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Positive Solutions for Nonlinear Caputo Type Fractional q-Difference Equations with Integral Boundary Conditions

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Abstract: In this paper, by applying some well-known fixed point theorems, we investigate the existence of positive solutions for a class of nonlinear Caputo type fractional q-difference equations with integral boundary conditions. Finally, some interesting examples are presented to illustrate the main results.

Keywords: fractional q-difference systems; integral boundary conditions; positive solution; Green's function; fixed point theorems

MSC: 34A08; 34A12; 34B15

1. Introduction

Since Al-Salam [1] and Agarwal [2] introduced the fractional q-difference calculus, the theory of fractional q-difference calculus itself and nonlinear fractional q-difference equation boundary value problems have been extensively investigated by many researchers. For some recent developments on fractional q-difference calculus and boundary value problems of fractional q-difference equations, see [3–16] and the references therein. For example, authors [17–20] considered some anti-periodic boundary value problems of nonlinear fractional q-difference equations. By applying the generalized Banach contraction principle, the monotone iterative method, and the Krasnoselskii's fixed point theorem. In [21], the authors investigated Caputo q-fractional initial value problems independently of the paper [3] where some open problems raised there. In [22], Mittag-Leffler stabilitry of q-fractional systems was investigated. In [23,24], some important q-fractional inequalities were proved. Those inequalities are necessary for the development of q-fractional systems. Zhao et al. [25] showed some existence results of positive solutions to nonlocal q-integral boundary value problems of nonlinear fractional q-derivative equations.

Under different conditions, Graef and Kong [26,27] investigated the existence of positive solutions for boundary value problems with fractional q-derivatives in terms of different ranges of λ, respectively. By applying some standard fixed point theorems, Agarwal et al. [28] and Ahmad et al. [29] showed some existence results for sequential q-fractional integrodifferential equations with q-antiperiodic boundary conditions and nonlocal four-point boundary conditions, respectively. In [30], by applying a mixed monotone method and the Guo-Krasnoselskii fixed point theorem, Zhao and Yang obtained the existence and uniqueness of positive solutions for the singular coupled integral boundary value problem of nonlinear higher-order fractional q-difference equations.

In [31], Ferreira considered the nonlinear fractional q-difference boundary value problem as follows:

$$\begin{aligned}&(D_q^\alpha u)(t)+f(u(t))=0, \quad t\in[0,1], \quad \alpha\in(2,3],\\&u(0)=(D_qu)(0)=0, \quad (D_qu)(1)=\beta\geq 0,\end{aligned}$$

where D_q^α is the q-derivative of Riemann-Liouville type of order α. By applying a fixed point theorem in cones, sufficient conditions for the existence of positive solutions were enunciated.

In [32], Ahmad et al. studied the following nonlocal boundary value problems of nonlinear fractional q-difference equations

$$\begin{aligned}&({}^cD_q^\alpha u)(t)=f(t,u(t)), \quad t\in[0,1], \quad \alpha\in(1,2],\\&a_1u(0)-b_1(D_qu)(0)=c_1u(\eta_1), \quad a_2u(1)+b_2(D_qu)(1)=c_2u(\eta_2),\end{aligned}$$

where ${}^cD_q^\alpha$ denotes the Caputo fractional q-derivative of order α, and $a_i,b_i,c_i,\eta_i\in\mathbb{R}$ $(i=1,2)$. The existence of solutions for the problem were shown by applying some well-known tools of fixed point theory such as the Banach contraction principle, the Krasnoselskii fixed point theorem, and the Leray-Schauder nonlinear alternative.

In [33], Zhou and Liu investigated the following fractional q-difference system

$$\begin{aligned}&({}^cD_q^\alpha u)(t)=f(t,v(t)), \quad ({}^cD_q^\beta v)(t)=f(t,u(t)), \quad t\in[0,1], \quad \alpha,\beta\in(1,2],\\&a_1u(0)-b_1(D_qu)(0)=c_1u(\eta_1), \quad a_2u(1)+b_2(D_qu)(1)=c_2u(\eta_2),\\&a_3u(0)-b_3(D_qu)(0)=c_3u(\eta_3), \quad a_4u(1)+b_4(D_qu)(1)=c_4u(\eta_4),\end{aligned}$$

where ${}^cD_q^\alpha$ and ${}^cD_q^\alpha$ denote the Caputo fractional q-derivative of order α and β, respectively. The uniqueness and existence of solution were obtained based on the nonlinear alternative of Leray-Schauder type and Banach's fixed-point theorem.

In [34], the author considered the following coupled integral boundary value problem for systems of nonlinear semipositone fractional q-difference equations

$$\begin{aligned}&(D_q^\alpha u)(t)+\lambda f(t,u(t),v(t))=0, \quad (D_q^\beta v)(t)+\lambda g(t,u(t),v(t))=0, \quad t\in[0,1], \quad \lambda>0,\\&(D_q^ju)(0)=(D_q^jv)(0)=0, \quad 0\leq j\leq n-2, \quad u(1)=\mu\int_0^1 v(s)d_qs, \quad v(1)=\nu\int_0^1 u(s)d_qs,\end{aligned}$$

where λ,μ,ν are three parameters with $0<\mu<[\beta]_q$ and $0<\nu<[\alpha]_q$, $\alpha,\beta\in(n-1,n]$ are two real numbers and $n\geq 3$, D_q^α, D_q^β are the fractional q-derivative of the Riemann-Liouville type, and f,g are sign-changing continuous functions. By applying the nonlinear alternative of Leray-Schauder type and the Krasnoselskii's fixed point theorems, sufficient conditions for the existence of one or multiple positive solutions were obtained.

In [35], Li and Yang considered the following nonlinear fractional q-difference equation with integral boundary conditions

$$\begin{aligned}&D_q^\alpha u(t)+h(t)f(t,u(t))=0, \quad t\in(0,1),\\&D_q^ju(0)=0, \quad 0\leq j\leq n-2, \quad u(1)=\mu\int_0^1 g(s)u(s)d_qs,\end{aligned}$$

where $\alpha\in(n-1,n]$ are a real number and $n\geq 3$ is an integer, D_q^α are the fractional q-derivative of the Riemann–Liouville type, $\mu>0$ and $0<q<1$ are two constants, g,h are two given continuous functions, and $f:[0,1]\times[0,\infty)\to[0,\infty)$is continuous and $f(t,0)\not\equiv 0$ on $[0,1]$. By applying monotone iterative method and some inequalities associated with the Green's function, the existence results of positive solutions and two iterative schemes approximating the solutions were established.

Motivated by the wide applications of coupled boundary value problems and the results mentioned above, we consider the following nonlinear Caputo type fractional q-difference equations with integral boundary conditions

$$
\begin{aligned}
&({}^cD_q^{\alpha}u)(t) = f(t,u(t)), \quad t \in [0,1], \quad \alpha \in (1,2], \\
&au(0) - bu(1) = \int_0^1 g(s)u(s)d_qs, \quad c(D_qu)(0) - d(D_qu)(1) = \int_0^1 h(s)u(s)d_qs,
\end{aligned} \tag{1}
$$

where ${}^cD^{\alpha}$ is the Caputo type fractional q-derivative of fractional order α, a,b,c,d are real constants with $a > b > 0$ and $c > d > 0$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R})$ and $g,h \in C([0,1], \mathbb{R}^+)$.

The main aim of this paper is to investigate the existence of positive solutions for a class of nonlinear Caputo type fractional q-difference equations with integral boundary conditions by means of the Guo-Krasnoselskii fixed point theorem and Leggett-Williams fixed point theorem. Furthermore, some examples are given to illustrate our main results.

2. Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of the q-fractional boundary value problem (1). These details can be found in the recent literature; see [36,37] and references therein.

Let $q \in (0,1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q-analogue of the power $(a-b)^n$ with $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ is

$$(a-b)^{(0)} = 1, \quad (a-b)^{(n)} = \prod_{k=0}^{n-1}(a - bq^k), \quad n \in \mathbb{N}_0, \ a,b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^{\alpha} \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$, then $a^{(\alpha)} = a^{\alpha}$. Here, we point out that the following equality holds

$$(a-b)^{(\alpha)} = (a - bq^{\alpha-1})(a-b)^{(\alpha-1)}.$$

The q-gamma function is defined by

$$\Gamma_q(x) = (1-q)^{(x-1)}(1-q)^{1-x}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}$$

and satisfies $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function f is defined by

$$(D_qf)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_qf)(0) = \lim_{x \to 0}(D_qf)(x),$$

and q-derivatives of higher order by

$$(D_q^0f)(x) = f(x) \quad \text{and} \quad (D_q^nf)(x) = D_q(D_q^{n-1}f)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [36]. We now point out five formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i)

$$\int_a^b f(s)(D_q g)(s) d_q s = [f(s)g(s)]_{s=a}^{s=b} - \int_a^b (D_q f)(s) g(qs) d_q s \quad (q\text{−integration by parts}),$$
$$[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}, \quad {}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$
$${}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q(t-qs)^{(\alpha-1)}, \quad \left({}_xD_q \int_0^x f(x,t) d_q t \right)(x) = \int_0^x {}_xD_q f(x,t) d_q t + f(qx, x).$$

Note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$ [13].

Definition 1 ([2]). *Let $\alpha \ge 0$ and f be function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and*

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha > 0, x \in [0, 1].$$

Definition 2 ([38]). *The fractional q-derivative of the Riemann-Liouville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and*

$$(D_q^{\alpha} f)(x) = (D_q^m I_q^{m-\alpha} f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 3 ([38]). *The fractional q-derivative of the Caputo type of order $\alpha \ge 0$ is defined by*

$$({}^c D_q^{\alpha} f)(x) = (I_q^{m-\alpha} D_q^m f)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 1 ([2]). *Let $\alpha, \beta \ge 0$ and f be a function defined on $[0, 1]$. Then, the next formulas hold:* (1) $(I_q^{\beta} I_q^{\alpha} f)(x) = I_q^{\alpha+\beta} f(x)$; (2) $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

Lemma 2 ([3]). *Let $\alpha > 0$ and $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, the following equality holds:*

$$(I_q^{\alpha c} D_q^{\alpha} f)(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0),$$

where m is the smallest integer greater than or equal to α.

Lemma 3 ([2]). *For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1, \infty)$, the following is valid:*

$$I_q^{\alpha}((x-a)^{(\lambda)}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (x-a)^{(\alpha+\lambda)}, \quad 0 < a < x < b.$$

Lemma 4. *Let $y \in C[0,1]$, and then the boundary value problem*

$$\begin{aligned} &({}^c D_q^{\alpha} u)(t) = y(t), \quad t \in (0,1), \quad \alpha \in (1,2], \\ &au(0) - bu(1) = 0, \quad c(D_q u)(0) - d(D_q u)(1) = 0, \end{aligned} \tag{2}$$

has a unique solution u in the form

$$u(t) = \int_0^1 G(t, qs) y(s) d_q s, \tag{3}$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{d[\alpha-1]_q t(1-s)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)} + \frac{b(1-s)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} + \frac{bd[\alpha-1]_q (1-s)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{d[\alpha-1]_q t(1-s)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)} + \frac{b(1-s)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} + \frac{bd[\alpha-1]_q (1-s)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. By applying Lemmas 1 and 2, we see that

$$u(t) = (I_q^{\alpha} y)(t) + d_1 + d_2 t = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s + d_1 + d_2 t. \tag{4}$$

Differentiating both sides of (4), we obtain

$$(D_q u)(t) = (I_q^{\alpha-1} y)(t) + d_2 = \int_0^t \frac{[\alpha-1]_q (t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} y(s) d_q s + d_2. \tag{5}$$

From (4) and (5), we get

$$\begin{aligned} &u(0) = d_1, \quad u(1) = \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s + d_1 + d_2, \\ &(D_q u)(0) = d_2, \quad (D_q u)(1) = \int_0^1 \frac{[\alpha-1]_q (1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} y(s) d_q s + d_2. \end{aligned} \tag{6}$$

Using (6) to the boundary conditions $au(0) - bu(1) = 0$ and $c(D_q u)(0) - d(D_q u)(1) = 0$, we obtain

$$\begin{aligned} d_1 &= \int_0^1 \frac{b(1-qs)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} y(s) d_q s + \int_0^1 \frac{bd[\alpha-1]_q (1-qs)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)} y(s) d_q s, \\ d_2 &= \int_0^1 \frac{d[\alpha-1]_q (1-qs)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)} y(s) d_q s. \end{aligned} \tag{7}$$

Substituting d_1 and d_2 in (7) into Equation (4), we find

$$u(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s + \int_0^1 \frac{d[\alpha-1]_q t(1-qs)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)} y(s) d_q s$$
$$+ \int_0^1 \frac{b(1-qs)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} y(s) d_q s + \int_0^1 \frac{bd[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)} y(s) d_q s = \int_0^1 G(t,qs) y(s) d_q s,$$

which implies that (2) has a unique solution (3). This completes the proof of the lemma. □

For the sake of simplicity, we always assume that the following condition (**H1**) holds:

(**H1**) $\kappa = \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$ and $\kappa_1, \kappa_4 \geq 0$, where $\phi(t) = \frac{b+(a-b)t}{(a-b)(c-d)}$, $t \in [0,1]$, and

$$\kappa_1 = 1 - \frac{1}{a-b}\int_0^1 g(t) d_q t, \quad \kappa_2 = \int_0^1 g(t)\phi(t) d_q t,$$
$$\kappa_3 = \frac{1}{a-b}\int_0^1 h(t) d_q t, \quad \kappa_4 = 1 - \int_0^1 h(t)\phi(t) d_q t.$$

Now, we will obtain the Green's function of the boundary value problem (1) and some of its properties.

Lemma 5. *Let $y \in C[0,1]$, and then the boundary value problem*

$$\begin{aligned} &({}^cD_q^\alpha u)(t) = y(t), \quad t \in (0,1), \quad \alpha \in (1,2], \\ &au(0) - bu(1) = \int_0^1 g(s)u(s) d_q s, \quad c(D_q u)(0) - d(D_q u)(1) = \int_0^1 h(s)u(s) d_q s, \end{aligned} \tag{8}$$

has a unique solution u in the form

$$u(t) = \int_0^1 H(t,qs) y(s) d_q s, \tag{9}$$

where

$$H(t,s) = G(t,s) + \frac{\kappa_4 + \kappa_3(a-b)\phi(t)}{\kappa(a-b)}\int_0^1 g(t)G(t,s) d_q t + \frac{\kappa_2 + \kappa_1(a-b)\phi(t)}{\kappa(a-b)}\int_0^1 h(t)G(t,s) d_q t.$$

Proof. By applying (6) to the boundary conditions $au(0) - bu(1) = \int_0^1 g(s)u(s) d_q s$ and $c(D_q u)(0) - d(D_q u)(1) = \int_0^1 h(s)u(s) d_q s$, we obtain

$$\begin{aligned} d_1 = &\int_0^1 \frac{b(1-qs)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} y(s) d_q s + \int_0^1 \frac{bd[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)} y(s) d_q s \\ &+ \frac{1}{a-b}\int_0^1 g(s)u(s) d_q s + \frac{b}{(a-b)(c-d)}\int_0^1 h(s)u(s) d_q s, \\ d_2 = &\int_0^1 \frac{d[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)} y(s) d_q s + \frac{1}{c-d}\int_0^1 h(s)u(s) d_q s. \end{aligned} \tag{10}$$

Substituting d_1 and d_2 in Equation (10) into the (4), we have

$$u(t) = \int_0^1 G(t,qs) y(s) d_q s + \frac{1}{a-b}\int_0^1 g(s)u(s) d_q s + \phi(t)\int_0^1 h(s)u(s) d_q s. \tag{11}$$

Multiplying both sides of the first and second equations of (11) by $g(t)$ and $h(t)$, respectively, and integrating the resulting equations obtained with respect to t from 0 to 1, we obtain

$$\begin{aligned}\int_0^1 g(t)u(t)d_qt &= \int_0^1 g(t)\int_0^1 G(t,qs)y(s)d_qsd_qt\\ &+\frac{1}{a-b}\int_0^1 g(t)d_qt\int_0^1 g(s)u(s)d_qs+\int_0^1 g(t)\phi(t)d_qt\int_0^1 h(s)u(s)d_qs,\\ \int_0^1 h(t)u(t)d_qt &= \int_0^1 h(t)\int_0^1 G(t,qs)y(s)d_qsd_qt\\ &+\frac{1}{a-b}\int_0^1 h(t)d_qt\int_0^1 g(s)u(s)d_qs+\int_0^1 h(t)\phi(t)d_qt\int_0^1 h(s)u(s)d_qs.\end{aligned}$$

Solving for $\int_0^1 g(s)u(s)d_qs$ and $\int_0^1 h(s)u(s)d_qs$ from the above equations, we have

$$\begin{aligned}\int_0^1 g(s)u(s)d_qs =&\frac{1}{\kappa}\left(\kappa_4\int_0^1 g(t)\int_0^1 G(t,qs)y(s)d_qsd_qt+\kappa_2\int_0^1 h(t)\int_0^1 G(t,qs)y(s)d_qsd_qt\right),\\ \int_0^1 h(s)u(s)d_qs =&\frac{1}{\kappa}\left(\kappa_3\int_0^1 g(t)\int_0^1 G(t,qs)y(s)d_qsd_qt+\kappa_1\int_0^1 h(t)\int_0^1 G(t,qs)y(s)d_qsd_qt\right).\end{aligned} \tag{12}$$

Substituting $\int_0^1 g(s)u(s)d_qs$ and $\int_0^1 h(s)u(s)d_qs$ in Equation (12) into the (11), we have

$$\begin{aligned}u(t) =&\frac{1}{\kappa(a-b)}\left(\kappa_4\int_0^1 g(t)\int_0^1 G(t,qs)y(s)d_qsd_qt+\kappa_2\int_0^1 h(t)\int_0^1 G(t,qs)y(s)d_qsd_qt\right)\\ &+\frac{\phi(t)}{\kappa}\left(\kappa_3\int_0^1 g(t)\int_0^1 G(t,qs)y(s)d_qsd_qt+\kappa_1\int_0^1 h(t)\int_0^1 G(t,qs)y(s)d_qsd_qt\right)\\ &+\int_0^1 G(t,qs)y(s)d_qs=\int_0^1 H(t,qs)x(s)d_qs,\end{aligned}$$

which implies that (8) has a unique solution (9). This completes the proof of the lemma. □

Lemma 6. *The function $H(t,s)$ has the following property:*

$$\mathfrak{M}_1\omega(qs)\le H(t,qs)\le\frac{a}{b}\mathfrak{M}_2\omega(qs),\quad \forall t\in[0,1],\ s\in(0,1),$$

where $\omega(s)=\frac{b(1-s)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)}+\frac{bd[\alpha-1]_q(1-s)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)}$ *and*

$$\begin{aligned}\mathfrak{M}_1 =&1+\frac{\kappa_4+\kappa_3(a-b)\phi(0)}{\kappa(a-b)}\int_0^1 g(t)d_qt+\frac{\kappa_2+\kappa_1(a-b)\phi(0)}{\kappa(a-b)}\int_0^1 h(t)d_qt,\\ \mathfrak{M}_2 =&1+\frac{\kappa_4+\kappa_3(a-b)\phi(1)}{\kappa(a-b)}\int_0^1 g(t)d_qt+\frac{\kappa_2+\kappa_1(a-b)\phi(1)}{\kappa(a-b)}\int_0^1 h(t)d_qt.\end{aligned}$$

Proof. Let

$$g_1(t,s)=\frac{(t-s)^{(\alpha-1)}}{\Gamma_q(\alpha)}+\frac{d[\alpha-1]_qt(1-s)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)}+\frac{b(1-s)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)}+\frac{bd[\alpha-1]_q(1-s)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)},$$

for $0\le s\le t\le 1$, and

$$g_2(t,s)=\frac{d[\alpha-1]_qt(1-s)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)}+\frac{b(1-s)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)}+\frac{bd[\alpha-1]_q(1-s)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)},\quad 0\le t\le s\le 1.$$

For given $s \in (0,1)$, g_1, g_2 are increasing with respect to t for $t \in [0,1]$. Hence, we have

$$\begin{aligned}\min_{t\in[0,1]} G(t,qs) =& \min\{\min_{t\in[s,1]} g_1(t,qs), \min_{t\in[0,s]} g_2(t,qs)\} = \min\{g_1(s,qs), g_2(0,qs)\} = g_2(0,qs)\\ =&\frac{b(1-qs)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} + \frac{bd[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)} := \omega(qs),\end{aligned}$$

and

$$\begin{aligned}\max_{t\in[0,1]} G(t,qs) =& \max\{\max_{t\in[s,1]} g_1(t,qs), \max_{t\in[0,s]} g_2(t,qs)\} = \max\{g_1(1,qs), g_2(s,qs)\} = g_1(1,qs)\\ =&\frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{d[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(c-d)\Gamma_q(\alpha)} + \frac{b(1-qs)^{(\alpha-1)}}{(a-b)\Gamma_q(\alpha)} + \frac{bd[\alpha-1]_q(1-qs)^{(\alpha-2)}}{(a-b)(c-d)\Gamma_q(\alpha)}\\ =&\frac{a}{b}\omega(qs).\end{aligned}$$

Therefore, we get

$$\omega(qs) \le G(t,qs) \le \frac{a}{b}\omega(qs), \quad \forall t \in [0,1], \; s \in (0,1).$$

Furthermore, we obtain

$$\begin{aligned}H(t,qs) =& G(t,qs) + \frac{\kappa_4 + \kappa_3(a-b)\phi(t)}{\kappa(a-b)} \int_0^1 g(t)G(t,qs)d_qt + \frac{\kappa_2 + \kappa_1(a-b)\phi(t)}{\kappa(a-b)} \int_0^1 h(t)G(t,qs)d_qt\\ \ge& \omega(qs) + \frac{\kappa_4 + \kappa_3(a-b)\phi(0)}{\kappa(a-b)} \int_0^1 g(t)d_qt\omega(qs) + \frac{\kappa_2 + \kappa_1(a-b)\phi(0)}{\kappa(a-b)} \int_0^1 h(t)d_qt\omega(qs)\\ :=& \mathfrak{M}_1\omega(qs),\end{aligned}$$

and

$$\begin{aligned}H(t,qs) =& G(t,qs) + \frac{\kappa_4 + \kappa_3(a-b)\phi(t)}{\kappa(a-b)} \int_0^1 g(t)G(t,qs)d_qt + \frac{\kappa_2 + \kappa_1(a-b)\phi(t)}{\kappa(a-b)} \int_0^1 h(t)G(t,qs)d_qt\\ \le& \frac{a}{b}\omega(qs) + \frac{\kappa_4 + \kappa_3(a-b)\phi(1)}{\kappa(a-b)} \int_0^1 g(t)d_qt\frac{a}{b}\omega(qs) + \frac{\kappa_2 + \kappa_1(a-b)\phi(1)}{\kappa(a-b)} \int_0^1 h(t)d_qt\frac{a}{b}\omega(qs)\\ :=& \frac{a}{b}\mathfrak{M}_2\omega(qs).\end{aligned}$$

This completes the proof of the lemma. □

Let the Banach space $\mathscr{E} = C[0,1]$ be endowed with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$. Define the cone $\mathscr{P} \subset \mathscr{E}$ by $\mathscr{P} = \{u \in \mathscr{E} : u(t) \ge \Delta\|u\|, \, t \in [0,1]\}$, where $\Delta = b\mathfrak{M}_1/(a\mathfrak{M}_2)$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are defined as in Lemma 6. In addition, define, for $0 < \mu < \nu$ two positive numbers, Ω_μ and $\overline{\Omega}_{\mu,\nu}$ by $\Omega_\mu = \{u \in E : \|u\| < \mu\}$ and $\overline{\Omega}_{\mu,\nu} = \{u \in \mathscr{E} : \mu \le u \le \nu\}$. Note that $\partial\Omega_\mu = \{u \in \mathscr{E} : \|u\| = \mu\}$.

Suppose that u is a solution of boundary value problem (1). Then,

$$u(t) = \int_0^1 H(t,qs)f(s,u(s))d_qs, \quad t \in [0,1].$$

We define an operator $\mathscr{T} : \mathscr{P} \to \mathscr{E}$ as follows

$$(\mathscr{T}u)(t) = \int_0^1 H(t,qs)f(s,u(s))d_qs, \quad t \in [0,1]. \tag{13}$$

By Lemma 6, we have

$$\|\mathscr{T}u\| \leq \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs)f(s,u(s))d_qs, \quad (\mathscr{T}u)(t) \geq \mathfrak{M}_1 \int_0^1 \omega(qs)f(s,u(s))d_qs \geq \Delta\|\mathscr{T}u\|.$$

Thus, $\mathscr{T}(\mathscr{P}) \subset \mathscr{P}$. Then, we have the following lemma.

Lemma 7. *$\mathscr{T}:\mathscr{P}\to\mathscr{P}$ is completely continuous.*

Proof. The operator $\mathscr{T}:\mathscr{P}\to\mathscr{P}$ is continuous in view of continuity of $H(t,s)$ and $f(t,u)$. By means of the Arzela-Ascoli theorem, $\mathscr{T}:\mathscr{P}\to\mathscr{P}$ is completely continuous. □

In order to obtain the main results in this paper, we will use the following cone compression and expansion fixed point theorem.

Lemma 8. (Guo-Krasnoselskii fixed point theorem, see [39]). *Let $\mathscr{P}$ be a cone of real Bananch space $\mathscr{E}$, Ω_1 and Ω_2 be two bounded open sets in $\mathscr{E}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let the operator $\mathscr{T}:\mathscr{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to\mathscr{P}$ be completely continuous. Suppose that one of the two conditions is satisfied:*

(i) $\|\mathscr{T}u\| \leq \|u\|, \forall u \in \mathscr{P}\cap\partial\Omega_1$ *and* $\|\mathscr{T}u\| \geq \|u\|, \forall u \in \mathscr{P}\cap\partial\Omega_2$,
(ii) $\|\mathscr{T}u\| \geq \|u\|, \forall u \in \mathscr{P}\cap\partial\Omega_1$ *and* $\|\mathscr{T}u\| \leq \|u\|, \forall u \in \mathscr{P}\cap\partial\Omega_2$.

Then, $\mathscr{T}$ has at least one fixed point in $\mathscr{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

Lemma 9. (Leggett-Williams fixed point theorem, see [40]). *Let $\mathscr{P}$ be a cone in a real Banach space $\mathscr{E}$, $\mathscr{P}_\gamma = \{x \in \mathscr{P} | \|x\| \leq \gamma\}$, θ be a nonnegative continuous concave functional on $\mathscr{P}$ such that $\theta(x) \leq \|x\|$ for $x \in \overline{\mathscr{P}}_\gamma$ and $\mathscr{P}(\theta,\nu,\delta) = \{u \in \mathscr{P} | \nu \leq \theta(x), \|x\| \leq \delta\}$. Suppose $\mathscr{T}:\overline{\mathscr{P}}_c \to \overline{\mathscr{P}}_c$ is completely continuous and there exist $0<\mu<\nu<\delta\leq\gamma$ such that*

(i) $\{x \in \mathscr{P}(\theta,\nu,\delta) | \theta(x) > \nu\} \neq \emptyset$ *and* $\theta(\mathscr{T}x) > b$, *for all* $x \in \mathscr{P}(\theta,\nu,\delta)$;
(ii) $\|\mathscr{T}x\| \leq \mu$, *for all* $\|x\| \leq \mu$;
(iii) $\theta(\mathscr{T}x) \geq \nu$, *for all* $x \in \mathscr{T}(\theta,\nu,\gamma)$ *with* $\|\mathscr{T}x\| > \delta$.

Then, $\mathscr{T}$ has at least three fixed points x_1, x_2 and x_3 satisfying

$$\|x_1\| < \mu, \quad \nu < \theta(x_2), \ \ and \ \ \|x_3\| > \mu, \quad \theta(x_3) < \nu.$$

In order to state our main results, we need to introduce the following notations.

$$f^0 = \limsup_{u\to 0} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f^\infty = \limsup_{u\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{u}, \quad f_0 = \liminf_{u\to 0} \min_{t\in[0,1]} \frac{f(t,u)}{u},$$
$$f_\infty = \liminf_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u}, \quad \mathfrak{L}_1 = \mathfrak{M}_1 \int_0^1 \omega(qs)d_qs, \quad \mathfrak{L}_2 = \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs)d_qs.$$

3. Main Results

In this section, we establish the existence of positive solutions for boundary value problem (1) by using the Guo-Krasnoselskii fixed point theorem and the Leggett-Williams fixed point theorem.

Theorem 1. *Assume **(H1)** holds. Furthermore, suppose one of the following conditions is satisfied.*

(**H2**) *There exist two constants μ and ν with $0 < \mu \leq (\mathfrak{L}_1/\mathfrak{L}_2)\nu$ such that*

$$f(t,u) \geq \mu\mathfrak{L}_1^{-1} \ \ for \ \ (t,u) \in [0,1]\times[0,\mu], \ \ and \ \ f(t,u) \leq \nu\mathfrak{L}_2^{-1} \ \ for \ \ (t,u) \in [0,1]\times[0,\nu];$$

(**H3**) $f_0 > \mathfrak{L}_1^{-1}$ *and* $f^\infty < \mathfrak{L}_2^{-1}$ *(particularly,* $f_0 = \infty$ *and* $f^\infty = 0$*);*

(**H4**) *$f^0 < \mathfrak{L}_2^{-1}$ and $f_\infty > \mathfrak{L}_1^{-1}$ (particularly, $f^0 = 0$ and $f_\infty = \infty$).*

Then, the problem (1) has at least one positive solution.

Proof. Let the operator $\mathscr{T}$ be defined by (13).
(**H2**) For $u \in \mathscr{P} \cap \partial\Omega_\mu$, we have $u \in [0, \mu]$, which implies $f(t, u) \geq \mu/\mathfrak{L}_1$. Hence, for $t \in [0, 1]$, by Lemma 6, we obtain

$$(\mathscr{T}u)(t) \geq \mathfrak{M}_1 \int_0^1 \omega(qs)\mu\mathfrak{L}_1^{-1} d_q s = \mathfrak{M}_1 \int_0^1 \omega(qs) d_q s \mathfrak{L}_1^{-1}\mu = \mu = \|u\|,$$

which implies that

$$\|\mathscr{T}u\| \geq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\mu. \tag{14}$$

Next, for $u \in \mathscr{P} \cap \partial\Omega_\nu$, we have $u \in [0, \nu]$, which implies $f(t, u) \leq \nu/\mathfrak{L}_2$. Hence, for $t \in [0, 1]$, by Lemma 6, we obtain

$$(\mathscr{T}u)(t) \leq \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs)\nu\mathfrak{L}_2^{-1} d_q s = \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs) d_q s \mathfrak{L}_2^{-1}\nu = \nu = \|u\|,$$

which implies that

$$\|\mathscr{T}u\| \leq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\nu. \tag{15}$$

(**H3**) Firstly, in view of $f_0 > \mathfrak{L}_1^{-1}$, there exists $\mu > 0$ such that $f(t, u) \geq (f_0 - \varepsilon_1)\|u\|$, for $t \in [0, 1]$, $\|u\| \in [0, \mu]$, where $\varepsilon_1 \geq 0$ satisfies $\mathfrak{L}_1(f_0 - \varepsilon_1) \geq 1$. Then, for $t \in [0, 1]$, $u \in \mathscr{P} \cap \partial\Omega_\mu$, which implies $\|u\| \leq \mu$, we have

$$(\mathscr{T}u)(t) \geq \int_0^1 H(t, qs)(f_0 - \varepsilon_1)\|u\| d_q s \geq \mathfrak{M}_1 \int_0^1 \omega(qs) d_q s (f_0 - \varepsilon_1)\|u\| = \mathfrak{L}_1(f_0 - \varepsilon_1)\|u\|,$$

which implies that

$$\|\mathscr{T}u\| \geq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\mu. \tag{16}$$

Nextly, turning to $f^\infty < \mathfrak{L}_2^{-1}$, there exists $\overline{\nu} > 0$ large enough such that $f(t, u) \leq (f^\infty + \varepsilon_2)\|u\|$, for $t \in [0, 1]$, $\|u\| \in (\overline{\nu}, \infty)$, where $\varepsilon_2 > 0$ satisfies $\mathfrak{L}_2^{-1} - f^\infty - \varepsilon_2 > 0$. Set $\overline{M} = \max_{\|u\| \leq \overline{\nu}, t \in [0,1]} f(t, u)$. Then, $f(t, u) \leq \overline{M} + (f^\infty + \varepsilon_2)\|u\|$. Choose $\nu > \max\left\{\mu, \overline{\nu}, \overline{M}(\mathfrak{L}_2^{-1} - f^\infty - \varepsilon_2)^{-1}\right\}$. Hence, for $u \in \mathscr{P} \cap \partial\Omega_\nu$, we have

$$\begin{aligned}(\mathscr{T}u)(t) \leq& \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs)(\overline{M} + (f^\infty + \varepsilon_2)\|u\|) d_q s \\ \leq& \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs) d_q s \left(\frac{\overline{M}}{\nu} + (f^\infty + \varepsilon_2)\right)\nu \leq \nu = \|u\|,\end{aligned}$$

which implies that

$$\|\mathscr{T}u\| \leq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\nu. \tag{17}$$

(**H$_4$**) Considering $f^0 < \mathfrak{L}_2^{-1}$, there exists $\mu > 0$ such that $f(t, u) \leq \eta u$, for any $u \in [0, \mu]$, $t \in [0, 1]$, where $\eta \leq \mathfrak{L}_2^{-1}$. Then, if Ω_μ is the ball in $\mathscr{E}$ centered at the origin with radius μ and if $u \in \mathscr{P} \cap \partial\Omega_\mu$, then we have

$$\|\mathscr{T}u\| = \max_{t \in [0,1]} \int_0^1 H(t, qs) f(s, u(s)) d_q s \leq \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs)\eta u(s) d_q s \leq \frac{a}{b}\mathfrak{M}_2 \int_0^1 \omega(qs) d_q s \eta\mu \leq \mu = \|u\|,$$

which implies that

$$\|\mathscr{T}u\| \leq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\mu. \tag{18}$$

On the other hand, we use the assumption $f_\infty > \mathfrak{L}_1^{-1}$. Then, there exists $\nu > 0$ large enough such that $f(t,u) \geq \varrho u$ for any $u \in [\nu, \infty)$, $t \in [0,1]$, where $\varrho \geq \mathfrak{L}_1^{-1}$. If we define $\Omega_\nu = \{u \in E : \|u\| < \nu\}$, for $t \in [0,1]$ and $u \in \mathscr{P} \cap \partial\Omega_\nu$, we get

$$(\mathscr{P}u)(t) \geq \mathfrak{M}_1 \int_0^1 \omega(qs)\varrho u(s) d_q s \geq \mathfrak{M}_1 \int_0^1 \omega(qs) d_q s \varrho\nu \geq \nu = \|u\|,$$

which implies that

$$\|\mathscr{T}u\| \geq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\nu. \tag{19}$$

Applying Lemma 8 to (14) and (15), (16) and (17) or (18) and (19) yields that $\mathscr{T}$ has a fixed point $u \in \mathscr{P} \cap \overline{\Omega}_{\mu,\nu}$ with $0 \leq \mu \leq \|u\| \leq \nu$. It follows from Lemma 8 that the problem (1) has at least one symmetric positive solution u. The proof is therefore complete. □

Theorem 2. *Assume* (**H1**) *holds. Furthermore, suppose the following conditions are satisfied.*

(**H5**) $f_0 > \mathfrak{L}_1^{-1}$ *and* $f_\infty > \mathfrak{L}_1^{-1}$ (*particularly,* $f_0 = f_\infty = \infty$),
(**H6**) *There exists* $\gamma > 0$ *satisfying* $f(t,u) < \gamma\mathfrak{L}_2^{-1}$, $(t,u) \in [0,1] \times [0,\gamma]$.

Then, the problem (1) *has at least two positive solutions* $u_1(t)$ *and* $u_2(t)$, *which satisfy* $0 < \|u_1\| < \gamma < \|u_2\|$.

Proof. At first, if $f_0 > \mathfrak{L}_1^{-1}$, it follows from the proof of (16) that we can choose μ with $0 < \mu < \gamma$ such that

$$\|Tu\| \geq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\mu. \tag{20}$$

If $f_\infty > \mathfrak{L}_1^{-1}$, then like in the proof of (19), we can choose b with $\gamma < \delta$ such that

$$\|\mathscr{T}u\| \geq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\nu. \tag{21}$$

Next, for $u \in \mathscr{P} \cap \partial\Omega_\gamma$, we have $u \in [0,\gamma]$, then from (**H6**), we obtain $f(t,u) < \gamma\mathfrak{L}_2^{-1}$. Thus, for $t \in [0,1]$, like in the proof of (17), we get

$$\|\mathscr{T}u\| \leq \|u\|, \text{ for } u \in \mathscr{P} \cap \partial\Omega_\gamma. \tag{22}$$

Applying Lemma 8 to (20) and (22), or (21) and (22) yields that $\mathscr{T}$ has a fixed point $u_1 \in \mathscr{P} \cap \overline{\Omega}_{\mu,\gamma}$, and a fixed point $u_2 \in \mathscr{P} \cap \overline{\Omega}_{\gamma,\nu}$. It follows from Lemma 8 that problem (1) has at least two positive solutions $u_1(t)$ and $u_2(t)$, which satisfy $0 < \|u_1\| < \gamma < \|u_2\|$. The proof is therefore complete. □

Theorem 3. *Assume* (**H1**) *holds. Furthermore, suppose the following conditions are satisfied.*

(**H7**) $f^0 < \mathfrak{L}_2^{-1}$ *and* $f^\infty < \mathfrak{L}_2^{-1}$ (*particularly,* $f^0 = f^\infty = 0$),
(**H8**) *There exists* $\delta > 0$ *satisfying* $f(t,u) > \delta\mathfrak{L}_1^{-1}$, $(t,u) \in [0,1] \times [0,d]$.

Then, problem (1) *has at least two positive solutions* $u_1(t)$ *and* $u_2(t)$, *which satisfy* $0 < \|u_1\| < \delta < \|u_2\|$.

Proof. It can be proved in a way similar to the third part of Theorems 1 and 2. □

Theorem 4. *Assume* (**H1**) *holds. In addition, there exist three positive constants* μ, ν *and* γ *with* $0 < \mu < \nu < \Delta\gamma$ *such that*

(**H9**) $f(t,u) < \mu\mathfrak{L}_2^{-1}$, *for all* $t \in [0,1]$, *and* $0 \le u \le \mu$,
(**H10**)$f(t,u) \ge \nu\mathfrak{L}_1^{-1}$, *for all* $t \in [0,1]$, *and* $\nu \le u \le \nu\Delta^{-1}$,
(**H11**)$f(t,u) \le \gamma\mathfrak{L}_2^{-1}$, *for all* $t \in [0,1]$, *and* $0 \le u \le \gamma$.

Then, the problem (1) *has at least three positive solutions* $u_1(t), u_2(t)$ *and* $u_3(t)$ *such that*

$$\|u_1\| < \mu, \quad \nu < \theta(u_2), \ \ \textit{and} \ \ \|u_3\| > \mu, \quad \theta(u_3) < \nu.$$

Proof. We show that all the conditions of Lemma 9 are satisfied. We first assert that there exists a positive number μ such that $\mathscr{T}(\overline{\mathscr{P}}_a) \subset \overline{\mathscr{P}}_\mu$. By (**H9**), we obtain

$$\|\mathscr{T}u\| = \max_{t\in[0,1]}(\mathscr{T}u)(t) = \max_{t\in[0,1]}\int_0^1 H(t,s)f(s,qu(s))d_qs < \frac{a}{b}\mathfrak{M}_2\int_0^1 G(qs)\mu\mathfrak{L}_2^{-1}d_qs = \mu.$$

Therefore, we have $\mathscr{T}(\overline{\mathscr{P}}_\mu) \subset \overline{\mathscr{P}}_\mu$. Especially, if $u \in \overline{\mathscr{P}}_\gamma$, then assumption (**H11**) yields $\mathscr{T}$: $\overline{\mathscr{P}}_\gamma \to \overline{\mathscr{P}}_\gamma$.

Next, we show that condition (i) of Lemma 9 is satisfied. Now, we define the nonnegative, continuous concave functional $\theta : \mathscr{P} \to [0,\infty)$ by $\theta(u) = \min_{t\in[0,1]} u(t)$. Obviously, for every $u \in \mathscr{P}$, we have $\theta(u) \le \|u\|$. Clearly, $\{u \in \mathscr{P}(\theta,\nu,\nu\Delta^{-1}) | \theta(u) > \nu\} \neq \emptyset$. Moreover, if $u \in \mathscr{P}(\theta,\nu,\nu\Delta^{-1})$, then $\theta(u) \ge \nu$, so $y \le \|u\| \le \nu\Delta^{-1}$. By the definition of θ and (**H10**), we obtain

$$\|\mathscr{T}u\| = \min_{t\in[0,1]}(\mathscr{T}u)(t) = \min_{t\in[0,1]}\int_0^1 H(t,qs)f(s,u(s))d_qs \ge \mathfrak{M}_1\int_0^1 \omega(qs)\nu\mathfrak{L}_1^{-1}d_qs = \nu.$$

Therefore, condition (i) of Lemma 9 is satisfied.

Finally, we address condition (iii) of Lemma 9. For this, we choose $u \in \mathscr{P}(\theta,\nu,\gamma)$ with $\|\mathscr{T}u\| > \nu\Delta^{-1}$. Then, from the definition of $\mathscr{P}$, we deduce

$$\varphi(\mathscr{T}u) = \min_{t\in[0,1]}(\mathscr{T}u)(t) \ge \Delta\|\mathscr{T}u\| > \nu.$$

This shows that (iii) of Lemma 9 holds. By Lemma 9, we then obtain the problem (1) has at least three positive solutions $u_1(t), u_2(t)$ and $u_3(t)$ such that

$$\|u_1\| < \mu, \quad \nu < \theta(u_2), \ \ \textit{and} \ \ \|u_3\| > \mu, \quad \theta(u_3) < \nu.$$

We have finished the proof of Theorem 4. □

4. Some Examples

In this section, as applications, we give some examples to illustrate the main results.

Example 1. *Consider the following q-fractional boundary value problem*

$$\begin{aligned} &({}^cD_{1/2}^{3/2}u)(t) = f(t,u(t)), \quad t \in [0,1], \\ &2u(0) - u(1) = \int_0^1 \frac{s^2}{2}u(s)d_qs, \quad 2(D_qu)(0) - (D_qu)(1) = \int_0^1 \frac{s^2}{2}u(s)d_qs, \end{aligned} \tag{23}$$

where

$$f(t,u(t)) = \begin{cases} (t+1)(2-t)u^3(t), & (t,u) \in [0,1]\times(0,3], \\ (t+1)(2-t)3u^2(t), & (t,u) \in [0,1]\times(3,\infty). \end{cases} \tag{24}$$

By simple calculations, we obtain that $\kappa_1 = 5/7$, $\kappa_2 = 58/105$, $\kappa_3 = 2/7$, $\kappa_4 = 47/105$, $\kappa = 119/735$, $\mathfrak{M}_1 = 3773/833$, $\mathfrak{M}_2 = 5243/833$, $\mathfrak{L}_1 \approx 3.131813$, $\mathfrak{L}_2 \approx 8.703999$ *and*

$$f^0 = \limsup_{u\to 0} \max_{t\in[0,1]} \frac{f(t,u)}{u} = \limsup_{u\to 0} \max_{t\in[0,1]} (t+1)(2-t)u^2 = \lim_{u\to 0} \frac{9}{4}u^2 = 0 < 0.1148897 \approx \frac{1}{\mathfrak{L}_2},$$

$$f_\infty = \liminf_{u\to\infty} \min_{t\in[0,1]} \frac{f(t,u)}{u} = \liminf_{u\to\infty} \min_{t\in[0,1]} (t+1)(2-t)3u = \lim_{u\to\infty} 6u = \infty > 0.3193039 \approx \frac{1}{\mathfrak{L}_1}.$$

*Then, condition (**H4**) holds. With the use of Theorem 1, the problem (23) with (24) has at least one positive solution.*

Example 2. *Consider the following q-fractional boundary value problem (23) with*

$$f(t,u(t)) = (1+\sin t)(1/144 + u^2(t)), \quad \forall (t,u) \in [0,1]\times[0,\infty), \tag{25}$$

and other conditions also hold. By simple calculations, we obtain that $f_0 = f_\infty = \infty$, *so the condition (**H5**) holds. On the other hand, choosing* $\gamma = 1/12$, *for* $(t,u) \in [0,1]\times[0,\infty)$, *we have*

$$f(t,u(t)) = (1+\sin t)\left(\frac{1}{144} + u^2(t)\right) \le 2\left(\frac{1}{144} + \frac{1}{144}\right) \approx 0.0277778 < 0.03829657 \approx \frac{\gamma}{\mathfrak{L}_2},$$

*thus condition (**H6**) holds. With the use of Theorem 2, the problem* (24) *with (25) has at least two positive solutions* $u_1(t)$ *and* $u_2(t)$, *which satisfy* $0 < \|u_1\| < \gamma < \|u_2\|$.

Example 3. *Consider the following q-fractional boundary value problem (23) with*

$$f(t,u) = \begin{cases} \dfrac{1}{t+9u} + \dfrac{1}{10}u^5, & 0 \le t \le 1, \quad u \le 1, \\ \dfrac{u}{t+9u} + \dfrac{u}{1000} + 1, & 0 \le t \le 1, \quad 1 < u, \end{cases} \tag{26}$$

and other conditions also hold. Choosing $\mu = \frac{1}{2}$, $\nu = 2$ *and* $\gamma = 10$, *then* $0 < \mu < \nu < \Delta\gamma$. *Now, we can verify the validity of conditions (**H9**)–(**H11**) in Theorem 4. Indeed, by direct computations, we have*

$f(t,u) \le \frac{1}{9} + \frac{1}{10}\left(\frac{1}{2}\right)^5 \approx 0.0557608 < 0.05744486 \approx \mu\mathfrak{L}_2^{-1}$, *for all* $t \in [0,1]$, *and* $0 \le u \le \mu$;

$f(t,u) \ge \frac{1}{10} + \frac{2}{1000} + 1 = 1.102 > 0.6386077 \approx \nu\mathfrak{L}_1^{-1}$, *for all* $t \in [0,1]$, *and* $\nu \le u \le \nu\mathfrak{L}^{-1}$;

$f(t,u) \le \frac{1}{9} + + \frac{10}{1000} + 1 \approx 1.121111 < 1.148897 \approx \gamma\mathfrak{L}_2^{-1}$, *for all* $t \in [0,1]$, *and* $0 \le u \le \gamma$.

Thus, according to Theorem 4, the problem (24) with (26) has at least three positive solutions $u_1(t)$, $u_2(t)$, *and* $u_3(t)$ *satisfying*

$$\|u_1\| < \frac{1}{2}, \quad 2 < \theta(u_2), \text{ and } \|u_3\| > \frac{1}{2}, \quad \theta(u_3) < 2.$$

5. Conclusions

In this paper, a class of nonlinear Caputo type fractional q-difference equations with integral boundary conditions are studied. By using some well-known fixed point theorems, the existence of one or multiple positive solutions are established for nonlinear Caputo type fractional q-difference equations with integral boundary conditions. Finally, two examples are presented to illustrate the effectiveness of the obtained results.

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References

1. Al-Salam, W.A. Some fractional q-integrals and q-derivatives. *Proc. Edinb. Math. Soc.* **1966**, *15*, 135–140.
2. Agarwal, R.P. Certain fractional q-integrals and q-derivatives. *Proc. Camb. Philos. Soc.* **1969**, *66*, 365–370.
3. Rajković, P.M.; Marinković, S.D.; Stanković, M.S. On q-analogues of Caputo derivative and Mittag-Leffler function. *Fract. Calc. Appl. Anal.* **2007**, *10*, 359–373.
4. Abdeljawad, T.; Benli, B.; Baleanu, D. A generalized q-Mittag-Leffler function by q-Captuo fractional linear equations. *Abstr. Appl. Anal.* **2012**, *2012*, 546062.
5. Yang, W. Positive solution for fractional q-difference boundary value problems with ϕ-Laplacian operator. *Bull. Malays. Math. Sci. Soc.* **2013**, *36*, 1195–1203.
6. Yang, W. Positive solutions for boundary value problems involving nonlinear fractional q-difference equations. *Differ. Equ. Appl.* **2013**, *5*, 205–219.
7. Yang, W. Positive solutions for three-point boundary value problem of nonlinear fractional q-difference equation. *Kyungpook Math. J.* **2016**, *56*, 419–430.
8. Yuan, Q.; Yang, W. Positive solutions of nonlinear boundary value problems for delayed fractional q-difference systems. *Adv. Differ. Equ.* **2014**, *2014*, 51.
9. Yuan, Q.; Yang, W. Positive solutions for q-fractional four-point boundary value problems with p-Laplacian operator. *J. Inequal. Appl.* **2014**, *2014*, 481.
10. Zhao, Y.; Ye, G.; Chen, H. Multiple positive solutions of a singular semipositone integral boundary value problem for fractional q-derivatives equation. *Abstr. Appl. Anal.* **2013**, *2013*, 643571.
11. Li, X.; Han, Z.; Sun, S.; Lu, H. Boundary value problems for fractional q-difference equations with nonlocal conditions. *Adv. Differ. Equ.* **2014**, *2014*, 57.
12. Jiang, M.; Zhong, S. Existence of extremal solutions for a nonlinear fractional q-difference system. *Mediterr. J. Math.* **2016**, *13*, 279–299.
13. Ferreira, R.A.C. Nontrivial solutions for fractional q-difference boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2010**, *2010*, 70.
14. Pongarm, N.; Asawasamrit, S.; Tariboon, J.; Ntouyas, S.K. Multi-strip fractional q-integral boundary value problems for nonlinear fractional q-difference equations. *Adv. Differ. Equ.* **2014**, *2014*, 193.
15. Ahmad, B.; Ntouyas, S.K.; Alsaedi, A.; Al-Hutami, H. Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions. *Electron. J. Qual. Theory Differ. Equ.* **2014**, *2014*, 26.
16. Ahmad, B.; Nieto, J.J.; Alsaedi, A.; Al-Hutami, H. Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. *Filomat* **2014**, *28*, 1719–1736.
17. Yang, W. Anti-periodic boundary value problems involving nonlinear fractional q-difference equations. *Malaya J. Mat.* **2013**, *4*, 107–114.
18. Li, X.; Han, Z.; Sun, S. Anti-periodic boundary value problems for fractional q-difference equations. *J. Appl. Math. Comput.* **2016**, *50*, 243–257.
19. Ahmad, B.; Etemad, S.; Ettefagh, M.; Rezapour, S. On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions. *Bull. Math. Soc. Sci. Math. Roumanie* **2016**, *59*, 119–134.
20. Abdallah, B.E.-D.; Abdeljawad, T. An anti-periodic Caputo q-fractional boundary value problem with a p-Laplacian operator. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **2014**, *63*, 91–99.
21. Abdeljawad, T.; Baleanu, D. Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function. *Commun. Nonlinear Sci. Numer. Simulat.* **2011**, *16*, 4682–4688.
22. Jarad, F.; Abdeljawad, T.; Gundodu, E.; Baleanu, D. On the Mittag-Leffler stability of q-fractional nonlinear dynamical systems. *Proc. Romanian Acad. A* **2011**, *12*, 309–314.
23. Abdeljawad, T.; Alzabut, J. The q-fractional analogue for Gronwall-type inequality. *J. Funct. Spaces Appl.* **2013**, *2013*, 543839.
24. Abdeljawad, T.; Alzabut, J.; Baleanu, D. A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems. *J. Inequal. Appl.* **2016**, *2016*, 240.
25. Zhao, Y.; Chen, H.; Zhang, Q. Existence and multiplicity of positive solutions for nonhomogeneous boundary value problems with fractional q-derivative. *Bound. Value Probl.* **2013**, *2013*, 103.

26. Graef, J.R.; Kong, L. Positive solutions for a class of higher order boundary value problems with fractional q-derivatives. *Appl. Math. Comput.* **2012**, *218*, 9682–9689.
27. Graef, J.R.; Kong, L. Existence of positive solutions to a higher order singular boundary value problem with fractional q-derivatives. *Fract. Calc. Appl. Anal.* **2013**, *16*, 695–708.
28. Agarwal, R.P.; Ahmad, B.; Alsaedi, A.; Al-Hutami, H. Existence theory for q-antiperiodic boundary value problems of sequential q-fractional integrodifferential equations. *Abstr. Appl. Anal.* **2014**, *2014*, 207547.
29. Ahmad, B.; Nieto, J.J.; Alsaedi, A.; Al-Hutami, H. Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. *J. Frankl. Inst.* **2014**, *351*, 2890–2909.
30. Zhao, Q.; Yang, W. Positive solutions for singular coupled integral boundary value problems of nonlinear higher-order fractional q-difference equations. *Adv. Differ. Equ.* **2015**, *2015*, 290.
31. Ferreira, R.A.C. Positive solutions for a class of boundary value problems with fractional q-differences. *Comput. Math. Appl.* **2011**, *61*, 367–373.
32. Ahmad, B.; Ntouyas, S.; Purnaras, I. Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. *Adv. Differ. Equ.* **2012**, *2012*, 140.
33. Zhou, W.; Liu, H. Uniqueness and existence of solution for a system of fractional q-difference equations. *Abstr. Appl. Anal.* **2014**, *2014*, 340159.
34. Yang, W. Positive solutions for nonlinear semipositone fractional q-difference system with coupled integral boundary conditions. *Appl. Math. Comput.* **2014**, *244*, 702–725.
35. Li, Y.; Yang, W. Monotone iterative method for nonlinear fractional q-difference equations with integral boundary conditions. *Adv. Differ. Equ.* **2015**, *2015*, 294.
36. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
37. Annaby, M.H.; Mansour, Z.S. *q-Fractional Calculus and Equations*; Springer: Berlin/Heidelberg, Germany, 2012.
38. Rajković, P.M.; Marinković, S.D.; Stanković, M.S. Fractional integrals and derivatives in q-calculus. *Appl. Anal. Discret. Math.* **2007**, *1*, 311–323.
39. Guo, D.; Lashmikanthan, V. *Nonlinear Problems in Abstract Cones*; Academic Press: San Diego, CA, USA, 1988.
40. Legget, R.W.; Williams, R.L. Multiple positive fixed points of nonlinear operators on ordered Banach spaces. *Indiana Univ. Math. J.* **1979**, *28*, 673–688.