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Coincidence Points of a Sequence of Multivalued Mappings in Metric Space with a Graph

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Abstract: In this article the coincidence points of a self map and a sequence of multivalued maps are found in the settings of complete metric space endowed with a graph. A novel result of Asrifa and Vetrivel is generalized and as an application we obtain an existence theorem for a special type of fractional integral equation. Moreover, we establish a result on the convergence of successive approximation of a system of Bernstein operators on a Banach space.

Keywords: graphic contraction; coincidence points; sequence of multivalued maps; Bernstein operators

1. Introduction and Preliminaries

For the metric space (X,d), using the notions of Nadler [1] and Hu [2], denote $CB(X), C(X)$ and 2^X by the collection of nonempty closed and bounded, compact and all nonempty subsets of X respectively. Consider $A,B \in CB(X)$ the distance between sets A and B is defined by $d(A,B) = \inf_{x\in A,\, y\in B} d(x,y)$, which does not allow to enjoy the properties of metric on $CB(X)$ therefore a well known idea of Hausdorff–Pompeiu distance H on $CB(X)$ induced by d is used to define a metric on $CB(X)$ as follows:

$$H(A,B) = \inf\{\epsilon > 0 : A \subseteq N(\epsilon,B), B \subseteq N(\epsilon,A)\},$$

where:

$$N(\epsilon,A) = \{x \in X : d(x,a) < \epsilon, \text{ for some } a \in A\}.$$

In 1969, Nadler [1] proved fixed point results for multivalued mappings in complete metric spaces, using the Hausdorff distance H, which was the generalization of Banach contraction principle in the settings of set-valued mappings. Covitz and Nadler [3] extended the idea of multivalued mappings in the generalized metric spaces. Reich [4] in 1972 published a fixed point result for the multivalued maps on the compact subsets of a complete metric space and posed the question, "can $C(X)$ be replaced by $CB(X)$?". In 1989, Mizoguchi and Takahashi answered this question in Theorem 5 of [5] and they also provide some Caristi type theorems for multivalued operators. Whereas Hu [2] in 1980 extended the multivalued fixed point results from complete metric space to complete ε-chainable metric space. Azam and Arshad [6] have extended the Theorem 6 of [1] by finding the fixed points of a sequence of locally contractive multivalued maps in ε-chainable metric space. Further Feng and Liu [7] used the concept of lower semi-continuity and a generalized contractive condition to extend the result of Nadler [1] and Caristi type theorems as defined in [5]. For more references the readers are referred to the work of Ciric [8], Klim and Wardowski [9,10] , Nicolae [11].

Jachymski [12] in 2007 unified and extended the work of Nieto [13] and Ran and Reuring [14] by defining a new class of contractions (G-contraction) on metric space (X,d) endowed with a graph. The connectivity of the graph brings more attractions regarding a necessary and sufficient condition for any G-contractive operator to be a Picard operator.

In the present article, fascinated by [6] the existence of coincidence points of a sequence of multivalued maps with a self map are taken into account with a generalized form of G-contraction. This provides a new way to generalize many existing results in the literature (see [1,6] and the references therein).

Let us recall some definitions from graph theory with the perspective of using them in our ideas and results. For a metric space (X,d) let $\triangle$ be the diagonal of the Cartesian product $X\times X$. Consider a directed graph G such that $X=V(G)$, where $V(G)$ is the set of vertices of G. The set $E(G)$ of edges of G contains all the loops. If G has no parallel edge then we can identify G with the pair $(V(G),E(G))$. Further, the graph G can be dealt with as a weighted graph if each edge is assigned by the distance between its edges.

Consider a directed graph G, then G^{-1} denote the graph obtained from G by reversing the direction of edges and if we ignore the direction of edges in graph G we get an undirected graph $\tilde{G}$. The pair (V',E') is said to be a subgraph of G if $V'\subseteq V(G)$ and $E'\subseteq E(G)$ and for any edge $(a,b)\in E'$ for all $a,b\in V'$.

Recall some fundamental definitions regarding the connectivity of graphs, which can be found in [15].

Definition 1. *A path in G from the vertex p to q of length K, is a sequence* $\{p_i\}$ *of* $K+1$ *vertices such that* $p_0=p,...,p_K=q$ *and* $(p_{j-1},p_j)\in E(G)$ *for* $j=1,2,...,K$.

Definition 2. *A graph G is called connected if there is a path between any two vertices. Graph G is weakly connected if* $\tilde{G}$ *is connected.*

Definition 3. *For* a,b *and* c *in* $V(G)$, $[a]_G$ *denote the equivalence class of the relation* $\sim$ *defined on* $V(G)$ *by the rule:*

$$b\sim c \text{ if there is a path in } G \text{ from } b \text{ to } c.$$

For $v\in V(G)$ *and* $K\in\mathbb{N}\cup\{0\}$ *by* $[v]_G^K$ *we denote the set*

$$[v]_G^K:=\{u\in V(G): \text{there is a path of length } K \text{ from } v \text{ to } u\}.$$

Following is the definition of G-contraction by Jachymski [12].

Definition 4. *[12] Let* (X,d) *be a metric space endowed with a graph G. We say that a mapping* $T:X\to X$ *is a G-contraction if T preserves edges of G i.e.,*

$$\underset{x,y\in X}{\forall}(x,y)\in E(G)\Rightarrow(Tx,Ty)\in E(G),$$

and there exists some $\alpha\in[0,1)$ *such that:*

$$\underset{x,y\in X}{\forall}(x,y)\in E(G)\Rightarrow d(Tx,Ty)\leq\alpha d(x,y).$$

Mizoguchi and Takahashi [5] had defined a $MT-$function as follows:

Definition 5. *[16] A function φ: $[0,+\infty) \to [0,1)$ is said to be a MT−function if it satisfies Mizoguchi and Takahashi's condition (i.e., $\limsup_{r\to t^+} \varphi(r) < 1$ for all $t \in [0,+\infty)$). Clearly, if φ: $[0,+\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then it is a MT−function.*

Now we state some results from the basic theory of multivalued mappings.

Lemma 1. *[17] Let (X,d) be a metric space and $A,B \in CB(X)$, with $H(A,B) < \epsilon$, then for each $a \in A$, there exists an element $b \in B$ such that:*

$$d(a,b) < \epsilon.$$

Lemma 2. *[18] Let (X,d) be a metric space and $A,B \in CB(X)$, then for each $a \in A$:*

$$d(a,B) \leq H(A,B).$$

Lemma 3. *[19] Let $\{A_n\}$ be a sequence in $CB(X)$ and there exists $A \in CB(X)$ such that $\lim_{n\to\infty} H(A_n, A) \to 0$. If $x_n \in A_n$ $(n = 1,2,3,...)$ and there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) \to 0$ then $x \in A$.*

2. Main Results

Definition 6. *[20] A multivalued mapping $F : X \to CB(X)$ is said to be Mizoguchi-Takahashi G-contraction if for all x, y in X, $x \neq y$ with $(x,y) \in E(G)$:*

(i) $H(F(x), F(y)) \leq \varphi(d(x,y))\, d(x,y)$;
(ii) *If $u \in F(x)$ and $v \in F(y)$ are such that $d(u,v) \leq d(x,y)$, then $(u,v) \in E(G)$.*

Motivated by the Definition 2.1 of [20], in a more general settings, we define the sequence of multivalued G_f-contraction as follows:

Definition 7. *Let $f : X \to X$ be a edge preserving surjection . A sequence of multivalued mappings $\{T_q\}_{q=1}^{\infty}$ from X into $CB(X)$ is said to be sequence of multivalued G_f-contraction if $(fu, fv) \in E(G)$, implies:*

$$H(T_q(u), T_r(v)) \leq \mu(d(fu, fv))d(fu, fv), \text{ for all } q, r \in \mathbb{N}. \tag{1}$$

For $x \in T_q(u)$ and $y \in T_r(v)$ satisfying $d(fx, fy) \leq d(fu, fv)$ implies $(fx, fy) \in E(G)$, where μ: $[0,\infty) \to [0,1)$ is a MT-function.

The next theorem provides the way to find the coincidence of a self map and a sequence of multivalued maps.

Theorem 1. *Let (X,d) a complete metric space,$\{T_q\}_{q=1}^{\infty}$ a sequence of multivalued G_f-contraction from X into $CB(X)$ and $f : X \to X$ a surjection. If there exist $m \in \mathbb{N}$ and $v_0 \in X$, such that:*

(i) $T_1(v_0) \cap [fv_0]_G^m \neq \phi$;
(ii) *For any sequence $\{v_n\}$ in X, if $v_n \to v$ and $v_n \in T_n(v_{n-1}) \cap [v_{n-1}]_G^m$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{v_{n_k}\}$ such that $(v_{n_k}, v) \in E(G)$ for all $k \in \mathbb{N}$.*

Then f and sequence of mappings $\{T_q\}_{q=1}^{\infty}$ have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* \in \bigcap_{q\in\mathbb{N}} T_q(v^*)$.

Proof. Choose any $v_1 \in X$ such that $fv_1 \in T_1(v_0) \cap [fv_0]_G^m$ then there exists a path from fv_0 to fv_1, i.e., $fv_0 = fu_0^{(1)}, ...\, fu_m^{(1)} = fv_1 \in T_1(v_0)$, and $\left(fu_i^{(1)}, fu_{i+1}^{(1)}\right) \in E(G)$ for all $i = 0,1,2,...,m-1$. □

Without any loss of generality, assume that $fu_k^{(1)} \neq fu_j^{(1)}$ for each $k, j \in \{0, 1, 2, ..., m\}$ with $k \neq j$. Since $(fu_0^{(1)}, fu_1^{(1)}) \in E(G)$, so:

$$\begin{aligned} H(T_1(u_0^{(1)}), T_2(u_1^{(1)})) &\leq \mu(d(fu_0^{(1)}, fu_1^{(1)}))d(fu_0^{(1)}, fu_1^{(1)}) \\ &< \sqrt{\mu(d(fu_0^{(1)}, fu_1^{(1)}))}d(fu_0^{(1)}, fu_1^{(1)}) \\ &< d(fu_0^{(1)}, fu_1^{(1)}) \end{aligned}$$

Rename fv_1 as $fu_0^{(2)}$. As $fu_0^{(2)} \in T_1(u_0^{(1)})$, and using Lemma 1 one can find some $fu_1^{(2)} \in T_2(u_1^{(1)})$ such that:

$$d(fu_0^{(2)}, fu_1^{(2)}) < d(fu_0^{(1)}, fu_1^{(1)}).$$

Since $(fu_1^{(1)}, fu_2^{(1)}) \in E(G)$, so:

$$\begin{aligned} H(T_2(u_1^{(1)}), T_2(u_2^{(1)})) &\leq \mu(d(fu_1^{(1)}, fu_2^{(1)}))d(fu_1^{(1)}, fu_2^{(1)}) \\ &< d(fu_1^{(1)}, fu_2^{(1)}). \end{aligned}$$

Similarly since $fu_1^{(2)} \in T_2(u_1^{(1)})$, again using Lemma 1 one can find some $fu_2^{(2)} \in T_2(u_2^{(1)})$ such that:

$$d(fu_1^{(2)}, fu_2^{(2)}) < d(fu_1^{(1)}, fu_2^{(1)}).$$

Thus we obtain $\{fu_0^{(2)}, fu_1^{(2)}, fu_2^{(2)}, \cdots, fu_m^{(2)}\}$ of $m+1$ vertices of X such that $fu_0^{(2)} \in T_1(u_0^{(1)})$ and $fu_s^{(2)} \in T_2(u_s^{(1)})$ for $s = 1, 2, \ldots, m$, with:

$$d(fu_s^{(2)}, fu_{s+1}^{(2)}) < d(fu_s^{(1)}, fu_{s+1}^{(1)}),$$

for $s = 0, 1, 2, \ldots, m-1$. As $(fu_s^{(1)}, fu_{s+1}^{(1)}) \in E(G)$ for all $s = 0, 1, 2, \ldots, m-1$, thus $(fu_s^{(2)}, fu_{s+1}^{(2)}) \in E(G)$ for all $s = 0, 1, 2, \ldots, m-1$.

Let $fu_m^{(2)} = fv_2$. Thus the set of points $fv_1 = fu_0^{(2)}, fu_1^{(2)}, fu_2^{(2)}, \cdots, fu_m^{(2)} = fv_2 \in T_2(v_1)$ is a path from fv_1 to fv_2. Rename fv_2 as $fu_0^{(3)}$. Then by the same procedure we obtain a path:

$$fv_2 = fu_0^{(3)}, fu_1^{(3)}, fu_2^{(3)}, \cdots, fu_m^{(3)} = fv_3 \in T_3(v_2)$$

from fv_2 to fv_3. Inductively, obtained:

$$fv_h = fu_0^{(h+1)}, fu_1^{(h+1)}, fu_2^{(h+1)}, \cdots, fu_m^{(h+1)} = fv_{h+1} \in T_{h+1}(v_h)$$

with:

$$d(fu_t^{(h+1)}, fu_{t+1}^{(h+1)}) < d(fu_t^{(h)}, fu_{t+1}^{(h)}), \tag{2}$$

hence $(fu_t^{(h+1)}, fu_{t+1}^{(h+1)}) \in E(G)$ for $t = 0, 1, 2, \ldots, m-1$.

Consequently, construct a sequence $\{fv_h\}_{h=1}^{\infty}$ of points of X with:

$$\begin{aligned} fv_1 &= fu_m^{(1)} = fu_0^{(2)} \in T_1(v_0), \\ fv_2 &= fu_m^{(2)} = fu_0^{(3)} \in T_2(v_1), \\ fv_3 &= fu_m^{(3)} = fu_0^{(4)} \in T_3(v_2), \\ &\vdots \\ fv_{h+1} &= fu_m^{(h+1)} = fu_0^{(h+2)} \in T_{h+1}(v_h), \end{aligned}$$

for all $h \in \mathbb{N}$.

For each $t \in \{0, 1, 2, ..., m-1\}$, and from (2), clearly $\{d(fu_t^{(h)}, fu_{t+1}^{(h)})\}_{h=1}^{\infty}$ is a decreasing sequence of non-negative real numbers and so there exists $a_t \geq 0$ such that:

$$\lim_{h\to\infty} d(fu_t^{(h)}, fu_{t+1}^{(h)}) = a_t.$$

By assumption, $\limsup_{t\to a_t^+} \mu(t) < 1$, so there exists $k_t \in \mathbb{N}$ such that $\mu(d(fu_t^{(h)}, fu_{t+1}^{(h)})) < \omega(a_t)$ for all $h \geq k_t$ where $\limsup_{t\to a_t^+} \mu(t) < \omega(a_t) < 1$.

Now put:

$$\Theta_t = \max\left\{\max_{r=1,\ldots,k_t} \sqrt{\mu(d(fu_t^{(r)}, fu_{t+1}^{(r)}))}, \sqrt{\omega(a_t)}\right\}.$$

Then, for every $h > k_t$, consider:

$$\begin{aligned} d(fu_t^{(h+1)}, fu_{t+1}^{(h+1)}) &< \sqrt{\mu(d(fu_t^{(h)}, fu_{t+1}^{(h)}))}d(fu_t^{(h)}, fu_{t+1}^{(h)}) \\ &< \sqrt{\omega(a_t)}d(fu_t^{(h)}, fu_{t+1}^{(h)}) \\ &\leq \Theta_t d(fu_t^{(h)}, fu_{t+1}^{(h)}) \\ &\leq (\Theta_t)^2 d(fu_t^{(h-1)}, fu_{t+1}^{(h-1)}) \\ &\leq ... \\ &\leq (\Theta_t)^h d(fu_t^{(1)}, fu_{t+1}^{(1)}). \end{aligned}$$

Putting $q = \max\{k_t : t = 0, 1, 2, ..., m-1\}$, gives:

$$\begin{aligned} d(fv_h, fv_{h+1}) &= d(fu_0^{(h+1)}, fu_m^{(h+1)}) \\ &\leq \sum_{t=0}^{m-1} d(fu_t^{(h+1)}, fu_{t+1}^{(h+1)}) \\ &< \sum_{t=0}^{m-1} (\Theta_t)^h d(fu_t^{(1)}, fu_{t+1}^{(1)}), \text{ for all } h > q. \end{aligned}$$

Now for $p > h > q$, consider:

$$\begin{aligned} d(fv_h, fv_p) &\leq d(fv_h, fv_{h+1}) + d(fv_{h+1}, fv_{h+2}) + \cdots + d(fv_{p-1}, fv_p) \\ &< \sum_{t=0}^{m-1} (\Theta_t)^h d(fu_t^{(1)}, fu_{t+1}^{(1)}) + \cdots + \sum_{t=0}^{m-1} (\Theta_t)^{p-1} d(fu_t^{(1)}, fu_{t+1}^{(1)}). \end{aligned} \tag{3}$$

Since $\Theta_t < 1$ for all $t \in \{0, 1, 2, ..., m-1\}$, it follows that $\{fv_h = fu_m^{(h)}\}$ is a Cauchy sequence. Using completeness of X, find $v^* \in X$ such that $fv_h \to fv^*$. Now using the fact that $fv_n \in T(v_{n-1}) \cap [fv_{n-1}]_G^m$ for all $n \in \mathbb{N}$, find a subsequence $\{fv_{n_k}\}$ of $\{fv_h\}$ such that $(fv_{n_k}, fv^*) \in E(G)$ for all $k \in \mathbb{N}$. Now for any $q \in \mathbb{N}$:

$$\begin{aligned} d(fv^*, T_q(v^*)) &\leq d(fv^*, fv_{h+1}) + d(fv_{h+1}, T_q(v^*)) \\ &\leq d(fv^*, fv_{h+1}) + H(T_{h+1}(v_h), T_q(v^*)) \\ &\leq d(fv^*, fv_{h+1}) + \mu(d(fv_h, fv^*))\, d(fv_h, fv^*). \end{aligned}$$

Letting $h \to \infty$ in the above inequality, gives $d(fv^*, T_q(v^*)) \to 0$, which implies $fv^* \in T_q(v^*)$ for all $q \in \mathbb{N}$. Hence, $fv^* \in \bigcap_{q\in\mathbb{N}} T_q(v^*)$ as required.

Example 1. *Let* $X = \{0\} \cup \left\{\frac{1}{q^n} : n \in \mathbb{N} \cup \{0\}\right\}$ *for* $q \in \mathbb{N}$. *Consider the graph* G *such that* $V(G) = X$ *and for all* x *and* y *in* X :

$$E(G) = \{(x, y) : x \neq y\}.$$

For $q \in \mathbb{N}$, *let* $T_q : X \to CB(X)$ *be defined by:*

$$T_q(x) = \begin{cases} \left\{0, \frac{1}{q} + 1, 1\right\} & if\ x = 0, \\ \left\{\frac{1}{q^{n+1}} + 1, 1\right\} & if\ x = \frac{1}{q^n},\ n \in \mathbb{N}, \\ \left\{\frac{1}{q} + 1\right\} & if\ x = 1. \end{cases}$$

If we assume $f : X \to X$ *as an identity map then sequence of multivalued mappings* $\{T_q\}_{q=1}^{\infty}$ *from* X *into* $CB(X)$ *is a sequence of multivalued* G_f*-contraction.*

It satisfies the conditions of Theorem 1 and $1 \in X$ is the fixed point of sequence of multivalued maps T_q for $q \in \mathbb{N}$.

The next theorem provides a way to find the coincidence point of a hybrid pair.

Theorem 2. *Let* (X, d) *be a complete metric space,* $T : X \to CB(X)$ *and* $f : X \to X$ *a surjection. If* $u, v \in X$ *(with* $u \neq v$*) such that* $(fu, fv) \in E(G)$*, implies:*

$$H(T(u), T(v)) \leq \mu(d(fu, fv))d(fu, fv), \tag{4}$$

where $\mu : [0, \infty) \to [0, 1)$ *is a MT-function, if there exist* $m \in \mathbb{N}$ *and* $v_0 \in X$*, such that:*

(i) $T(v_0) \cap [fv_0]_G^m \neq \phi$;
(ii) *For any sequence* $\{v_n\}$ *in X, if* $v_n \to v$ *and* $v_n \in T(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$ *and* $j = 1, 2, ...,$ *then there exists a subsequence* $\{v_{n_k}\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then f and T have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* \in T(v^*)$.

Proof. Take $T_q := T$ for all $q \in \mathbb{N}$ in Theorem 1 and proof is following the same procedure. □

Corollary 1. *Let* (X, d) *be a complete metric space,*$\{T_q\}_{q=1}^{\infty}$ *a sequence of the self mappings on* X *and* $f : X \to X$ *a surjection. If* $u, v \in X$ *(with* $u \neq v$*) such that* $(fu, fv) \in E(G)$*, implies:*

$$d(T_q(u), T_r(v)) \leq \mu(d(fu, fv))d(fu, fv), \tag{5}$$

for all $q, r \in \mathbb{N}$*, where* $\mu : [0, \infty) \to [0, 1)$ *is a MT function, if there exist* $m \in \mathbb{N}$ *and* $v_0 \in X$*, such that:*

(i) $T_1(v_0) \cap [fv_0]_G^m \neq \phi$;
(ii) *For any sequence* $\{v_n\}$ *in X, if* $v_n \to v$ *and* $v_n = T_n(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$,

then there exists a subsequence $\{v_{n_k}\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then f and sequence of mappings $\{T_q\}_{q=1}^{\infty}$ have a coincidence point, i.e., there exists $v^* \in X$ such that $fv^* = \bigcap\limits_{q \in \mathbb{N}} T_q(v^*)$.

Corollary 2. *Let* (X, d) *be a complete metric space,* $T : X \to CB(X)$ *and if* $u, v \in X$ *(with* $u \neq v$*) such that* $(u, v) \in E(G)$*, implies:*

$$H(T(u), T(v)) \leq \mu(d(u, v))d(u, v), \tag{6}$$

where μ*:* $[0, \infty) \to [0, 1)$ *is a MT-function, if there exist* $m \in \mathbb{N}$ *and* $v_0 \in X$*, such that:*

(i) $T(v_0) \cap [v_0]_G^m \neq \phi$;

(ii) *For any sequence* $\{v_n\}$ *in X, if* $v_n \to v$ *and* $v_n \in T(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$ *and* $j = 1, 2, ...,$ *then there exists a subsequence* $\{v_{n_k}\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then T has a fixed point, i.e., $v^* = T(v^*)$.

The following are the consequence of the Theorem 1 and Theorem 2 for the case of self mappings.

Corollary 3. *Let* (X, d) *be a complete metric space,* $T : X \to X$ *and* $f : X \to X$ *a surjection. If* $u, v \in X$ *(with* $u \neq v$*) such that* $(fu, fv) \in E(G)$*, implies:*

$$d(T(u), T(v)) \leq \mu(d(fu, fv))d(fu, fv), \tag{7}$$

where $\mu : [0, \infty) \to [0, 1)$ *is a MT function, if there exist* $m \in \mathbb{N}$ *and* $v_0 \in X$*, such that:*

(i) $T(v_0) \cap [fv_0]_G^m \neq \phi$;
(ii) *For any sequence* $\{v_n\}$ *in X, if* $v_n \to v$ *and* $v_n = T(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$ *and* $j = 1, 2, ...,$ *then there exists a subsequence* $\{v_{n_k}\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then f and T have a coincidence point, i.e., there exists $v^* \in X$ *such that* $fv^* = T(v^*)$.

Corollary 4. *Let* (X, d) *be a complete metric space,* $T : X \to X$ *and if* $u, v \in X$ *(with* $u \neq v$*) such that* $(u, v) \in E(G)$*, implies:*

$$d(T(u), T(v)) \leq \mu(d(u, v))d(u, v), \tag{8}$$

where $\mu : [0, \infty) \to [0, 1)$ *is a MT-function, if there exist* $m \in \mathbb{N}$ *and* $v_0 \in X$*, such that:*

(i) $T(v_0) \cap [v_0]_G^m \neq \phi$;
(ii) *For any sequence* $\{v_n\}$ *in X, if* $v_n \to v$ *and* $v_n = T(v_{n-1}) \cap [v_{n-1}]_G^m$ *for all* $n \in \mathbb{N}$ *and* $j = 1, 2, ...,$ *then there exists a subsequence* $\{v_{n_k}\}$ *such that* $(v_{n_k}, v) \in E(G)$ *for all* $k \in \mathbb{N}$.

Then T has a fixed point, i.e., $v^* = T(v^*)$.

The next remark highlights the applications of all the above results in settings of complete metric spaces, complete metric spaces endowed with partial order and ε-chainable complete metric spaces.

Remark 1. *Consider the following cases:*

R1. *Let* (X, d) *be a complete metric space, consider the graph* G_0 *with:*

$$E(G_0) = X \times X.$$

R2. *Let* (X, d) *be a complete metric space with partial order* $\preceq$ *on X, consider the graphs* G_1 *and* G_2 *with:*

$$E(G_1) = \{(x, y) \in X \times X : x \preceq y\},$$

and:

$$E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

R3. *Let* $\varepsilon > 0$ *and* (X, d) *be a complete* ε*-chainable metric space, consider the graph:*

$$G_3 := \{(x, y) \in X \times X : 0 < d(x, y) < \varepsilon, \text{ for } \varepsilon > 0\}.$$

We remark that all above results are valid under the above construction of remarks $(R1)$, $(R2)$ *and* $(R3)$.

Further, in an application of Theorem 1 we generalize the Theorem 6 of [20]. It establishes the convergence of successive approximations of operators on a Banach space, which consequently yields the Kelisky-Rivlin theorem on iterates of Bernstein operators on the space $C(I)$, where I is the closed unit interval.

Theorem 3. *Let X be a Banach space and X_0 be a closed subspace of X. Let $T, f : X \to X$ be maps such that f is surjection and:*

$$\|Tx - Ty\| \leq \varphi(\|fx - fy\|) \|fx - fy\| \text{ whenever } fx - fy \in X_0,\ x \neq y. \tag{9}$$

If $(I - f)(X) \subseteq X_0$ and $(f - T)(X) \subseteq X_0$, then for all $x \in X$, $\{T^n x\}$ converges to $Coin\{T, f\}$, where $Coin\{T, f\} = \{x \in X : Tx = fx\}$.

Proof. Consider the graph $G = (V(G), E(G))$ where $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x - y \in X_0\}$. Clearly, $\triangle \subseteq E(G)$, $\tilde{G} = G$ and G has no parallel edges. Consider $(x, y) \in E(G)$, then $fx - fy = (y - fy) - (x - fx) + (x - y) \in X_0$, since $(I - f)(X) \subseteq X_0$. Hence and by given contractive condition (9), we see that $\forall (x, y) \in E(G)$ with $x \neq y$, (6) holds. Also $Tx - Ty = (fy - Ty) - (fx - Tx) + (fx - fy) \in X_0$, since $(f - T)(X) \subseteq X_0$.

The use of $(f - T)(X) \subseteq X_0$, implies that $(fx, Tx) \in E(G)$ for x in X. Therefore condition (i) of Corollary 4 holds with $x = v_0 = x_0$ and $N = 1$. Thus we are able to generate a sequence such that $Tx_{n-1} = fx_n$ for all $n \in \mathbb{N}$. Assume that $Tx_n \to v^* \in X$ but since f is surjection so there exists some v in X such that $v^* = fv$. Here also $Tx_n \in [Tx_{n-1}]_G^1$ for all $n \in \mathbb{N}$, which implies that $(Tx_n, Tx_{n-1}) \in E(G)$ for all $n \in \mathbb{N}$. Now using the outline of the proof of Theorem 4.1 of [12], $(Tx_n, fv) \in E(G)$ for all $n \in \mathbb{N}$. Now assume:

$$\begin{aligned} \|fv - Tv\| &\leq \|fv - fx_{n+1}\| + \|fx_{n+1} - Tv\| \\ &= \|fv - fx_{n+1}\| + \|Tx_n - Tv\|. \end{aligned} \tag{10}$$

Since$(Tx_n, fv) \in E(G)$ for all $n \in \mathbb{N}$, thus from (9) and (10) we have:

$$\|fv - Tv\| \leq \|fv - fx_{n+1}\| + \varphi(\|fx_n - fv\|) \|fx_n - fv\|.$$

As $n \to \infty$, we get $fv = Tv$. Thus v is the coincidence point of f and T, by using Corollary 4. For the uniqueness of the coincidence point we let two coincidence points u, v of f and T, then:

$$\begin{aligned} \|Tu - Tv\| &\leq \varphi(\|fu - fv\|) \|fu - fv\| \\ (1 - \varphi(\|fu - fv\|)) \|Tu - Tv\| &\leq 0. \end{aligned}$$

This implies that $Tu = Tv$. □

In the next result, we discussed the generalization of fractional differential equation described in [21]. For the closed interval $I = [0, 1]$, assume function $g \in C(I, \mathbb{R})$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The fractional differential equation is given as follows:

$$D^\alpha x(t) + f(t, g(x(t))) = 0 \quad (0 \leq t \leq 1,\ \alpha > 1) \tag{11}$$

with boundary conditions $x(0) = x(1) = 0$. It is to be noted that associated Green's function with the problem (11) is:

$$G(t, s) = \begin{cases} (t(1 - s))^{\alpha - 1} - (t - s)^{\alpha - 1} & 0 \leq s \leq t \leq 1, \\ \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1. \end{cases},$$

where $\Gamma(.)$ represents the Gamma function.

Theorem 4. *Consider the surjective function $g \in C(I, \mathbb{R})$ and $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies:*

(i) $|(f(s, g(x(s))) - f(s, g(y(s))))| \leq |g(x(s)) - g(y(s))|$ *for all $s \in I$;*

(ii) $\sup_{t\in I}\int_0^1 G(t,s)\,ds \le k < 1.$

Then, problem (11) *has a unique solution.*

Proof. Assume space $X = C(I,\mathbb{R})$, and we have $d(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|$ for x and y in X. It is well known that $x \in X$ is a solution of (11) if and only if it is a solution of the integral equation:

$$x(t) = \int_0^1 G(t,s)\, f(s,(gx)(s))\, ds \text{ for all } t \in I.$$

Define the operator $F : X \to X$ by:

$$Fx(t) = \int_0^1 G(t,s)\, f(s,(gx)(s))\, ds \text{ for all } t \in I,$$

and $S : X \to X$ by:

$$Sx = gx, \text{ with } (Sx)(t) = (gx)(t) \text{ for } t \in I.$$

Thus, for finding a solution of (11), it is sufficient to show that F has a coincidence point with g. Now let $x, y \in C(I)$ for all $s \in I$. Here we have:

$$\begin{aligned}
|Fx(t) - Fy(t)| &= \left|\int_0^1 G(t,s)\left(f(s,(gx)(s)) - f(s,(gy)(s))\right) ds\right| \\
&\le \int_0^1 G(t,s)\left|\left(f(s,(gx)(s)) - f(s,(gy)(s))\right)\right| ds \\
&\le \int_0^1 G(t,s)\left|(gx)(s) - (gy)(s)\right| ds \\
&\le \int_0^1 G(t,s)\left|(Sx)(s) - (Sy)(s)\right| ds \\
&\le \int_0^1 G(t,s)\, d(Sx,Sy)\, ds \\
&\le d(Sx,Sy) \sup_{t\in I}\int_0^1 G(t,s)\, ds \\
&\le kd(Sx,Sy).
\end{aligned}$$

This implies that for each $x, y \in X$, we have:

$$d(Fx,Fy) \le kd(Sx,Sy).$$

Now the use of Corollary 3 with graph $G = G_0$, we have $x^* \in X$ such that $Fx^* = Sx^*$ with $(Sx^*)(t) = (gx^*)(t)$ for $t \in I$. Thus x^* is the required coincidence point of F and g. □

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