mathematics

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The Split Common Fixed Point Problem for a Family of Multivalued Quasinonexpansive Mappings and Totally Asymptotically Strictly Pseudocontractive Mappings in Banach Spaces

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Academic Editor: Hari M. Srivastava

Received: 29 September 2016; Accepted: 6 February 2017; Published: 11 February 2017

Abstract: In this paper, we introduce an iterative algorithm for solving the split common fixed point problem for a family of multi-valued quasinonexpansive mappings and totally asymptotically strictly pseudocontractive mappings, as well as for a family of totally quasi-ϕ-asymptotically nonexpansive mappings and k-quasi-strictly pseudocontractive mappings in the setting of Banach spaces. Our results improve and extend the results of Tang et al., Takahashi, Moudafi, Censor et al., and Byrne et al.

Keywords: split common fixed point problem; totally asymptotically strictly pseudocontractive mapping; quasinonexpansive mapping; k-quasi-strictly pseudocontractive mapping

MSC Classification: 47H05; 47H09; 47J25

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. For nonlinear operators $T : H_1 \longrightarrow H_1$ and $U : H_2 \longrightarrow H_2$, the split fixed point problem (SFPP) is to find a point:

$$x \in Fix(T) \quad \text{such that} \quad Ax \in Fix(U) \tag{1}$$

It is often desirable to consider the above problem for finitely many operators. Given n nonlinear operators $T_i : H_1 \longrightarrow H_1$ and m nonlinear operators $U_j : H_2 \longrightarrow H_2$, the split common fixed point problem (SCFPP) is to find a point:

$$x \in \cap_{i=1}^{n} Fix(T_i) \quad \text{such that} \quad Ax \in \cap_{j=1}^{m} Fix(U_j)$$

In particular, if $T_i = P_{C_i}$ and $U_j = P_{Q_j}$, then the SCFPP reduces to the multiple sets split feasibility problem (MSSFP); that is, to find $x \in \cap_{i=1}^{n} C_i$, such that $Ax \in \cap_{j=1}^{m} Q_j$, where $\{C_i\}_{i=1}^{n}$ and $\{Q_j\}_{j=1}^{m}$ are nonempty closed convex subsets in H_1 and H_2, respectively.

In the Hilbert space setting, the split feasibility problem and the split common fixed point problem have been studied by several authors; see, for instance, [1–3]. In [4], Censor and Segal introduced the iterative scheme:

$$x_{n+1} = U(I - \rho_n A^*(I - T)A)x_n$$

which solves the problem (1) for directed operators. This algorithm was then extended to the case of quasinonexpansive mappings [5], as well as to the case of demicontractive mappings [6]. Recently, Takahashi in [7,8] extended the split feasibility problem in Hilbert spaces to the Banach space setting.

Then, Alsulami et al. [1] established some strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. Using the shrinking projection method of [8], Takahashi proved the strong convergence theorem for finding a solution of the split feasibility problem in Banach spaces. In this direction, Byrne et al. [2] studied the split common null point problem for multi-valued mappings in Hilbert spaces. Consider finitely many multi-valued mappings $F_i : H_1 \to 2^{H_1}, 1 \le i \le n$, and $B_j : H_2 \to 2^{H_2}, 1 \le j \le m$, and let $A_j : H_1 \to H_2$ be bounded linear operators. The split common null point problem is to find a point:

$$z \in H_1 \quad \text{such that} \quad z \in (\cap_{i=1}^{n} F_i^{-1} 0) \cap (\cap_{j=1}^{m} A_j B_j^{-1} 0)$$

Very recently, using the hybrid method and the shrinking projection method in mathematical programming, Takahashi et al. [9] proved two strong convergence theorems for finding a solution of the split common null point problem in Banach spaces. In [10], Tang et al. proved a theorem regarding the split common fixed point problem for a k-quasi-strictly pseudocontractive mapping and an asymptotical nonexpansive mapping. In this paper, motivated by [11], we use the hybrid method to study the split common fixed point problem for an infinite family of multi-valued quasinonexpansive mappings and an infinite family of L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\})$-totally asymptotically strictly pseudocontractive mappings. Compared to the Theorem of Tang et al. [10], we remove an extra condition and present a strong convergence theorem, which is more desirable than the weak convergence. The point is that the authors of [10] considered a semi-compact mapping, that is a mapping T on a set X having the property that if $\{x_n\}$ is a bounded sequence in X such that $\|Tx_n - x_n\|$ tends to zero, then $\{x_n\}$ has a convergent subsequence. We will not assume that our mappings are semi-compact, and at the same time, we propose a different algorithm; instead, we impose some restrictions on the control sequences to get the strong convergence. We also present an algorithm for solving the split common fixed point problem for totally quasi-ϕ-asymptotically nonexpansive mappings and for k-quasi-strictly pseudocontractive mappings. Under some mild conditions, we establish the strong convergence of these algorithms in Banach spaces. As applications, we consider the algorithms for a split variational inequality problem and a split common null point problem. Our results improve and generalize the result of Tang et al. [10], Takahashi [12], Moudafi [5], Censor et al. [13] and Byrne et al. [2].

2. Preliminaries

Let E be a real Banach space and C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be $\{k_n\}$-asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$, such that:

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \qquad \forall x, y \in C,\ n \ge 1$$

The mapping $T : C \to C$ is said to be k-quasi-strictly pseudocontractive if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1]$, such that:

$$\|Tx - p\|^2 \le \|x - p\|^2 + k\|x - Tx\|^2 \quad \forall x \in C, p \in F(T)$$

The mapping $T : C \to C$ is said to be $(k, \{\mu_n\}, \{\xi_n\})$-totally asymptotically strictly pseudocontractive if there exist a constant $k \in [0, 1]$ and null sequences $\{\mu_n\}$ and $\{\xi_n\}$ in $[0, \infty)$ and a continuous strictly increasing function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$, such that for all $x, y \in H$ and $n \ge 1$:

$$\|T^n x - T^n y\|^2 \le \|x - y\|^2 + k\|(x - y) - (Tx - Ty)\|^2 + \mu_n \zeta(\|x - y\|) + \xi_n$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, where $\rho_E(t)$ is the modulus of smoothness of E. Let $q > 1$; then, E is called q-uniformly smooth if there exists a constant $c > 0$, such that $\rho_E(t) \le ct^q$ for all $t > 0$. Throughout, J will stand for the duality mapping of E. We recall that a Banach space E is smooth if and only if the duality mapping J is single valued.

Lemma 1. *[14] If E is a two-uniformly smooth Banach space, then for each $t > 0$ and each $x, y \in E$:*

$$\|x + ty\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2\|ty\|^2$$

For a smooth Banach space E, Alber [15] defined:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in E$$

It follows that $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$ for each $x, y \in E$. Moreover, if we denote by $\Pi_C x$ the generalized projection from E onto a closed convex subset C in E, then we have:

Lemma 2. *[15] Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then:*

(a) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y)$, *for all* $x \in C$ *and* $y \in E$;
(b) *For* $x, y \in E, \phi(x, y) = 0$ *if and only if* $x = y$;
(c) *For* $x, y, z \in E, \phi(x, y) \le \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$;
(d) *For* $x, y, z \in E, \lambda \in [0, 1], \phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \le \lambda\phi(x, y) + (1-\lambda)\phi(x, z)$.

Lemma 3. *[16] If E is a uniformly-smooth Banach space and $r > 0$, then there exists a continuous, strictly-increasing convex function $g : [0, 2r] \to [0, \infty)$, such that $g(0) = 0$ and:*

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \le \lambda\phi(x, y) + (1-\lambda)\phi(x, z) - \lambda(1-\lambda)g(\|Jy - Jz\|)$$

for all $\lambda \in [0, 1], x \in E$ and $y, z \in B_r = \{u \in E : \|u\| \le r\}$.

We denote by $N(C)$, $CB(C)$ and $P(C)$ the collection of all nonempty subsets, nonempty closed bounded subsets and nonempty proximal bounded subsets of C, respectively. Let $T : E \to N(E)$ be a multivalued mapping. An element $x \in E$ is said to be a fixed point of T if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$.

Definition 1. *Let C be a closed convex subset of a smooth Banach space E and $T : C \to N(C)$ be a multivalued mapping. We set:*

$$\Phi(Tx, Tp) = \max\{\sup_{q\in Tp}\inf_{y\in Tx}\phi(y, q), \sup_{y\in Tx}\inf_{q\in Tp}\phi(y, q)\}$$

We call T a quasinonexpansive multivalued mapping if $F(T) \neq \emptyset$ and:

$$\Phi(Tx, Tp) \le \phi(x, p), \quad \forall p \in F(T), \forall x \in C$$

Definition 2. *A multivalued mapping T is called demi-closed if $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$ and $x_n \rightharpoonup w$ imply that $w \in Tw$.*

Let C be a nonempty closed convex subset of E and $T := \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C. We use $Fix(T)$ to denote the common fixed point set of the semigroup T. It is well known that $Fix(T)$ is closed and convex. A nonexpansive semigroup T on C is said to be uniformly asymptotically regular (u.a.r.) if for all $h \ge 0$ and any bounded subset D of C:

$$\lim_{n\to\infty}\sup_{x\in D}\|T(h)(T(t)x) - T(t)x\| = 0$$

For each $h \geq 0$, define $\sigma_t(x) = \frac{1}{t}\int_0^t T(s)xds$. Then, $\lim_{t\to\infty}\sup_{x\in D}\|T(h)(\sigma_t(x)) - \sigma_t(x)\| = 0$ provided that D is a closed bounded convex subset of C. It is known that the set $\{\sigma_t(x) : t > 0\}$ is a u.a.r. nonexpansive semigroup; see [17].

A mapping $T : E \to E$ is said to be α-averaged if $T = (1-\alpha)I + \alpha S$ for some $\alpha \in (0,1)$; here, I is the identity operator, and $S : E \to E$ is a nonexpansive mapping (see [18]). It is known that in a Hilbert space setting, every firmly-nonexpansive mapping (in particular, a projection) is a $\frac{1}{2}$-averaged mapping (see Proposition 11.2 in the book [19]).

Lemma 4. *[20] (i) The composition of finitely many averaged mappings is averaged. In particular, if T_i is α_i-averaged, where $\alpha_i \in (0,1)$ for $i = 1,2$, then the composition T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$. (ii) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then $\cap_{i=1}^N F(T_i) = F(T_1 \cdots T_N)$. (iii) In case E is a uniformly-convex Banach space, every α-averaged mapping is nonexpansive.*

Lemma 5. *[21] Let E be a uniformly-convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 6. *[15] Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$, then $x_0 = \Pi_C x$ if and only if for all $y \in C$, $\langle x_0 - y, Jx - Jx_0\rangle \geq 0$.*

Lemma 7. *[22] Let E be a uniformly-convex Banach space, and let $B_r(0) = \{x \in E : \|x\| \leq r\}$, for $r > 0$, then there exists a continuous, strictly-increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, such that, for any given sequence $\{x_n\}_{n=1}^\infty \subset B_r(0)$ and for any given sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers with $\sum_{n=1}^\infty a_n = 1$ and for any positive integers i, j with $i < j$:*

$$\|\sum_{n=1}^{\infty}\alpha_n x_n\|^2 \leq \sum_{n=1}^{\infty}\alpha_n\|x_n\|^2 - \alpha_i\alpha_j g(\|x_i - x_j\|).$$

Lemma 8. *[23] Let $\{\alpha_n\}$ be a sequence in $[0,1]$, δ_n and $\{\gamma_n\}$ be sequences in $\mathbb{R}$, such that (i) $\sum_{n=1}^\infty \alpha_n = \infty$, (ii) $\limsup_{n\to\infty}\delta_n \leq 0$ and (iii) $\gamma_n \geq 0$ and $\sum_{n=1}^\infty \gamma_n < \infty$. If $\{a_n\}$ is a sequence of nonnegative real numbers, such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n$, for each $n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 9. *[24] Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$, such that $s_{n_i} \leq s_{n_{i+1}}$ for all $i \geq 0$. For every $n \in \mathbb{N}$, define an integer sequence $\{\tau(n)\}$ as $\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}$. Then, $\tau(n) \to \infty$ and $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.*

Lemma 10. *[25] Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be nonnegative and $\{\alpha_n\}$ be positive real numbers, such that $\lambda_{n+1} \leq \lambda_n - \alpha_n\lambda_n + \gamma_n$, $n \geq 0$. Let for all $n > 1$, $\frac{\lambda_n}{\alpha_n} \leq c_1$ and $\alpha_n \leq \alpha$. Then, $\lambda_n \leq \max\{\lambda_1, K^*\}$, where $K^* = (1+\alpha)c_1$.*

Definition 3. *(1) A mapping $T : C \to C$ is said to be a k-quasi-strictly pseudocontractive mapping if there exists $k \in [0,1)$, such that $\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \quad \forall x \in C, p \in F(T)$. (2) A mapping $T : C \to C$ is called quasinonexpansive if $F(T) \neq \emptyset$; and $\phi(p, Tx) \leq \phi(p, x) \forall x \in C, p \in F(T)$. (3) A countable family of mappings $\{T_i\} : C \to C$ is said to be totally uniformly quasi-ϕ-asymptotically nonexpansive, if $\Im = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and there exist nonnegative real sequences $\{\mu_n\}$,$\{\nu_n\}$ with $\mu_n \to 0, \nu_n \to 0 (as \quad n \to \infty)$ and a strictly-increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$, such that $\phi(p, T_i^n x) \leq \phi(p, x) + \nu_n\zeta(\phi(p, x)) + \mu_n$, $n \geq 1, i \geq 1, x \in C, p \in \Im$. (4) A mapping $T : C \to C$ is said to be uniformly L-Lipschitzian continuous, if there exists a constant $L > 0$, such that $\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in C, n \geq 1$.*

Lemma 11. *[11] Let E be a real uniformly-smooth and uniformly-convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and totally quasi-ϕ-asymptotically nonexpansive mapping* *with nonnegative real sequences* $\{\mu_n\}, \{\nu_n\}$ *and a strictly-increasing continuous function* $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$, *such that* $\mu_n \to 0$, $\nu_n \to 0$ *and* $\zeta(0) = 0$. *If* $\mu_1 = 0$, *then the fixed point set of* T *is closed and convex.*

Lemma 12. *[26] Let* C *be a nonempty closed convex subset of a real Banach space* E, *and let* $T : C \to C$ *be a* k*-quasi-strictly pseudocontractive mapping. If* $F(T) \neq \emptyset$, *then* $F(T)$ *is closed and convex.*

3. Main Results

This section is devoted to the main results of this paper.

Theorem 1. *Let* E_1 *be a real uniformly-convex and two-uniformly-smooth Banach space with the best smoothness constant* t *satisfying* $0 < t < \frac{1}{\sqrt{2}}$, *and let* E_2 *be a real smooth Banach space. Let* $A : E_1 \to E_2$ *be a bounded linear operator and* A^* *be its adjoint. Suppose* $T : E_2 \to E_2$ *is a uniformly* L*-Lipschitzian continuous and* $(k, \{\mu_n\}, \{\xi_n\})$*-totally asymptotically strictly pseudocontractive mapping satisfying the following conditions:*

(1) $\sum_{n=1}^{\infty} \mu_n < \infty, \sum_{n=1}^{\infty} \xi_n < \infty$,
(2) $\{r_n\}$ *is a real sequence in* $(0,1)$, *such that* $\mu_n = o(r_n), \xi_n = o(r_n), \lim r_n = 0, \sum_{n=1}^{\infty} r_n = \infty$,
(3) *there exist constants* $M_0 > 0, M_1 > 0$, *such that* $\zeta(\lambda) \leq M_0\lambda^2, \forall \lambda > M_1$.

Let $\{S_n\}_{n=1}^{\infty} : E_1 \to CB(E_1)$ *be a family of multivalued quasinonexpansive mappings, such that for each* $i \geq 1, S_i$ *is demi-closed at zero, and for each* $p \in Fix(S_i), S_i(p) = \{p\}$. *Suppose:*

$$\Omega = \left\{ x \in \bigcap_{i=1}^{\infty} F(S_i) : Ax \in F(T) \right\} \neq \emptyset$$

and $\{x_n\}$ *is the sequence generated by* $x_1 \in E_1$:

$$\begin{cases} u_n = (1 - r_n)x_n \\ y_n = J_1^{-1}(\alpha_n J_1 u_n + (1 - \alpha_n)\gamma A^* J_2(T^n - I)Au_n) \\ x_{n+1} = J_1^{-1}(\beta_{n,0} J_1 y_n + \sum_{i=1}^{\infty} \beta_{n,i} J_1 w_{n,i}) \quad w_{n,i} \in S_i y_n \end{cases} \tag{2}$$

where $\gamma \in (0, \frac{1-k}{2\|A\|^2})$; *the sequences* $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0,1)$ *satisfy the following conditions:*

(a) $\sum_{i=0}^{\infty} \beta_{n,i} = 1, \liminf_n \beta_{n,0}\beta_{n,i} > 0$,
(b) $\lim_{n\to\infty} \alpha_n = 1, \sum_{n=1}^{\infty}(1 - \alpha_n) < \infty$, $(1 - \alpha_n) = o(r_n)$.

Then, $\{x_n\}$ *converges strongly to an element of* Ω.

Proof. Since ζ is continuous, ζ attains its maximum in $[0, M_1]$, and by assumption, $\zeta(\lambda) \leq M_0\lambda^2$, $\forall \lambda > M_1$. In either case, we have $\zeta(\lambda) \leq M + M_0\lambda^2$, $\forall \lambda \in [0, \infty)$. Let $p \in \Omega$, then:

$$\phi(p, u_n) \leq (1 - r_n)\phi(p, x_n) + r_n\|p\|^2 \tag{3}$$

From (2) and Lemma 2(d,c), we have:

$$
\begin{aligned}
\phi(p,y_n) &\leq \alpha_n\phi(p,u_n) + (1-\alpha_n)\phi(p, J_1^{-1}(\gamma A^* J_2(T^n - I)Au_n)) \\
&\leq \alpha_n\phi(p,u_n) + (1-\alpha_n)[\phi(p,u_n) + \phi(u_n, J_1^{-1}(\gamma A^* J_2(T^n - I)Au_n)) \\
&+ 2\langle p - u_n, J_1 u_n - \gamma A^* J_2(T^n - I)Au_n\rangle] \\
&= \phi(p,u_n) + (1-\alpha_n)[\|u_n\|^2 + \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 - 2\langle u_n, \gamma A^* J_2(T^n - I)Au_n\rangle \\
&+ 2\langle p - u_n, J_1 u_n\rangle + 2\langle p - u_n, \gamma A^* J_2(T^n - I)Au_n\rangle] \\
&\leq \phi(p,u_n) + (1-\alpha_n)[\|p\|^2 + \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 - 2\langle u_n, \gamma A^* J_2(T^n - I)Au_n\rangle \\
&+ 2\langle p - u_n, \gamma A^* J_2(T^n - I)Au_n\rangle]
\end{aligned} \tag{4}
$$

From Lemma 1, we have:

$$
\begin{aligned}
-2\langle u_n, \gamma A^* J_2(T_n - I)Au_n\rangle &\leq \|\gamma A^* J_2(T^n - I)Au_n\|^2 + 2\|tu_n\|^2 - \|u_n + \gamma A^* J_2(T_n - I)Au_n\|^2 \\
&\leq \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 + \|u_n\|^2 \\
&= \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 + 4\|\frac{1}{2}u_n - \frac{1}{2}p + \frac{1}{2}p\|^2 \\
&\leq \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 + 4(\frac{1}{2}\|u_n - p\|^2 + \frac{1}{2}\|p\|^2) \\
&= \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 + 2\|u_n - p\|^2 + 2\|p\|^2)
\end{aligned} \tag{5}
$$

Since $Ap \in F(T)$ and T is a totally quasi-asymptotically strictly pseudocontractive mapping, we obtain:

$$
\begin{aligned}
\langle u_n - p, \gamma A^* J_2(T^n - I)Au_n &= \gamma\langle A(u_n - p), J_2(T^n - I)Au_n\rangle \\
&= \gamma\langle A(u_n - p) + (T^n - I)Au_n - (T^n - I)Au_n, J_2(T^n - I)Au_n\rangle \\
&= \gamma(\langle T^n A(u_n) - Ap, J_2(T^n - I)Au_n\rangle - \|(T^n - I)Au_n\|^2) \\
&\leq \gamma(\frac{1}{2}[\|(T^n - I)Au_n\|^2 + 2\|t(T^n Au_n - Ap)\|^2 \\
&- \|Ap - Au_n\|^2] - \|(T^n - I)Au_n\|^2) \\
&\leq \gamma(\frac{1}{2}[\|(T^n - I)Au_n\|^2 + \|(T^n Au_n - Ap)\|^2 \\
&- \|Ap - Au_n\|^2] - \|(T^n - I)Au_n\|^2) \\
&\leq \gamma(\frac{1}{2}[\|Au_n - Ap\|^2 + k\|(T^n - I)Au_n\|^2 + \mu_n\zeta(\|Au_n - Ap\|) + \xi_n]) \\
&- \frac{1}{2}(\|(T^n - I)Au_n\|^2 + \|Ap - Au_n\|^2) \\
&= \gamma(\frac{k-1}{2}\|(T^n - I)Au_n\|^2 + \frac{\mu_n}{2}[M + M_0\|Au_n - Ap\|^2] + \frac{\xi_n}{2})
\end{aligned} \tag{6}
$$

Substituting (5) and (6) into (4), we have:

$$
\begin{aligned}
\phi(p,y_n) &\leq \alpha_n\phi(p,u_n) + (1-\alpha_n)\phi(p, J_1^{-1}(\gamma A^* J_2(T^n - I)Au_n)) \\
&\leq \phi(p,u_n) + (1-\alpha_n)[3\|p\|^2 + 2\gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 + 2\|u_n - p\|^2 \\
&+ \gamma(k-1)\|(T^n - I)Au_n\|^2 + \gamma\mu_n[M + M_0\|A\|^2\|u_n - p\|^2] + \gamma\xi_n \\
&\leq \phi(p,u_n) + 3(1-\alpha_n)\|p\|^2 - \gamma(1 - k - 2\gamma\|A\|^2)\|(T^n - I)Au_n\|^2 \\
&+ \gamma\mu_n M + (\gamma\mu_n M_0\|A\|^2 + 2)\|u_n - p\|^2 + \gamma\xi_n
\end{aligned} \tag{7}
$$

From Lemma 1 and the fact that $0 < t < \frac{1}{\sqrt{2}}$, we have:

$$\begin{aligned}
\phi(p,y_n) &\leq \alpha_n\phi(p,u_n)+(1-\alpha_n)\phi(p,J_1^{-1}(\gamma A^*J_2(T^n-I)Au_n)) \\
&\leq \phi(p,u_n)+3(1-\alpha_n)\|p\|^2-\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+\gamma\mu_nM+(\gamma\mu_nM_0\|A\|^2+2)\|u_n-p\|^2+\gamma\xi_n \\
&\leq \phi(p,u_n)+3(1-\alpha_n)\|p\|^2-\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+\gamma\mu_nM+(\gamma\mu_nM_0\|A\|^2+2)[\|u_n\|^2-\langle p,Ju_n\rangle+2\|tp\|^2]+\gamma\xi_n \\
&\leq \phi(p,u_n)+3(1-\alpha_n)\|p\|^2-\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+\gamma\mu_nM+(\gamma\mu_nM_0\|A\|^2+2)\phi(p,u_n)+\gamma\xi_n
\end{aligned} \tag{8}$$

Putting (3) and (8) into (2), we obtain:

$$\begin{aligned}
\phi(p,x_{n+1}) &= \phi(p,J_1^{-1}(\beta_{n,0}J_1y_n+\sum_{i=1}^{\infty}\beta_{n,i}J_1w_{n,i})) \\
&\leq \beta_{n,0}\phi(p,y_n)+\sum_{i=1}^{\infty}\beta_{n,i}\phi(p,w_{n,i}) \\
&= \beta_{n,0}\phi(p,y_n)+\sum_{i=1}^{\infty}\beta_{n,i}\inf_{t\in S_i(p)}\phi(p,w_{n,i}) \\
&\leq \beta_{n,0}\phi(p,y_n)+\sum_{i=1}^{\infty}\beta_{n,i}\Phi(p,w_{n,i})=\phi(p,y_n) \\
&\leq \phi(p,u_n)+3(1-\alpha_n)\|p\|^2-\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+\gamma\mu_nM+(\gamma\mu_nM_0\|A\|^2+2)\phi(p,u_n)+\gamma\xi_n \\
&\leq (1-r_n)\phi(p,x_n)+r_n\|p\|^2+3(1-\alpha_n)\|p\|^2-\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+\gamma\mu_nM+(\gamma\mu_nM_0\|A\|^2+2)((1-r_n)\phi(p,x_n)+r_n\|p\|^2)+\gamma\xi_n \\
&\leq \phi(p,x_n)-(r_n-\gamma\mu_nM_0\|A\|^2+2)(1-r_n)\phi(p,x_n) \\
&+(3(1-\alpha_n)+r_n+\mu_n\gamma M_0\|A\|^2r_n)\|p\|^2+\gamma\mu_nM+\gamma\xi_n \\
&\leq \phi(p,x_n)-(r_n-(\gamma\mu_nM_0\|A\|^2+2))(1-r_n)\phi(p,x_n)+\sigma_n
\end{aligned} \tag{9}$$

where $\sigma_n=(3(1-\alpha_n)+r_n+\mu_n\gamma M_0\|A\|^2r_n)\|p\|^2+\mu_n\gamma M+\gamma\xi_n$. Since $\mu_n=o(r_n)$, $(1-\alpha_n)=o(r_n)$ and $\xi_n=o(r_n)$, we may assume without loss of generality that there exist constants $k_0\in(0,1)$ and $M_2>0$, such that for all $n\geq 1$:

$$\frac{\mu_n}{r_n}\leq\frac{r_n(1-k_0+2)-2}{r_n(1-r_n)\gamma M_0\|A\|^2}\quad \textit{and}\quad \frac{\sigma_n}{r_n}\leq M_2$$

Thus, we obtain:

$$\phi(p,x_{n+1})\leq\phi(p,x_n)-r_nk_0\phi(p,x_n)+\sigma_n \tag{10}$$

According to Lemma 10, $\phi(p,x_{n+1})\leq\max\{\phi(p,x_1),(1+k_0)M_2\}$. Therefore, $\{\phi(p,x_n)\}$ and $\{x_n\}$ are bounded. Furthermore, the sequences $\{y_n\}$ and $\{u_n\}$ are bounded, as well. We now consider two cases.

Case 1. *Suppose that there exists $n_0\in\mathbb{N}$, such that $\{\phi(p,x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. Then, $\{\phi(p,x_n)\}_{n=1}^{\infty}$ converges, and $\phi(p,x_n)-\phi(p,x_{n+1})\to 0$ as $n\to\infty$. Since E_1 is a uniformly smooth Banach space, it follows from Lemma 3 and Equations (8) and (10) that:*

$$\begin{aligned}
\phi(p,x_{n+1}) &\le \phi(p,y_n) \\
&\le \alpha_n\phi(p,u_n) + (1-\alpha_n)\phi(p,J_1^{-1}(\gamma A^*J_2(T^n-I)Au_n)) - \alpha_n(1-\alpha_n)g(\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\|) \\
&\le \phi(p,u_n) + 3(1-\alpha_n)\|p\|^2 - \gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+ \gamma\mu_n M + (\gamma\mu_n M_0\|A\|^2+2)\phi(p,u_n) + \gamma\xi_n - \alpha_n(1-\alpha_n)g(\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\| \\
&\le \phi(p,x_n) - (r_n - (\gamma\mu_n M_0\|A\|^2+2))\phi(p,u_n) + (3(1-\alpha_n)+r_n)\|p\|^2 \\
&+ \gamma\xi_n - \gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 - \alpha_n(1-\alpha_n)g(\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\|) \\
&\le \phi(p,x_n) - r_nk_0\phi(p,x_n) + \sigma_n - \gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&- \alpha_n(1-\alpha_n)g(\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\|)
\end{aligned} \tag{11}$$

Hence, from (10)*, we have:*

$$\alpha_n(1-\alpha_n)g(\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\|) \le \phi(p,x_n) - \phi(p,x_{n+1}) - r_nk_0\phi(p,x_n) + \sigma_n$$

and:

$$\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \le \phi(p,x_n) - \phi(p,x_{n+1}) - r_nk_0\phi(p,x_n) + \sigma_n$$

Therefore, $\alpha_n(1-\alpha_n)g(\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\|)$ *and* $\gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2$ *tend to zero as* $n\to\infty$*. Since* $\liminf \alpha_n(1-\alpha_n) > 0$ *and* $\gamma \in (0, \frac{1-k}{2\|A\|^2})$*, we obtain:*

$$\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\| \longrightarrow 0 \quad n\to\infty \tag{12}$$

$$\|(T^n-I)Au_n\|^2 \longrightarrow 0 \quad n\to\infty \tag{13}$$

Furthermore, we observe that $\|J_1y_n - J_1u_n\| = (1-\alpha_n)\|J_1u_n-\gamma A^*J_2(T^n-I)Au_n\| \to 0$*. Since* J_1^{-1} *is uniformly norm-to-norm continuous on bounded subsets, we conclude that:*

$$\lim_{n\to\infty}\|y_n-u_n\| = 0 \tag{14}$$

Using (7) *and Lemma 3 in* (2)*, we have:*

$$\begin{aligned}
\phi(p,x_{n+1}) &= \phi(p,J_1^{-1}(\beta_{n,0}J_1y_n + \sum_{i=1}^{\infty}\beta_{n,i}J_1w_{n,i})) \\
&\le \beta_{n,0}\phi(p,y_n) + \sum_{i=1}^{\infty}\beta_{n,i}\phi(p,w_{n,i}) - \beta_{n,0}\beta_{n,i}g(\|J_1y_n - J_1w_{n,i}\|) \\
&\le \phi(p,y_n) - \beta_{n,0}\beta_{n,i}g(\|J_1y_n - J_1w_{n,i}\|) \\
&\le \phi(p,u_n) + 3(1-\alpha_n)\|p\|^2 - \gamma(1-k-2\gamma\|A\|^2)\|(T^n-I)Au_n\|^2 \\
&+ \gamma\mu_n M + (\gamma\mu_n M_0\|A\|^2+2)\|u_n-p\|^2 + \gamma\xi_n - \beta_{n,0}\beta_{n,i}g(\|J_1y_n - J_1w_{n,i}\|)
\end{aligned} \tag{15}$$

It now follows from (3) *and* $\gamma \in (0, \frac{1-k}{2\|A\|^2})$ *that:*

$$\begin{aligned}
\beta_{n,0}\beta_{n,i}g(\|J_1y_n - J_1w_{n,i}\|) &\le \phi(p,x_n) - \phi(p,x_{n+1}) - (r_n - (\gamma\mu_n M_0\|A\|^2+2))\phi(p,u_n) \\
&+ (3(1-\alpha_n)+r_n)\|p\|^2 + \gamma\xi_n \\
&\le \phi(p,x_n) - \phi(p,x_{n+1}) - r_nk_0\phi(p,x_n) + \sigma_n
\end{aligned}$$

From Condition (a), we have $\lim_{n\to\infty} g(\|J_1y_n - J_1w_{n,i}\|) = 0$*. Since* g *is continuous and* $g(0)=0$*, we obtain* $\lim_{n\to\infty}\|J_1y_n - J_1w_{n,i}\| = 0$*. Since* J_1^{-1} *is uniformly norm-to-norm continuous on bounded subsets, we have:*

$$\lim_{n\to\infty}\|y_n - w_{n,i}\| = 0 \qquad \forall i\in\mathbb{N} \tag{16}$$

which implies that $\lim_{n\to\infty} dist(y_n, S_iy_n) \le \lim_{n\to\infty}\|y_n - w_{n,i}\| = 0$*,* $\forall i\in\mathbb{N}$*. From* (2)*, we obtain:*

$$\|J_1x_{n+1} - J_1y_n\| = (1-\beta_{n,0})\|J_1y_n - J_1w_{n,i}\| \longrightarrow 0 \quad n\to\infty$$

Since J is uniformly norm-to-norm continuous on bounded subsets, we have:

$$\|x_{n+1} - y_n\| \longrightarrow 0 \quad n \to \infty \tag{17}$$

From (14), (17) *and* $\lim_{n\to\infty} r_n = 0$, *we have:*

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - u_n\| + \|u_n - x_n\| \\ &= \|x_{n+1} - y_n\| + \|y_n - u_n\| + r_n\|x_n\| \longrightarrow 0 \quad n \to \infty\end{aligned}$$

Consequently:

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|(1 - r_{n+1})x_{n+1} - (1 - r_n)x_n)\| \\ &\leq |r_{n+1} - r_n|\|x_{n+1}\| + (1 - r_n)\|x_{n+1} - x_n\| \longrightarrow 0 \quad n \to \infty\end{aligned} \tag{18}$$

Using the fact that T is uniformly L-Lipschitzian, we have:

$$\begin{aligned}\|TAu_n - Au_n\| \leq\ & \|TAu_n - T^{n+1}Au_n\| + \|T^{n+1}Au_n - T^{n+1}Au_{n+1}\| \\ &+ \|T^{n+1}Au_{n+1} - Au_{n+1}\| + \|Au_{n+1} - Au_n\| \\ &\leq L\|Au_n - T^nAu_n\| + (1+L)\|Au_{n+1} - Au_n\| + \|T^{n+1}Au_{n+1} - Au_{n+1}\| \\ &\leq L\|Au_n - T^nAu_n\| + (1+L)\|A\|\|u_{n+1} - u_n\| + \|T^{n+1}Au_{n+1} - Au_{n+1}\|\end{aligned}$$

From (13) *and* (18), *we obtain:*

$$\|(T - I)Au_n\| \longrightarrow 0, \quad n \to \infty \tag{19}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, such that $x_{n_j} \rightharpoonup z$. Using the fact that $x_{n_j} \rightharpoonup z$ and $\|y_n - x_n\| \to 0$, $n \to \infty$, we have that $y_{n_j} \rightharpoonup z$. Similarly, $u_{n_j} \rightharpoonup z$, since $\|u_n - x_n\| \to 0$, $n \to \infty$. Now, we show that $z \in \Omega$. Since $y_{n_j} \rightharpoonup z$ and $\lim_{n\to\infty} dist(y_n, S_i(y_n)) = 0$ and by the demi-closedness of each S_i, we have $z \in \bigcap_{i\in\mathbb{N}} F(S_i)$. On the other hand, since A is a bounded operator, it follows from $u_{n_j} \rightharpoonup z$ that $Au_{n_j} \rightharpoonup Az$. Hence, from (13), *we have $\|TAu_{n_j} - Au_{n_j}\| \to 0$ as $j \to \infty$. Since T is demi-closed at zero, we have that $Az \in F(T)$. Hence, $z \in \Omega$. Next, we prove that $\{x_n\}$ converges strongly to z. From* (7), *Lemma 1 and $\gamma \in (0, \frac{1-k}{2\|A\|^2})$, we have:*

$$\begin{aligned}\phi(z, x_{n+1}) \leq \phi(z, y_n) &\leq \alpha_n\phi(z, u_n) + (1 - \alpha_n)\phi(z, J_1^{-1}(\gamma A^* J_2(T^n - I)Au_n)) \\ &\leq \alpha_n\phi(z, u_n) + (1 - \alpha_n)[\phi(z, u_n) + \phi(u_n, J_1^{-1}(\gamma A^* J_2(T^n - I)Au_n)) \\ &+ 2\langle z - u_n, J_1u_n - \gamma A^* J_2(T^n - I)Au_n\rangle] \\ &\leq \phi(z, u_n) + (1 - \alpha_n)[\|z\|^2 + \|u_n - z + z\|^2 + 2\gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 \\ &+ \gamma(k-1)\|(T^n - I)Au_n\|^2 + \gamma\mu_n[M + M_0\|A\|^2\|u_n - z\|^2] + \gamma\xi_n \\ &\leq \phi(z, u_n) + (1 - \alpha_n)[\|z\|^2 + \|u_n - z\|^2 + \|z\|^2 + 2\langle u_n - z, Jz\rangle \\ &+ 2\gamma^2\|A\|^2\|(T^n - I)Au_n\|^2 + \gamma(k-1)\|(T^n - I)Au_n\|^2 \\ &+ \gamma\mu_n[M + M_0\|A\|^2\|u_n - z\|^2] + \gamma\xi_n \\ &\leq \phi(z, u_n) + (1 - \alpha_n)(\|u_n - z\| + 2\langle u_n, J_1z\rangle) + \mu_nM^* + \gamma\xi_n \\ &\leq (1 - r_n)\phi(z, x_n) - 2r_n\langle x_n - z, J_1z\rangle + (1 - \alpha_n)(\|u_n - z\| \\ &+ 2\langle x_n, J_1z\rangle) + \mu_nM^* + \gamma\xi_n \\ &\leq (1 - r_n)\phi(z, x_n) - 2r_n\langle x_n - z, J_1z\rangle + (1 - \alpha_n)(\|u_n - z\|^2 \\ &+ 2\langle x_n, J_1z\rangle + \mu_nM^* + \gamma\xi_n\end{aligned} \tag{20}$$

where $M^* > \gamma \sup_{n\geq 0}(M + M_0\|A\|^2\|u_n - z\|^2) > 0$. *It is clear that* $-2\langle u_n - z, z\rangle \to 0$, $n \to \infty$, *and* $\sum_{n=1}^{\infty} M^*\mu_n < \infty, \sum_{n=1}^{\infty} \gamma\xi_n < \infty$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n)(\|u_n - z\|^2 + 2\langle x_n, J_1 z\rangle < \infty$. *Now, using Lemma 8 in* (20), *we have* $\phi(z, x_n) \to 0$. *Therefore,* $x_n \to z$ *as* $n \to \infty$.

Case 2. *Assume that there exists a subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$, *such that* $\phi(z, x_{n_j}) < \phi(z, x_{n_j+1})$, $\forall j \in \mathbb{N}$. *By Lemma 9, there exists a nondecreasing sequence* $\{\tau(n)\}$ *of* $\mathbb{N}$, *such that for all* $n \geq n_0$ *(for some* n_0 *large enough)* $\tau(n) \to \infty$ *as* $n \to \infty$ *and such that the following inequalities hold:*

$$\phi(z, x_n) < \phi(z, x_{\tau(n)+1}), \quad \phi(z, x_{\tau(n)}) < \phi(z, x_{\tau(n)+1})$$

By a similar argument as in Case 1, we obtain:

$$\begin{aligned}\phi(z, x_{\tau(n)+1}) &\leq (1 - r_{\tau(n)})\phi(z, x_{\tau(n)}) - 2r_{\tau(n)}\langle x_{\tau(n)} - z, J_1 z\rangle \\ &\quad + (1-\alpha_{\tau(n)})(\|u_{\tau(n)} - z\|^2 + 2\langle x_{\tau(n)}, J_1 z\rangle) + \gamma\mu_{\tau(n)}M^* + \gamma\xi_{\tau(n)}\end{aligned} \tag{21}$$

and $\lim\langle x_{\tau(n)} - z, J_1 z\rangle = 0$. *Since* $\phi(z, x_{\tau(n)}) \leq \phi(z, x_{\tau(n)+1})$, *we have:*

$$\begin{aligned}r_{\tau(n)}\phi(z, x_{\tau(n)}) &\leq \phi(z, x_{\tau(n)}) - \phi(z, x_{\tau(n)+1}) - 2r_{\tau(n)}\langle x_{\tau(n)} - z, J_1 z\rangle \\ &\quad + (1-\alpha_{\tau(n)})(\|u_{\tau(n)} - z\|^2 + 2\langle x_{\tau(n)}, J_1 z\rangle) + \gamma\mu_{\tau(n)}M^* + \gamma\xi_{\tau(n)}\end{aligned}$$

By our assumption that $r_{\tau(n)} > 0$, *we obtain:*

$$\phi(z, x_{\tau(n)}) \leq -2r_{\tau(n)}\langle x_{\tau(n)} - z, J_1 z\rangle + (1-\alpha_{\tau(n)})(\|u_{\tau(n)} - z\|^2 + 2\langle x_{\tau(n)}, J_1 z\rangle) + \gamma\mu_{\tau(n)}M^* + \gamma\xi_{\tau(n)}$$

which implies that $\lim_{n\to\infty}\phi(\bar{x}, x_{\tau(n)}) = 0$. *It now follows from* (21) *that* $\lim_{n\to\infty}\phi(\bar{x}, x_{\tau(n)+1}) = 0$. *Now, since* $\phi(\bar{x}, x_n) < \phi(\bar{x}, x_{\tau(n)+1})$, *we obtain that* $\phi(\bar{x}, x_n) \to 0$. *Finally, we conclude from Lemma 5 that* $\{x_n\}$ *converges strongly to* $\bar{x}$.

□

Theorem 2. *Let* E_1 *be a real uniformly-convex and two-uniformly-smooth Banach space with the best smoothness constant t satisfying* $0 < t < \frac{1}{\sqrt{2}}$, *and let* E_2 *be a real smooth Banach space. Let* $A : E_1 \to E_2$ *be a bounded linear operator and* A^* *be its adjoint. Let* $T_i : E_2 \to E_2$ $(i \in \mathbb{N})$ *be an infinite family of k-quasi-strict pseudocontractive mappings and* $\{S_i\}_{i=1}^{\infty} : E_1 \to E_1$ *be an infinite family of uniformly* L_i*-Lipschitzian continuous and totally quasi-*ϕ*-asymptotically nonexpansive mappings. Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in E_1$:

$$\begin{cases} u_n = J_1^{-1}(\alpha_{n,0}J_1 x_n + \sum_{i=1}^{\infty}\alpha_{n,i}(\gamma A^* J_2(T_i - I)Ax_n)) \\ y_{n,m} = J_1^{-1}(\beta_n J_1 x_1 + (1-\beta_n)J_1 S_m^n x_n) \\ C_{n+1} = \{z \in C_n : sup_{m\geq 1}\phi(z, y_{n,m}) \leq \beta_n\phi(z, x_1) + (1-\beta_n)(\phi(z, x_n) + \|x_n\|^2 + \|z\|^2) + \xi_n\} \\ x_{n+1} = \Pi_{C_{n+1}}x_1 \end{cases} \tag{22}$$

where $\xi_n = \nu_n \sup_{z\in\Omega}\zeta(\phi(z, u_n)) + \mu_n, \gamma \in (0, \frac{1-k}{2\|A\|^2})$, *and* $\Pi_{C_{n+1}}$ *is the generalized projection of E onto* C_{n+1}; *and the sequences* $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0,1)$ *and satisfy the following conditions:*

(a) $\{\beta_n\} \subset [0,1]$ *and* $\lim_{n\to\infty}\beta_n = 0$
(b) $\{\alpha_{n,i}\} \subseteq [0,1], \sum_{i=0}^{\infty}\alpha_{n,i} = 1$ *and* $\lim_{n\to\infty}\alpha_{n,0} = 1$

If $\Omega = \{x \in \cap_{m=1}^{\infty}F(S_m) : Ax \in \cap_{i=1}^{\infty}F(T_i)\}$ *is nonempty and bounded and* $\mu_1 = 0$, *then* $\{x_n\}$ *converges strongly to:* $\Pi_\Omega u$.

Proof. (I) Both Ω and C_n, $n \geq 1$, are closed and convex.

We know from Lemma 11 and Lemma 12 that $F(T_i)$ and $F(S_i)$, $i \geq 1$, are closed and convex. This implies that Ω is closed and convex. Again, by the assumption, $C_1 = E_1$ is closed and convex. Now, suppose that C_n is closed and convex for some $n \geq 1$. In view of the definition of ϕ, we have:

$$\begin{aligned}
C_{n+1} &= \{z \in C_n : \sup_{m\geq 1} \phi(z, y_{n,m}) \leq \beta_n \phi(z, x_1) + (1-\beta_n)(\phi(z, x_n) + 2\langle z, J_1 x_n\rangle) + \xi_n\} \\
&= \cap_{m\geq 1}\{z \in E_1 : \phi(z, y_{n,m}) \leq \beta_n \phi(z, x_1) + (1-\beta_n)(\phi(z, x_n) + 2\langle z, J_1 x_n\rangle) + \xi_n\} \cap C_n \\
&= \cap_{m\geq 1}\{z \in E_1 : 2\beta_n\langle z, J_1 x_1\rangle + 2(1-\beta_n)\langle z, J_1 x_n\rangle - 2\langle z, y_{n,m}\rangle \leq \beta_n\|x_1\|^2 + 2(1-\beta_n)\|x_n\|^2 \\
&\quad - \|y_{n,m}\|^2 + \|z\|^2\} \cap C_n
\end{aligned}$$

from which, it follows that C_{n+1} is closed and convex.

(II) $\Omega \subset C_n, n \geq 1$.

It is clear that $\Omega \subset E_1$. Suppose that $\Omega \subset C_n$ for some $n \geq 1$. Let $u \in \Omega \subset C_n$, then we have:

$$\begin{aligned}
\phi(u, u_n) &= \phi(u, J_1^{-1}(\alpha_{n,0} J_1 x_n + \sum_{i=1}^{\infty} \alpha_{n,i}(\gamma A^* J_2(T_i - I)Ax_n))) \\
&\leq \alpha_{n,0}\phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}\phi(u, J_1^{-1}(\gamma A^* J_2(T_i - I)Ax_n)) \\
&\leq \phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}[\phi(x_n, J_1^{-1}(\gamma A^* J_2(T_i - I)Ax_n) \\
&\quad + 2\langle u - x_n, J_1 x_n - \gamma A^* J_2(T_i - I)Ax_n\rangle \\
&\leq \phi(u, x_n) + \sum_{i=1}^{\infty} \alpha_{n,i}[\|x_n\|^2 + 2\langle u - x_n, J_1 x_n\rangle + \gamma^2\|A\|^2\|(T_i - I)Ax_n\|^2 \\
&\quad - 2\langle x_n, J_1^{-1}(\gamma A^* J_2(T_i - I)Ax_n)\rangle + 2\langle u - x_n, \gamma A^* J_2(T_i - I)Ax_n\rangle
\end{aligned} \tag{23}$$

From Lemma 1, we have:

$$\begin{aligned}
-2\langle x_n, \gamma A^* J_2(T_i - I)Ax_n\rangle &\leq \|\gamma A^* J_2(T_i - I)Ax_n\|^2 + 2\|tx_n\|^2 - \|x_n + \gamma A^* J_2(T_i - I)Ax_n\|^2 \\
&\leq \gamma^2\|A\|^2\|(T_i - I)Ax_n\|^2 + \|x_n\|^2
\end{aligned} \tag{24}$$

Since $Au \in \cap_{i=1}^{\infty} F(T_i)$ and T_i is a k-quasi-strictly pseudocontractive mapping:

$$\begin{aligned}
\langle x_n - u, \gamma A^* J_2(T_i - I)Ax_n\rangle &= \gamma\langle A(x_n - u), J_2(T_i - I)Ax_n\rangle \\
&= \gamma\langle A(x_n - u) + (T_i - I)Ax_n - (T_i - I)Ax_n, J_2(T_i - I)Ax_n\rangle \\
&= \gamma(\langle T_i A(x_n) - Au, J_2(T_i - I)Ax_n\rangle - \|(T_i - I)Ax_n\|^2) \\
&\leq \gamma(\frac{1}{2}(\|T_i Ax_n - Au\|^2 + \|(T_i - I)Ax_n\|^2)) - \gamma\|(T_i - I)Ax_n\|^2 \\
&= \frac{\gamma}{2}(\|T_i Ax_n - Au\|^2 - \|(T_i - I)Ax_n\|^2) \\
&\leq \frac{\gamma}{2}(\|Ax_n - Au\|^2 + (k-1)\|(T_i - I)Ax_n\|^2) \\
&\leq \frac{1}{2}\|x_n - u\|^2 + \frac{\gamma}{2}(k-1)\|(T_i - I)Ax_n\|^2
\end{aligned} \tag{25}$$

Substituting (24) and (25) into (23), we obtain:

$$\begin{aligned}\phi(u,u_n) &\leq \alpha_{n,0}\phi(u,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}\phi(u, J_1^{-1}(\gamma A^* J_2(T_i - I)Ax_n)) \\ &\leq \phi(u,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}[2\langle u, J_1 x_n\rangle - \gamma(1-k-2\gamma\|A\|^2)\|(T_i - I)Ax_n\|^2 + \|x_n - u\|^2] \\ &\leq \phi(u,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}(\|x_n\|^2 + \|u\|^2) - \gamma(1-k-2\gamma\|A\|^2)\|(T_i - I)Ax_n\|^2\end{aligned} \tag{26}$$

It now follows from Lemma 2(d) and Equation (22):

$$\begin{aligned}\phi(u,y_{n,m}) &\leq \beta_n\phi(u,x_1) + (1-\beta_n)\phi(u, S_n^m u_n) \\ &\leq \beta_n\phi(u,x_1) + (1-\beta_n)[\phi(u,u_n) + \nu_n\zeta(\phi(u,u_n)) + \mu_n] \\ &\leq \beta_n\phi(u,x_1) + (1-\beta_n)[\phi(u,u_n) + \nu_n \sup_{u\in\Omega}\zeta(\phi(u,u_n)) + \mu_n] \\ &= \beta_n\phi(u,x_1) + (1-\beta_n)(\phi(u,u_n) + \xi_n) \quad \forall m \geq 1 \\ &\leq \beta_n\phi(u,x_1) + (1-\beta_n)(\phi(u,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}(\|x_n\|^2 + \|u\|^2) \\ &+ \xi_n) - \gamma(1-2\gamma\|A\|^2)\|(T_i - I)Ax_n\|^2 \quad \forall m \geq 1 \\ &\leq \beta_n\phi(u,x_1) + (1-\beta_n)(\phi(u,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}(\|x_n\|^2 + \|u\|^2) + \xi_n) \quad \forall m \geq 1\end{aligned} \tag{27}$$

Therefore, we have:

$$\begin{aligned}\sup_{m\geq 1}\phi(u,y_{n,m}) &\leq \beta_n\phi(u,x_1) + (1-\beta_n)(\phi(u,x_n) + \sum_{i=1}^{\infty}\alpha_{n,i}(\|x_n\|^2 + \|u\|^2) + \xi_n) \\ &\leq \beta_n\phi(u,x_1) + (1-\beta_n)(\phi(u,x_n) + \|x_n\|^2 + \|u\|^2 + \xi_n)\end{aligned} \tag{28}$$

This argument shows that $u \in C_{n+1}$, and so, $F \subset C_{n+1}$.

(III) $\{x_n\}$ converges strongly to some point $p^* \in E_1$.

Since $x_n = \Pi_{C_n}x_1$, from Lemma 6, we have $\langle x_n - y, J_1x_1 - J_1x_n\rangle \geq 0$, $\forall y \in C_n$. Again, since $\Omega \subset C_n$, we obtain $\langle x_n - u, J_1x_1 - J_1x_n\rangle \geq 0$, $\forall u \in \Omega$. It now follows from Lemma 2(a) that for each $u \in \Omega$ and each $n \geq 1$:

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \leq \phi(u,x_1) - \phi(u,x_n) \leq \phi(u,x_1) \tag{29}$$

Therefore, $\{\phi(x_n,x_1)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_n+1}x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n,x_1) \leq \phi(x_{n+1},x_1)$, $n \geq 1$. This implies that $\{\phi(x_n,x_1)\}$ is nondecreasing. Hence, $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. Since E is reflexive, there exists a subsequence $x_{n_i} \subset x_n$, such that $x_{n_i} \rightharpoonup p^*$ (some point in E_1). Since C_n is closed and convex and $C_{n+1} \subset C_n$, it follows that C_n is weakly closed and $p^* \in C_n$ for each $n \geq 1$. Now, in view of $x_{n_i} = \Pi_{C_{n_i}}x_1$, we have $\phi(x_{n_i},x_1) \leq \phi(p^*,x_1)$, $\forall n_i \geq 1$. Since the norm $\|.\|$ is weakly lower semicontinuous, we have:

$$\liminf_{n_i\to\infty}\phi(x_{n_i},x_1) = \liminf_{n_i\to\infty}\{\|x_{n_i}\|^2 + \|x_1\|^2 - 2\langle x_{n_i}, J_1x_1\rangle\} \geq \|p^*\|^2 + \|x_1\|^2 - 2\langle p^*, x_1\rangle = \phi(p^*,x_1)$$

and so:

$$\phi(p^*,x_1) \leq \liminf_{n_i\to\infty}\phi(x_{n_i},x_1) \leq \limsup_{n_i\to\infty}\phi(x_{n_i},x_1) \leq \phi(p^*,x_1)$$

This implies that $\lim_{n_i}\phi(x_{n_i},x_1) = \phi(x_1,p^*)$, and so, $\|x_{n_i}\| \to \|p^*\|$. Since $x_{n_i} \rightharpoonup p^*$ and E_1 is uniformly convex, we obtain $\lim_{n_i\to\infty}x_{n_i} = p^*$. Now, the convergence of $\{\phi(x_n,x_1)\}$, together with $\lim_{n_i\to\infty}\phi(x_{n_i},x_1)=\phi(p^*,x_1)$, implies that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(p^*,x_1)$. If there exists some subsequence $\{x_{n_j}\}\subset\{x_n\}$, such that $x_{n_j}\to q$, then from Lemma 2(a), we have:

$$\begin{aligned}\phi(p^*,q)&=\lim_{n_i,n_j\to\infty}\phi(x_{n_i},x_{n_j})=\lim_{n_i,n_j\to\infty}\phi(x_{n_i},\Pi_{C_j}x_1)\le\lim_{n_i,n_j\to\infty}(\phi(x_{n_i},x_1)-\phi(\Pi_{C_j}x_1,x_1))\\&\le\lim_{n_i,n_j\to\infty}(\phi(x_{n_i},x_1)-\phi(x_{n_j},x_1))=\phi(p^*,q)-\phi(p^*,q)=0\end{aligned}$$

i.e., $p^*=q$, and so:

$$\lim_{n\to\infty}x_n=p^* \tag{30}$$

By the way, it follows from from (26) that $\phi(u,u_n)$ is bounded, so:

$$\lim_{n\to\infty}\xi_n=\lim_{n\to\infty}\{\nu_n\sup_{p\in\Omega}\zeta(\phi(p,u_n))+\mu_n\}=0 \tag{31}$$

(IV) $p^*\in\Omega$. Since $x_{n+1}\in C_{n+1}$, from (28), (30) and (31):

$$\sup_{m\ge1}\phi(x_{n+1},y_{n,m})\le\beta_n\phi(x_{n+1},x_1)+(1-\beta_n)[\phi(x_{n+1},x_n)+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}(\|x_n\|^2+\|x_{n+1}\|^2)+\xi_n]\to0 \tag{32}$$

Since $x_{n+1}\in C_{n+1}$, from (27) and (32) we have:

$$\begin{aligned}\gamma(1-k-2\gamma\|A\|^2)\|(T_i-I)Ax_n\|^2&\le\beta_n\phi(x_{n+1},x_1)+(1-\beta_n)(\phi(x_{n+1},x_n)\\&\quad+\sum_{i=1}^{\infty}\alpha_{n,i}(\|x_{n+1}\|^2+\|x_n\|^2)+\xi_n)-\phi(x_{n+1},y_{n,m})\to0\quad n\to\infty\end{aligned} \tag{33}$$

Since $\gamma\in(0,\frac{1-k}{2\|A\|^2})$, we have:

$$\|(T_i-I)Ax_n\|\to0\quad n\to\infty \tag{34}$$

Since $x_n\to p^*$, it follows from (32) and Lemma 5 that for each $m\ge1$:

$$\lim_{n\to\infty}y_{n,m}=p^* \tag{35}$$

Since $\{x_n\}$ is a bounded sequence and $\{S_m\}_{m=1}^{\infty}$ is uniformly totally quasi-asymptotically nonexpansive, $\{S_m^nx_n\}_{m,n=1}^{\infty}$ is uniformly bounded. In view of $\beta_n\to0$ and (22), we conclude that for each $m\ge1$:

$$\|J_1y_{n,m}-J_1S_m^nx_n\|=\lim_{n\to\infty}\beta_n\|J_1x_1-J_1S_m^nx_n\|=0 \tag{36}$$

Since for each $m\ge1$, $J_1y_{n,m}\to J_1p^*$, it follows that for each $m\ge1$, $\lim_{n\to\infty}J_1S_m^nx_n=J_1p^*$. Since J_1 is continuous on each bounded subset of E_1, for each $m\ge1$:

$$\lim_{n\to\infty}S_m^nx_n=p^* \tag{37}$$

On the other hand, by the assumption that for each $m\ge1$, S_m is uniformly L_m-Lipschitzian continuous, we have:

$$\begin{aligned}\|S_m^{n+1}x_n-S_m^nx_n\|&\le\|S_m^{n+1}x_n-S_m^{n+1}x_{n+1}\|+\|S_m^{n+1}x_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-S_m^nx_n\|\\&\le(L_m+1)\|x_{n+1}-x_n\|+\|S_m^{n+1}x_{n+1}-x_{n+1}\|+\|x_n-S_m^nx_n\|\end{aligned} \tag{38}$$

From (37) and $x_n\to p^*$, we have that $\lim_{n\to\infty}\|S_m^{n+1}x_n-S_m^nx_n\|=0$ and $\lim_{n\to\infty}S_m^{n+1}x_n=p^*$, i.e., $\lim_{n\to\infty}S_mS_m^nx_n=p^*$. In view of the closedness of S_m, it follows that $S_mp^*=p^*$, i.e., for each

$m \geq 1, p^* \in F(S_m)$. By the arbitrariness of $m \geq 1$, we have $p^* \in \cap_{m=1}^{\infty} F(S_m)$. On the other hand, since A is bounded, it follows from $x_{n_i} \rightharpoonup p^*$ that $Ax_{n_i} \rightharpoonup Ap^*$. Hence, from (34), we have that:

$$\|T_i A x_{n_i} - A x_{n_i}\| \longrightarrow 0, \quad i \to \infty$$

Since T_i is demi-closed at zero, we have that $Az \in F(T_i)$. Hence, $z \in \Omega$.

(V) Finally, $p^* \in \Pi_\Omega x_1$, and so, $x_n \to \Pi_\Omega x_1$.

Let $w = \Pi_\Omega x_1$. Since $w \in \Omega \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$, $n \geq 1$. This implies that $\phi(p^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_{1)} \leq \phi(w, x_1)$. Since $w = \Pi_\Omega x_1$, it follows that $p^* = w$, and hence, $x_n \to p^* = \Pi_\Omega x_1$. □

Corollary 1. *Let E_1 be a real uniformly-convex and two-uniformly-smooth Banach space with the best smoothness constant t satisfying $0 < t < \frac{1}{\sqrt{2}}$, and let E_2 be a real smooth Banach space. Let $A : E_1 \to E_2$ be a bounded linear operator and A^* be its adjoint. Let $T : E_2 \to E_2$ be a k-quasi-strict pseudocontractive mapping and T be demi-closed at zero. Let $\{S_n\}_{n=1}^{\infty} : E_1 \to CB(E_1)$ be a family of multivalued quasinonexpansive mappings, such that for each $i \geq 1$, S_i is demi-closed at zero. Assume that for each $p \in Fix(S_i)$, $S_i(p) = \{p\}$. Let $\{x_n\}$ be the sequence generated by $x_1 \in E_1$:*

$$\begin{cases} u_n = (1 - r_n)x_n \\ y_n = J_1^{-1}(\alpha_n J_1 u_n + (1 - \alpha_n)\gamma A^* J_2(T - I)Au_n) \\ x_{n+1} = J_1^{-1}(\beta_{n,0} J_1 y_n + \sum_{i=1}^{\infty} \beta_{n,i} J_1 w_{n,i}) \quad w_{n,i} \in S_i y_n \end{cases}$$

where $\gamma \in (0, \frac{1-k}{2\|A\|^2})$; the sequences $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0,1)$ satisfy the following conditions:

(a) $\sum_{i=0}^{\infty} \beta_{n,i} = 1$ *and* $\liminf_n \beta_{n,0}\beta_{n,i} > 0$,
(b) $\lim_{n\to\infty} \alpha_n = 1$, $\sum_{n=1}^{\infty}(1 - \alpha_n) < \infty$ *and* $(1 - \alpha_n) = o(r_n)$.

Then, $\{x_n\}$ converges strongly to an element of Ω.

Proof. Since every k-quasi-strictly pseudocontractive mapping is clearly $(k, 0, 0)$-totally asymptotically strictly pseudocontractive, the result follows. □

Corollary 2. *Let E_1 be a real uniformly-convex and two-uniformly-smooth Banach space with the best smoothness constant t satisfying $0 < t < \frac{1}{\sqrt{2}}$, and let E_2 be a real smooth Banach space. Let $A : E_1 \to E_2$ be a bounded linear operator and A^* be its adjoint. Let $T : E_2 \to E_2$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\})$-totally asymptotically strictly pseudocontractive mapping satisfying the following conditions:*

(a) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \xi_n < \infty$,
(b) *$\{r_n\}$ is a real sequence in $(0,1)$, such that $\mu_n = o(r_n), \xi_n = o(r_n), \lim r_n = 0, \sum_{n=1}^{\infty} r_n = \infty$,*
(c) *there exist constants $M_0 > 0, M_1 > 0$, such that $\zeta(\lambda) \leq M_0\lambda^2, \forall \lambda > M_1$.*

Let $\mathfrak{F} = \{S(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on E_1. Suppose further that $\Omega = \{x \in \cap_{t\geq 0} F(S(t)) : Ax \in F(T)\} \neq \emptyset$, and $\{x_n\}$ is the sequence generated by $x_1 \in E_1$:

$$\begin{cases} u_n = (1 - r_n)x_n \\ y_n = J_1^{-1}(\alpha_n J_1 u_n + (1 - \alpha_n)\gamma A^* J_2(T^n - I)Au_n) \\ x_{n+1} = J_1^{-1}(\beta_n J_1 y_n + (1 - \beta_n)(\frac{1}{t_n}\int_0^{t_n} S(u)du J_1 y_n) \end{cases}$$

where $\gamma \in (0, \frac{1-k}{2\|A\|^2})$; the sequence $\{\alpha_n\} \subset (0,1), 0 < \epsilon \leq \beta_n \leq b < 1$, and $\lim_{n\to\infty} \alpha_n = 1$, $\sum_{n=1}^{\infty}(1 - \alpha_n) < \infty$ and $(1 - \alpha_n) = o(r_n)$. Then, $\{x_n\}$ converges strongly to to an element of Ω.

Proof. Since $\{\sigma_t(x) = \frac{1}{t}\int_0^t S(u)xdu : t \geq 0\}$ is a u.a.r. nonexpansive semigroup, the result follows from Corollary 1. □

In the following, we shall provide an example to illustrate the main result of this paper.

Example 1. *Let C be the unit ball of the real Hilbert space l^2, and let $T : C \to C$ be a mapping defined by:*

$$T(x_1, x_2, ...) = (0, x_1, a_2x_2, a_3x_3, ...)$$

where $\{a_i\}$ is a sequence in $(0,1)$, such that $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. It was shown in [27] that T is a $(0, k_n - 1, \xi_n)-$totally asymptotically strictly pseudocontractive mapping and $F(T) = \{0\}$, where $k_n = 2\prod_{i=2}^{n} a_i$. Let B be the unit interval in $\mathbb{R}$, and let $S_i : B \to B$ be a mapping defined by:

$$S_i(x) = \begin{cases} \dfrac{1}{2^i}x & x \in [0, \dfrac{1}{2}] \\ 0 & x \in (\dfrac{1}{2}, 1] \end{cases}$$

Then, $\cap_{i=1}^{\infty} Fix(S_i) = \{0\}$ and:

$$|S_ix - 0| = |\frac{1}{2^i}x - 0| = \frac{1}{2^i}|x| \leq |x|$$

Therefore, each S_i is a quasinonexpansive mapping. Let $A : B \to C$ be the linear operator defined by:

$$A(x) = (0, x, a_2x, a_3a_2x, a_4a_3a_2x, ...), \quad x \in B \subset \mathbb{R}.$$

Then, A is bounded and $\|A\| = 1 + a_2^2 + (a_3a_2)^2 + (a_4a_3a_2)^2 + \cdots$. It now follows that:

$$A^* : C \to B, \quad A^*(x_1, x_2, \cdots) = x_2 + a_2x_3 + a_3a_2x_4 + a_4a_3a_2x_5 + \cdots.$$

We now put, for $n \in \mathbb{N}$, $\alpha_n = \frac{1}{3}$, $r_n = \frac{1}{n}$, $\beta_{n,0} = \frac{1}{2}$, $\beta_{n,0} = \frac{1}{3^i}$ and $\lambda = \frac{1}{4}(1 + a_2^2 + \cdots + (a_n \cdots a_2)^2)$. Furthermore, we have:

$$\Omega = \{x \in F(T) : Ax \in \cap_{i=1}^{\infty} F(S_i)\} = \{0\}$$

Now, all of the assumptions in Theorem 1 are satisfied. Let us consider the following numerical algorithm:

$$T^n(x_1, x_2, ...) = (0, 0, ..., 0, a_n...a_2x_1^2, a_{n+1}...a_2x_2, ...)$$

$$T^n(Au_n) - Au_n = (0, -u_n, -a_2u_n, -a_3a_2u_n, ..., -a_n...a_2u_n, 0, 0, ...$$

$$A^*(T^n(Au_n) - Au_n) = -u_n(1 + a_2^2 + (a_3a_2)^2 + ... + (a_n...a_2)^2)$$

$$y_n = \frac{1}{6}u_n = \frac{1}{6}(1 - \frac{1}{n})x_n, \quad x_{n+1} = \frac{1}{2}y_n + \sum_{i=1}^{\infty}\frac{1}{3^i}(\frac{1}{2^i}y_n) = \frac{1}{10}y_n$$

$$x_{n+1} = \frac{1}{60}(1 - \frac{1}{n})x_n$$

By Theorem 1, the sequence $\{x_n\}$ converges to the unique element of Ω.

4. Application

Let E be a uniformly-smooth Banach space, E^* be the dual of E, J be the duality mapping on E and $F : E \to 2^{E^*}$ be a multi-valued operator. Recall that F is called monotone if $\langle u - v, x - y\rangle \geq 0$, for any $(x, u), (y, v) \in G(F)$, where $G(F) = \{(x, u) : x \in D(F), u \in F(x)\}$. A monotone operator F is said to be maximally monotone if its graph $G(F)$ is not properly contained in the graph of any other monotone operator. For a maximally-monotone operator $F : E \to 2^{E^*}$ and $r > 0$, we can define a single-valued operator:

$$J_r^F = (J + rF)^{-1} J : E \to E$$

It is known that for any $r > 0$, J_r^F is firmly nonexpansive, and its domain is all of E, also $0 \in F(x)$ if and only if $x \in Fix(J_r^F)$.

Theorem 3. *Let E_1 be a real uniformly-convex and two-uniformly-smooth Banach space with the best smoothness constant t satisfying $0 < t < 1/\sqrt{2}$, and let E_2 be a real smooth Banach space and $T : E_1 \to E_2$ be a bounded linear operator. Let $A : E_2 \to 2^{E_2^*}$ and $B_i : E_1 \to 2^{E_1^*}$, for $i = 1, 2, ...$, be maximal monotone mappings, such that $A^{-1}0 \neq \emptyset$ and $\cap_{i=1}^{\infty} B_i^{-1}0 \neq \emptyset$. Suppose:*

$$\Omega = \{x \in E_1 : 0 \in \cap_{i=1}^{\infty} B_i(x) \quad \text{such that} \quad 0 \in A(Tx)\} \neq \emptyset$$

Let $\{x_n\}$ be a sequence generated by $x_0 \in E_1$ and:

$$\begin{cases} u_n = (1 - r_n)x_n \\ y_n = J_1^{-1}(\alpha_n J_1 u_n + (1 - \alpha_n)\gamma T^* J_2(J_r^A T u_n - T u_n) \\ x_{n+1} = J_1^{-1}(\beta_{n,0} J_1 y_n + \sum_{i=1}^{\infty} \beta_{n,i} J_1 J_\mu^{B_i} y_n \end{cases}$$

where $r, \mu > 0, \gamma \in (0, \frac{1-k}{2\|T\|^2})$, and the sequences $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0, 1)$ satisfy the following conditions:

(1) $\sum_{i=0}^{\infty} \beta_{n,i} = 1$ *and* $\liminf_n \beta_{n,0}\beta_{n,i} > 0$,

(2) $\lim_{n\to\infty} \alpha_n = 1$, $\sum_{n=1}^{\infty}(1 - \alpha_n) < \infty$ *and* $(1 - \alpha_n) = o(r_n)$.

Then, $\{x_n\}$ converges strongly to an element of Ω.

Proof. Since J_r^A and $J_\mu^{B_i}$ are nonexpansive, the result follows from Corollary 1. □

Remark 1. *Set $S_i = J_r^{B_i}$ in Corollary 1, where B_i is a maximal monotone mapping, then Corollary 1 improves Theorem 4.2 of Takahashi et al. [12].*

Moudafi [28] introduced the split monotone variational inclusion (SMVIP) in Hilbert spaces. We present the SMVIP in a Banach space. Let E_1 and E_2 be two real Banach spaces and J_1 and J_2 be the duality mapping of E_1 and E_2, respectively. Given the operators $f : E_1 \to E_1$, $g : E_2 \to E_2$, a bounded linear operator $A : E_1 \to E_2$ and two multi-valued mappings $B_1 : E_1 \to 2^{E_1^*}$ and $B_2 : E_2 \to 2^{E_2^*}$, the SMVI is formulated as follows:

$$\text{find a point} \quad x \in C \quad \text{such that} \quad 0 \in J_1(f(x)) + B_1(x)$$

$$\text{and such that the point:}$$

$$y = A(x) \in E_2 \quad \text{solves} \quad 0 \in J_2(g(y)) + B_2(y)$$

Note that if C and Q are nonempty closed convex subsets of E_1 and E_2, (resp.) and $B_1 = N_C$ and $B_2 = N_Q$, where N_C and N_Q are normal cones to C and Q (resp.), then the split monotone variational inclusion problem reduces to the split variational inequality problem (SVIP), which is formulated as follows: find a point:

$$x \in C \quad \text{such that} \quad \langle J_1(f(x)), w - x\rangle \geq 0 \quad \text{for all} \quad w \in C$$

$$\text{and such that the point:}$$

$$y = Ax \in Q \quad \text{solves} \quad \langle J_2(g(y)), z - y\rangle \geq 0 \quad \text{for all} \quad z \in Q$$

SVIP is quite general and enables the split minimization between two spaces in such a way that the image of a solution of one minimization problem, under a given bounded linear operator, is a solution of another minimization problem.

Let $h : C \to E$ be an operator, and let $C \subset E$. The operator h is called inverse strongly monotone with constant $\beta > 0$, or in brief $(\beta - ism)$, on E if:

$$\langle h(x) - h(y), Jx - Jy\rangle \geq \beta\|h(x) - h(y)\|^2, \quad \forall x, y \in C$$

Remark 2. *If $h : E \to E$ is an $\alpha - ism$ operator on E and $B : E \to 2^{E^*}$ is a maximal monotone mapping, then $J_\lambda^B(I - \lambda h)$ is averaged for each $\lambda \in (0, 2\alpha)$.*

Theorem 4. *Let E_1 be a real uniformly-convex and two-uniformly-smooth Banach space with the best smoothness constant t satisfying $0 < t < 1/\sqrt{2}$, and let E_2 be a real smooth Banach space and $T : E_1 \to E_2$ be a bounded linear operator. Let $A : E_2 \to 2^{E_2^*}$ and, for $i = 1, 2, ...,$ $B_i : E_1 \to 2^{E_1^*}$ be maximal monotone mappings, such that $A^{-1}0 \neq \emptyset$ and $\cap_{i=1}^\infty B_i^{-1}0 \neq \emptyset$; and that $h : E_2 \to E_2$ is an $\alpha - ism$ operator and $g_i : E_1 \to E_1$ is a $\gamma_i - ism$ operator. Assume that $\rho = \alpha\, inf_{i\in\mathbb{N}}\gamma_i > 0$ and $\tau \in (0, 2\rho)$. Suppose SMVI:*

$$\begin{cases} x \in \cap_{i=1}^\infty B_i^{-1}0 & 0 \in J_1(g_i(x)) + B_i(x) \qquad \forall i \in \mathbb{N} \\ Tx \in A^{-1}0 & 0 \in J_2(h(Tx)) + A(Tx) \end{cases}$$

has a nonempty solution set Ω. Let $\{x_n\}$ be a sequence generated by $x_0 \in E_1$ and:

$$\begin{cases} u_n = (1 - r_n)x_n \\ y_n = J_1^{-1}(\alpha_n J_1 u_n + (1 - \alpha_n)\gamma T^* J_2((J_r^A(I - \tau h) - I)Tu_n)) \\ x_{n+1} = J_1^{-1}(\beta_{n,0} J_1 y_n + \sum_{i=1}^\infty \beta_{n,i} J_1 J_\mu^{B_i}(I - \tau g_i)y_n) \end{cases}$$

where $\gamma \in (0, \frac{1-k}{2\|T\|^2})$; the sequences $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0, 1)$ satisfy the following conditions:

(1) $\sum_{i=0}^\infty \beta_{n,i} = 1$ *and* $\liminf_n \beta_{n,0}\beta_{n,i} > 0,$

(2) $\lim_{n\to\infty} \alpha_n = 1, \sum_{n=1}^\infty(1 - \alpha_n) < \infty$ *and* $(1 - \alpha_n) = o(r_n).$

Then, $\{x_n\}$ converges strongly to an element of Ω.

Proof. The results follow from Remark 2, Lemma 4(iii) and Corollary 1. □

We mention in passing that the above theorem improves and extends Theorems 6.3 and 6.5 of [13] to Banach spaces. Indeed, we removed an extra condition and obtained a strong convergence theorem, which is more desirable than the weak convergence already obtained by the authors.

Acknowledgments: We wish to thank the academic editor for his right choice of reviewers, and the anonymous reviewers for their comments and criticisms.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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