varsigma)|_{\varsigma=\tau_i}. \end{aligned}$$

We know that $\psi_i(\tau) \in V_2^{11}[0,1]$. For each fixed $h(\tau) \in V_2^{11}[0,1]$, let

$$\langle h(\tau), \psi_i(\eta) \rangle = 0, \quad (i = 1, 2, \ldots),$$

$$\langle h(\tau), (Y^* \varphi_i)(\tau) \rangle = \langle Yh(\cdot), \varphi_i(\cdot) \rangle = (Yh)(\tau_i) = 0.$$

$\{\tau_i\}_{i=1}^{\infty}$ is dense in $[0,1]$. Thus, $(Yh)(\tau) = 0$. $h \equiv 0$ by the Y^{-1}. □

Theorem 3. *If $h(\tau)$ is the exact solution of Equation (4), then*

$$h = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(\tau_k) \overline{\psi}_i(\tau), \tag{6}$$

where $\{(\tau_i)\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof. We obtain

$$\begin{aligned} h(\tau) &= \sum_{i=1}^{\infty} \langle h(\tau), \overline{\psi}_i(\tau) \rangle_{V_2^{11}} \overline{\psi}_i(\tau) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle h(\tau), \Psi_k(\tau) \rangle_{V_2^{11}} \overline{\psi}_i(\tau) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle h(\tau), Y^* \varphi_k(\tau) \rangle_{V_2^{11}} \overline{\psi}_i(\tau) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Yh(\tau), \varphi_k(\eta) \rangle_{V_2^1} \overline{\psi}_i(\tau) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} Yh(\tau_k) \overline{\psi}_i(\tau) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(\tau_k) \overline{\psi}_i(\eta), \end{aligned}$$

by Equation (5) and the uniqueness of solution of Equation (4). □

The approximate solution h_n can be obtained by:

$$h_n = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} M(\tau_k)\overline{\psi}_i(\tau). \tag{7}$$

4. Numerical Results

Example 1. *We consider the following tenth-order equation as:*

$$q^{(10)}(\tau) = -(80 + 19\tau + \tau^2)\exp(\tau), \quad 0 \le \tau \le 1,$$

with boundary conditions:

$$\begin{aligned} q(0) &= 0, & q(1) &= 0, \\ q''(0) &= 0, & q''(1) &= -4\exp(1), \\ q^{(iv)}(0) &= -8, & q^{(iv)}(1) &= -16\exp(1), \\ q^{(vi)}(0) &= -24, & q^{(vi)}(1) &= -36\exp(1), \\ q^{(viii)}(0) &= -48, & q^{(viii)}(1) &= -64\exp(1). \end{aligned}$$

The exact solution to the above boundary value problem is given by [1]:

$$q(\tau) = \tau(1-\tau)\exp(\tau).$$

We need to homogenize the boundary conditions to apply the reproducing kernel Hilbert space method:

$$\begin{gathered} r_1(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 \\ + c_6 x^5 + c_7 x^6 + c_8 x^7 + c_9 x^8 + c_{10} x^9 \\ s_1 = c_1 = 0, \\ s_2 = 2c_3 = 0, \\ s_3 = 24c_5 = -8, \\ s_4 = 720c_7 = -24, \\ s_5 = 40,320c_9 = -48, \\ s_6 = c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} = 0, \\ s_7 = 72c_{10} + 56c_9 + 42c_8 + 30c_7 + 20c_6 + 12c_5 + 6c_4 + 2c_3 = -4e, \\ s_8 = 3024c_{10} + 1680c_9 + 840c_8 + 360c_7 + 120c_6 + 24c_5 = -16e, \\ s_9 = 60,480c_{10} + 20,160c_9 + 5040c_8 + 720c_7 = -36e, \\ s_{10} = 362,880c_{10} + 40,320c_9 = -64e, \end{gathered}$$

$$\left\{ \begin{array}{l} c_1 = 0, c_2 = -\frac{24}{175} + \frac{1123}{2700}e, c_3 = 0, \\ \\ c_4 = \frac{352}{945} - \frac{719}{2268}, c_5 = -\frac{1}{3}, c_6 = \frac{28}{225} - \frac{253}{2700}e, \\ \\ c_7 = -\frac{1}{30}, c_8 = \frac{1}{126} - \frac{19}{3780}e, \\ c_9 = -\frac{1}{840}, c_{10} = \frac{1}{7560} - \frac{1}{5670}e. \end{array} \right\}$$

Therefore, we obtain

$$
\begin{aligned}
r_1(x) = -\frac{1}{113,400}x \quad &(20x^8e - 15x^8, \\
+570x^6e + 135x^7 - 900x^6 + 10,626x^4e, \\
+3780x^5 - 14,112x^4 + 35,950x^2e, \\
+37,800x^3 - 42,240x^2 - 47,166e + 15,552).
\end{aligned}
$$

We obtain Table 1 by our accurate technique.

Table 1. Exact solution (ES) and absolute errors (AE) of Example 1.

τ	ES	AE (NPCSM) [1]	AE (PCSM) [1]	AE (RKHSM)
0.2	0.195424441	2.433×10^{-7}	3.982×10^{-4}	3.330×10^{-8}
0.4	0.358037927	3.986×10^{-7}	6.663×10^{-4}	7.031×10^{-8}
0.6	0.358037927	4.428×10^{-7}	7.598×10^{-4}	6.076×10^{-8}
0.8	0.356086549	3.328×10^{-7}	5.885×10^{-4}	2.682×10^{-8}

5. Conclusions

In this paper, we used an accurate technique for investigating tenth order boundary value problems. An example was chosen to prove the computational accuracy. As shown in Table 1, our method is very accurate. We acquired some significant reproducing kernel functions to get approximate solutions and absolute errors.

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