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Article

Solution of the Master Equation for Quantum Brownian Motion Given by the Schrödinger Equation

R. Sinuvasan [1,†], Andronikos Paliathanasis [2,3,*,†], Richard M. Morris [4,†] and Peter G. L. Leach [4,5,†]

1 Department of Mathematics, Pondicherry University, Kalapet Puducherry 605 014, India; rsinuvasan@gmail.com
2 Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia 5090000, Chile
3 Institute of Systems Science, Durban University of Technology, PO Box 1334, Durban 4000, South Africa
4 Department of Mathematics and Institute of Systems Science, Research and Postgraduate Support, Durban University of Technology, PO Box 1334, Durban 4000, South Africa; rmcalc85@gmail.com (R.M.M.); leach@ucy.ac.cy (P.G.L.L.)
5 School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa
* Correspondence: anpaliat@phys.uoa.gr; Tel.: +56-63-2221200
† These authors contributed equally to this work.

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Abstract: We consider the master equation of quantum Brownian motion, and with the application of the group invariant transformation, we show that there exists a surface on which the solution of the master equation is given by an autonomous one-dimensional Schrödinger Equation.

Keywords: quantum Brownian motion; master equation; group invariant transformations

1. Introduction

With the method of path integrals—specifically the Feynman–Vernon influence functional [1]—Haake and Reibold [2] and few years later Hu, Paz, and Zhang derived an equation that inherits the properties of quantum Brownian motion for a harmonic oscillator interacting with a linear passive heat bath of oscillators [3,4]. An alternate derivation of that master equation has been performed by Halliwell and Yu by tracing the evolution equation for the Wigner function of the system [5].

The master equation of quantum Brownian motion is a $(1+2)$ linear nonautonomous evolution equation given by

$$Z_{,t} = -\frac{x}{m} Z_{,y} + m\Omega^2(t) y Z_{,x} + 2\Gamma(t)(xZ)_{,x} + \hbar m\Gamma(t) h(t) Z_{,xx} + \hbar\Gamma(t) f(t) Z_{,xy} \tag{1}$$

where m is the mass of the Brownian particle, and $Z = Z(t,x,y)$ is the Wigner function of the density matrix (x denotes the momentum of the oscillator and y its position). Furthermore, the coefficients, $\Omega^2(t)$, $\Gamma(t)$, $h(t)$, and $f(t)$ in general are time-dependent and related to the natural frequency of the Brownian motion and the terms interacting with the heat bath of oscillators. The derivation of the coefficients is given in [5,6].

A general analytical solution of the master Equation (1) with the use of the Langevin Equation has been derived in [6], whereas in [7], some solutions of the master equations for quantum Brownian motion are given. The analysis of open quantum systems does not stop in Equation (1). A relation of the exact master equation with the nonequilibrium Green functions for non-Markovian open quantum systems was derived in [8], while new phenomena concerning the thermal-state of a quantum system were predicted for a strongly non-Markovian enviroment in [9].

In this work, we are interested in the existence of solutions for Equation (1), which follow from the method of group invariant transformations; in particular, we are interested in the one-parameter point transformations which were introduced by S. Lie [10], where the generator of the infinitesimal transformation is called a Lie (point) symmetry. The importance of Lie symmetries is that they provide a systematic method to facilitate the solution of differential equations, because they provide first-order invariants which can be used to reduce the order of differential equations. Moreover, Lie symmetries can be used for the classification of differential equations, and important information for the differential equation can be extracted from the admitted group of invariant transformations. The method of group invariant transformations has been applied in various systems of quantum mechanics (see [11–16] and references therein).

By applying the Lie theory for differential equations, we show that Equation (1) is invariant under a group of one-parameter point transformations in which the generators form the $\{A_1 \oplus_s W_5\} \oplus_s \infty A_1$, Lie algebra, where W_5 denotes the five-element Weyl–Heisenberg algebra and ∞A_1 is the infinite-dimensional abelian algebra of the solutions of the linear $(1+2)$ evolution equation and follows from the linearity of (1). Furthermore, from the Lie symmetries, we can define a surface in which Equation (1) is independent of one of the independent variables, and with the use of the zeroth-order invariants, we can reduce Equation (1) in a nonautonomous one-dimensional evolution equation. We study the Lie point symmetries of this equation and show that is maximally symmetric. Hence, it is invariant under a group of transformations which form the $\{sl(2,R) \oplus_s W_3\} \oplus_s \infty A_1$ Lie algebra (algebra $sl(2,R)$ is the $A_{3,8}$, and W_3 is the $A_{3,3}$ in the Mubarakzyanov Classification Scheme [17–20]) . From S. Lie's theorem, this indicates that there exists a "coordinate" transformation in which the reduced equation is equivalent to the equation. Hence, solutions of the Schrödinger equation are also solutions of the master Equation (1). The plan of the paper is as follows.

In Section 2, we give the basic properties and definitions of Lie symmetries, and we study the existence of Lie symmetries for the master Equation (1). Furthermore, we apply the zeroth-order invariants of the Lie symmetries and we reduce the original equation to a one-dimensional evolution equation. In Section 3, we study the relationship between the reduced equation and the Schrödinger equation. Finally, we draw our conclusions and give an example in Section 4.

2. Lie Point Symmetries of the Master Equation

For the convenience of the reader we present the basic properties and definitions of Lie symmetries of differential equations.

Consider a differential equation $\Theta\left(x^k, u, u_{,i}, u_{,ij}\right) = 0$, where x^k are the independent variables, and $u = u\left(x^k\right)$ is the dependent variable. Then the differential operator,

$$X = \xi^i\left(x^k, u\right)\partial_i + \eta\left(x^k, u\right)\partial_u \tag{2}$$

is called a Lie symmetry of Θ, if there exists a function λ such that $\mathcal{L}_{X^{[2]}}\Theta = \lambda\Theta$, where $X^{[2]}$ is the second prolongation/extension of the vector field X in the space $\left\{x^k, u, u_{,i}, u_{ij}\right\}$ [21,22].

Lie symmetries of differential equations can be used to determine invariant solutions or transform solutions to solutions [22]. From the Lie symmetry condition, one defines the associated Lagrange's system

$$\frac{dx^i}{\xi^i} = \frac{du}{\eta} = \frac{du_i}{\eta_{[i]}} = \ldots = \frac{du_{ij..i_n}}{\eta_{[ij\ldots i_n]}} \tag{3}$$

the solution of which provides the characteristic functions

$$\Lambda^{[0]}\left(x^k, u\right),\ \Lambda^{[1]i}\left(x^k, u, u_i\right), \ldots, \Lambda^{[n]}\left(x^k, u, u_{,i}, \ldots, u_{ij\ldots i_n}\right) \tag{4}$$

The solution $\Lambda^{[k]}$ is called the kth-order invariant of the Lie symmetry vector, (2). These invariants can be used in order to reduce the order or the number of the independent variables of the differential equations. Another important feature of Lie symmetries of differential equations is that they span the Lie algebra G_L. The application of a Lie symmetry to Θ leads to a new differential equation $\bar{\Theta}$ which is different from Θ, and possibly admits Lie symmetries which are not Lie symmetries of Θ. This means that the reduced equation can have properties different from the original equation. However, the solutions of these equations are related through the point transformation which transformed Θ to $\bar{\Theta}$.

2.1. The Master Equation

In order to simplify the presentation of the calculations, we rewrite Equation (1) in the following form (it is also possible to apply a coordinate transformation, $(x,y) \to (\bar{x},\bar{y})$, which "diagonalises" the second derivatives in Equation (1). However, we prefer to work on the original physical system):

$$-\frac{x}{m}Z_{,y} + p(t)yZ_{,x} + q(t)\left(xZ\right)_{,x} + r(t)Z_{,xx} + s(t)Z_{,xy} - Z_{,t} = 0 \tag{5}$$

where $p\left(t\right) = m\Omega^2(t)$, $q\left(t\right) = 2\Gamma(t)$, $r\left(t\right) = \hbar m\Gamma(t)h(t)$, $s\left(t\right) = \hbar\Gamma(t)f(t)$.

We assume the generator of the one-parameter infinitesimal point transformation to be

$$X = \xi^t\partial_t + \xi^x\partial_x + \xi^y\partial_y + \eta\partial_Z \tag{6}$$

in which ξ^t, ξ^x, ξ^y, and η are functions of $\{t,x,y,Z\}$. Furthermore, because Equation (5) is a linear equation, we have that $\eta = G\left(t,x,y\right)Z + G_0 Z + b\left(t,x,y\right)$, where $b\left(t,x,y\right)$ are solutions of Equation (5) and form the infinite-dimensional Lie algebra ∞A_1, [23].

Hence, from the Lie symmetry condition, we have that (in this work, we used the symbolic package Sym for Mathematica [24]):

$$\xi^t = a(t)\ ,\ \xi^y = f_1(t) \tag{7}$$

$$\xi^x = m\frac{-f_1 ps + 2rf_1' + m\left(qsf_1' + f_1's' - sf_1''\right)}{\left(2r + m\left(2qs + s'\right)\right)} \tag{8}$$

and

$$\begin{aligned} G \;=\; & -\left(2r + m\left(2qs + s'\right)\right)^{-2}\left[f_1\left(-myp'\left(2r + m\left(2qs\right.\right.\right.\right. \\ & \left.\left. + s'\right)\right) + p\left(2(x + myq)r + m\left(2xqs + 2myq^2 s\right.\right. \\ & \left.\left.\left. + 2mysq' + 2yr' + xs' + 3myqs' + mys''\right)\right)\right) \\ & + m\left(2m^2yq^3sf_1' - 2myrf_1'q' - m^2yf_1'q's'\right. \\ & + mq^2f_1'\left(2yr + 2xs + 3mys'\right) - ypf_1'\left(2r + m\left(2qs\right.\right. \\ & \left.\left. + s'\right)\right) + 2xrf_1'' + 2m^2ysq'f_1'' + 2myr'f_1'' + mxs'f_1'' + m^2yf_1''s'' \\ & - 2myrf_1''' - m^2ys'f_1''' + q\left(2xrf_1' + m\left(f_1'\left(2yr' + xs' + mys''\right)\right.\right. \\ & \left.\left.\left.\left. + 2\left(xsf_1'' + mys'f_1'' - mysf_1'''\right)\right)\right)\right)\right] \end{aligned} \tag{9}$$

where G_0 is a constant and prime means differentiation with respect to time, "t", and functions $a\left(t\right)$ and $f_1\left(t\right)$ are related to p,q,r,s by a system of ordinary differential equations which we omit. We can see that $f_1\left(t\right)$ satisfies a linear fourth-order differential equation, which means that it provides us with four symmetries. Another symmetry vector arises from the unique solution of $a\left(t\right)$. Therefore, from Equations (7)–(10), it is easy to see that the Lie symmetries of the master equation form the $\{A_1 \oplus_s W_5\} \oplus_s \infty A_1$ Lie algebra.

Indeed, the form of the symmetry vector (6) it is not a closed form. The reason for this is that we have considered arbitrary functions $\Omega^2(t)$, $\Gamma(t)$, $h(t)$, and $f(t)$. In a case for specific functional forms of the coefficients, one can calculate the symmetry vector in closed-form. For instance, in the case for which the coefficients, $p, q, r,$ and s are constants, the Lie symmetries are

$$Y_1 = a_1\partial_t\ ,\ Y_Z = Z\partial_Z\ ,\ Y_b = b\partial_z \tag{10}$$

$$X_1 = e^{\frac{\lambda-q}{2}t}\left[m(\lambda-q)\partial_x + 2\partial_y\right]\ ,\ X_2 = e^{-\frac{\lambda+q}{2}t}\left[m(\lambda+q)\partial_x - 2\partial_y\right] \tag{11}$$

$$X_3 = e^{-\frac{(\lambda-q)}{2}t}\left[2rm(q-\lambda)\partial_x + 4(r+sqm)\partial_y + \left(2mq(\lambda-q)x + m^2q\left(\lambda^2-q^2\right)y\right)Z\partial_Z\right] \tag{12}$$

and

$$X_4 = e^{\frac{\lambda+q}{2}t}\left[2rm(\lambda+q)\partial_x + (r+sqm)\partial_y + \left(-2mq(\lambda+q)x + m^2q\left(\lambda^2-q^2\right)y\right)Z\partial_Z\right] \tag{13}$$

where $\lambda = \sqrt{4p - mq^2}$.

Below, we apply the zeroth-order invariants of the Lie symmetry which corresponds to the solution of the function $f_1(t)$.

2.2. Application of the Lie Invariants

Consider now the Lie symmetry vector (6) for which $G_0 = 0$ and $a(t) = 0$. The characteristic functions are

$$Z(t,x,y) = U(t, B(t)f_1(t)x - A(t)y)\exp[J(t,x)(B(t)f_1(t)x - A(t)y)] \tag{14}$$

where

$$J = \frac{x}{2A^2}\left[\begin{array}{c} A^2(2G + xH) + \\ +B(-2(Bf_1x - A(t)y) + xBf_1)K \end{array}\right] \tag{15}$$

and the functions $A(t), B(t), K(t), G(t)$, and $H(t)$ are the coefficients of the symmetry vector (6).

Hence, the application of (14) to (5) gives the reduced equation

$$S(t)U_{,ww} - wR(t)U_{,w} + q(t)U - U_{,t} = 0 \tag{16}$$

where $w = B(t)f_1(t)x - A(t)y$, $S(t) = Bf_1(Bf_1r - As)$, and $R(t) = A^{-1}(Bf_1p - A')$.

This means that the Lie symmetries provide us with a solution for the master Equation (1), which is (14) and the function $U(t,w)$ is given by (16). Below we study the Lie symmetries of (16), and we show that it is maximally symmetric. This means that it is equivalent with the elementary one-dimensional Schrödinger Equation.

3. Equivalence with the Schrödinger Equation

For simplicity in the following, we consider a "time" rescaling, $t \to T$, such that $S(T) = 1$. Hence, Equation (16) becomes

$$U_{,ww} - wR(T)U_{,w} + q(T)U - U_{,T} = 0 \tag{17}$$

We apply the Lie symmetry condition to the this equation, and we derive the following symmetry vector field

$$Y = \alpha(T)\partial_t + \left[\frac{\dot{\alpha}(T)v}{2} + \beta(T)\right]\partial_v + \left(F(T,w)U + \bar{b}(t,w)\right)\partial_U \tag{18}$$

where

$$F(T,v)=\phi(T)+\frac{1}{4}\left[2v\beta(T)R(T)+v^2R(T)\dot{\alpha}-2v\dot{\beta}+v^2\alpha(T)\dot{R}-\frac{1}{2}v^2\ddot{\alpha}\right] \quad (19)$$

in which overdot denotes differentiation with respect to T and the functions $\phi(T)$, $\beta(T)$, and $\alpha(T)$ are solutions of the equations

$$\phi = \phi_0+\alpha(\left(q+\frac{1}{2}R\right)-\frac{1}{4}\dot{\alpha} \quad (20)$$

$$\ddot{\beta} = \left(\dot{R}+R^2\right)\beta \quad \text{and} \quad (21)$$

$$\dddot{\alpha} = 4\dot{\alpha}\left(\dot{R}+R^2\right)+2\alpha\frac{d}{dT}\left(\dot{R}+R^2\right) \quad (22)$$

Furthermore $\bar{b}\,(t,w)$ satisfies the original equation, Equation (17).

Equation (21) is a maximally symmetric linear second-order differential equation. In this case, by application of the Riccati transformation $R=\frac{\dot{L}}{L}$, in (21) we find the solution,

$$\beta\,(T)=\beta_0L\,(T)+\beta_1L\,(T)\int L^{-2}\,(T)\,dT \quad (23)$$

Equation (22) is a nonautonomous third-order differential equation. We multiply with $\alpha\,(T)$ and integrate to obtain

$$\alpha(T)\ddot{\alpha}-\frac{1}{2}\dot{\alpha}^2-2\alpha^2(T)\left(\dot{R}+R^2(T)\right)=2K$$

where K is a constant. We substitute $\alpha=\gamma^2$ into this equation, and hence we find the well-known Ermakov–Pinney equation [25,26]

$$\ddot{\rho}-\rho(T)\left(\dot{R}+R^2(T)\right)=\frac{K}{\rho^3(T)} \quad (24)$$

The solution of (24) is given in [26], and it is related with the solution of the linear equation

$$\ddot{\sigma}-\left(\dot{R}+R^2(T)\right)\sigma=0 \quad (25)$$

Therefore, we conclude that Equation (17) admits as Lie symmetries the vector fields which form the $\{sl(2,R)\oplus_s W_3\}\oplus_s\infty A_1$ Lie algebra. Hence, from S. Lie's theorem [10], we have that there exists a transformation $(T,w,U)\rightarrow(\tau,\chi,\Psi)$, in which (17) becomes

$$-\frac{\hbar}{2M}\frac{\partial^2\Psi}{\partial\chi^2}=i\hbar^2\frac{\partial\Psi}{\partial\tau} \quad (26)$$

which is the Schrödinger equation for a free particle. That is possible because Equations (17) and (26) are both maximally symmetric.

4. Discussion

In this work, with the application of the group invariant transformations, we proved that there exists a surface in the space of the dependent and independent variables in which the master Equation (1) can be seen as a one-dimensional equation. That means that solutions of the latter generate solutions for the master equation given by the expression (14); that is, there is class of solutions which describe the two different systems, but the solutions are given in different representations.

We remark that in our analysis, we considered that the coefficients of the master equation are arbitrary functions of time, which means that the result holds when the coefficients are constants. For instance, consider the application of the Lie symmetry X_3, (11), in Equation (5) for constant coefficients. Hence, we have that $Z = U(t,w)$, where $w = \frac{ym(L-q)-2x}{m(L-q)}$, and $U(t,w)$ satisfies the equation

$$\bar{s}U_{,ww} - \frac{(\lambda+q)}{2}wU_{,w} + 2qU - U_{,t} = 0 \tag{27}$$

and $\bar{s} = 2\frac{-2r+sm(\lambda-q)}{m^2(\lambda-q)^2}$. Therefore under the coordinate transformation,

$$U(t,w) = e^{2qt}\Psi(\tau,\chi)\ ,\ w = \left(\frac{2Ms}{\hbar}\right)^{1/2}\chi e^{\left(\frac{\lambda+q}{2}t\right)}\ ,\ d\tau = -i\hbar e^{-(\lambda+q)t}dt \tag{28}$$

The latter equation takes the form of the one-dimensional Schrödinger equation. A similar result holds, and for the remaining Lie symmetry vectors, $X_2 - X_4$, or any linear combination of them.

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