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Article

Complete Classification of Cylindrically Symmetric Static Spacetimes and the Corresponding Conservation Laws

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Abstract: In this paper we find the Noether symmetries of the Lagrangian of cylindrically symmetric static spacetimes. Using this approach we recover all cylindrically symmetric static spacetimes appeared in the classification by isometries and homotheties. We give different classes of cylindrically symmetric static spacetimes along with the Noether symmetries of the corresponding Lagrangians and conservation laws.

Keywords: Noether symmetries; cylindrically symmetric spacetime; first integrals; solutions of Einstein field equations

1. Introduction

Symmetries play an important role in different areas of research, including differential equations and general relativity. The Einstein field equations (EFE),

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{1}$$

are the building blocks of the theory of general relativity. These are non-linear partial differential equations and it is not easy to obtain exact solutions of these equations. Symmetries help a lot in finding solutions of these equations. These solutions (spacetimes) have been classified by using different spacetime symmetries [1–7]. Among different spacetime symmetries isometries, the Killing vectors (KVs) are important becuase they help in understanding the geometric properties of spaces, and, there is some conserved quantity corresponding to each isometry. It is well known fact that isometries or KVs are a subset of the Noether symmetries (NS) i.e.,

$$KVs \subseteq NS$$

This relation shows that the KVs do not lead to all the conserved quantities or the first integrals. Therefore, it reasonable to look for Noether symmetries of the Lagrangians of spacetimes. Instead of taking the Lagrangian from—the most general form of the spacetime—and solving a set of partial differential equations involving unknown metric coefficients, one may adopt an easy approach and directly look for the NS of Lagrangian of all known spacetimes obtained through classification by KVs and homotheties given in [7,8]. However, we have adopted here the longer route, so that we also have a counter check on the spacetimes obtained through the classification by the KVs. Using the Lagrangians of plane and spherical symmetric static spacetimes, complete lists of Noether symmetries and first integrals have been obtained [9,10]. Some new solutions have also been found in the cases of plane and spherical symmetry. Here, we classify cylindrically symmetric static spacetimes. These spacetimes are axisymmetric (symmetric about an infinite axis) and translationally symmetric about the given infinite axis (in our case z axis) and time. The most general form of cylindrically symmetric static space-time is [11–13].

$$ds^2 = e^{\nu(r)}dt^2 - dr^2 - e^{\mu(r)}b^2d\theta^2 - e^{\lambda(r)}dz^2 \tag{2}$$

where b is a constant used to make the dimension of the metric homogeneous. The spacetime metric is obtained from the metric given in Equation (22.1) of reference [11] by taking $A = 0$ and $\rho \to r$ to make the coefficient of dr^2 equal to 1. Also, we take the static case that is all the functions ν, μ and λ are function of radial coordinate r only, this type of spacetime is given in Equation (22.20) in the same reference. The ranges of the coordinates are $-\infty < t < \infty, 0 < r < \infty, 0 \leq \theta \leq 2\pi$ and $-\infty < z < \infty$. By definition, these spacetimes admit the following set of three isometries (minimum number).

$$\mathbf{X_0} = \frac{\partial}{\partial t}, \qquad \mathbf{X_1} = \frac{\partial}{\partial \theta}, \qquad \mathbf{X_2} = \frac{\partial}{\partial z} \tag{3}$$

2. The Noether Symmetry Governing Equation

A symmetry

$$\mathbf{X} = \xi\frac{\partial}{\partial s} + \eta^i\frac{\partial}{\partial x^i}, \quad (i = 1, 2, ...n) \tag{4}$$

is Noether symmetry if it leaves the action

$$W = \int L\left(s, x^i(s), \dot{x}^i(s)\right) ds \tag{5}$$

invariant up to some gauge function A, i.e., under the transformation

$$\begin{aligned} &\tilde{s} \to s + \epsilon\xi(s, x^i) \\ &\tilde{x}^i \to x^i + \epsilon\eta^i(s, x^i) \\ &\tilde{\dot{x}}^i \to \dot{x}^i + \epsilon\eta^i(s, x^i, \dot{x}^i) \end{aligned}$$

the action (5) takes the form

$$\tilde{W} = \int L\left(\tilde{s}, \tilde{x}^i(s), \tilde{\dot{x}}^i(s)\right) ds \tag{6}$$

The variation in the action up to the gauge function is

$$\tilde{W} - W = \int \mathbf{D}A\left(s, x^i(s)\right) ds \tag{7}$$

where L is the Lagrangian, s the independent variable, x^i are the dependent variables and $\dot{x}^i$ their derivatives with respect to s and $\mathbf{D}$ is the standard total derivative operator given by

$$\mathbf{D} = \frac{\partial}{\partial s} + \dot{x}^i\frac{\partial}{\partial x^i} \tag{8}$$

After simplification Equation (7) takes the form

$$\mathbf{X}^{(1)}L + \mathbf{D}(\xi)L = \mathbf{D}A \tag{9}$$

where

$$\mathbf{X}^{(1)} = \mathbf{X} + \eta^i_{,s}\frac{\partial}{\partial \dot{x}^i} \tag{10}$$

is the first order prolonged generator. The coefficients of Noether symmetry, namely, ξ and η^i are functions of (s, x^i). The coefficients of prolonged operator $\mathbf{X}^{(1)}$, namely, $\eta^i_{,s}$, are functions of $(s, x^i(s), \dot{x}^i(s))$ and are defined as

$$\eta^i_{,s} = \mathbf{D}(\eta^i) - \dot{x}^i \mathbf{D}(\xi) \tag{11}$$

where x^i refers to the space of dependent variables. The importance of these symmetries is that corresponding to each Noether symmetry there is a conservation law/first integrals. For example, the time translation $\partial/\partial t$ and rotation $\partial/\partial\theta$ respectively give conservation of energy and angular momentum, respectively [14,15]. From differential geometry we know that for the general cylindrically symmetric static space-times given by Equation (2), the usual Lagrangian is

$$L = e^{\nu(r)}\dot{t}^2 - \dot{r}^2 - b^2 e^{\mu(r)}\dot{\theta}^2 - e^{\lambda(r)}\dot{z}^2 \tag{12}$$

Using Equation (12) in Equation (9) we get the following system of 19 partial differential equations (PDEs)

$$\begin{aligned}
&\xi_t = 0, \quad \xi_r = 0, \quad \xi_\theta = 0, \quad \xi_z = 0, \quad A_s = 0 \\
&2e^{\nu(r)}\eta^0_s = A_t, \quad -2\eta^1_s = A_r \\
&-2b^2 e^{\mu(r)}\eta^2_s = A_\theta, \quad -2e^{\lambda(r)}\eta^3_s = A_z \\
&\mu_r(r)\eta^1 + 2\eta^2_\theta - \xi_s = 0, \quad b^2 e^{\mu(r)}\eta^2_z - e^{\lambda(r)}\eta^3_\theta = 0 \\
&\eta^1_\theta + b^2 e^{\mu(r)}\eta^2_r = 0, \quad \eta^1_z + e^{\lambda(r)}\eta^3_r = 0 \\
&e^{\nu(r)}\eta^0_r - \eta^1_t = 0, \quad e^{\nu(r)}\eta^0_\theta - b^2 e^{\mu(r)}\eta^2_t = 0 \\
&e^{\nu(r)}\eta^0_z - e^{\lambda(r)}\eta^3_t = 0, \quad \nu_r(r)\eta^1 + 2\eta^0_t - \xi_s = 0 \\
&\lambda_r(r)\eta^1 + 2\eta^3_z - \xi_s = 0, \quad 2\eta^1_r - \xi_s = 0
\end{aligned} \tag{13}$$

This system consists of nine unknowns, $\xi, \eta^i (i = 0, 1, 2, 3), \lambda, \mu, \nu$, and A. Solutions of this system give the Lagrangian along with the Noether symmetries. Corresponding to these Lagrangians, one may easily write spacetimes, which are the exact solutions of EFE. In the following sections, a list of metric coefficients, Noether symmetries and corresponding first integrals are given.

3. Five Symmetries

The minimal set of Noether symmetries for cylindrically symmetric static spacetimes consists of three isometries given in Equation (3) and $Y_0 = \frac{\partial}{\partial s}$, this makes a set of four Noether symmetries

$$\mathbf{X_0} = \frac{\partial}{\partial t}, \qquad \mathbf{X_1} = \frac{\partial}{\partial \theta}, \qquad \mathbf{X_2} = \frac{\partial}{\partial z}, \quad \mathbf{Y}_0 = \frac{\partial}{\partial s} \tag{14}$$

This is the minimal set of Noether symmetries for cylindrically symmetric static spacetime and is a solution of system (13) for arbitrary values of $\mu(r)$, $\nu(r)$ and $\lambda(r)$. Corresponding to each Noether symmetry a first integral can be obtained using the relation

$$\phi = \frac{\partial L}{\partial(\dot{x}^i)}(\eta^i - \xi\dot{x}^i) + L\xi - A \tag{15}$$

where x^i denotes the dependent variables. First integrals (conservation laws) corresponding to the minimal set of Noether symmetries are given in Table 1.

Table 1. First integrals for the minimal set of Noether symmetries.

Gen	First Integrals
$\mathbf{X_0}$	$\phi_0 = -2e^{\nu(r)}\dot{t}$
$\mathbf{X_1}$	$\phi_1 = 2e^{\mu(r)}b^2\dot{\theta}$
$\mathbf{X_2}$	$\phi_2 = 2e^{\lambda(r)}\dot{z}$
$\mathbf{Y_0}$	$\phi_3 = e^{\nu(r)}\dot{t}^2 - \dot{r}^2 - e^{\mu(r)}b^2\dot{\theta}^2 - e^{\lambda(r)}\dot{z}^2$

Except for the first class in Table 2, all the classes with five Noether symmetries admit only the minimal set of isometries (only three isometries), and we see from the same table that in some cases the values of the functions $\mu(r)$, $\nu(r)$ or $\lambda(r)$ are arbitrary; therefore, there may be infinitely many classes for five Noether symmetries. However, some examples of metrics with five Noether symmetries are given in Table 2.

Table 2. Metric coefficients for five symmetries.

No.	$\nu(r)$	$\mu(r)$	$\lambda(r)$
1.	$k\ln\frac{r}{\alpha}$	$\frac{r}{\alpha}$	const
2.	$k\ln\frac{r}{\alpha}$	$const \neq \mu(r) \neq k\ln\frac{r}{\alpha}$	const
3.	$\frac{r}{\alpha}$	$2\ln\frac{r}{\alpha}$	const
4.	$const \neq \nu(r) \neq k\ln\frac{r}{\alpha}$	$k\ln\frac{r}{\alpha}$	const
5.	$k\ln\frac{r}{\alpha}$	const	$\frac{r}{\alpha}$
6.	$k\ln\frac{r}{\alpha}$	const	$const \neq \mu(r) \neq k\ln\frac{r}{\alpha}$
7.	$\frac{r}{\alpha}$	const	$k\ln\frac{r}{\alpha}$
8.	$const \neq \nu(r) \neq k\ln\frac{r}{\alpha}$	const	$k\ln\frac{r}{\alpha}$
9.	const	$k\ln\frac{r}{\alpha}$	$const \neq \mu(r) \neq k\ln\frac{r}{\alpha}$
10.	const	$k\ln\frac{r}{\alpha}$	$\frac{r}{\alpha}$
11.	const	$\frac{r}{\alpha}$	$k\ln\frac{r}{\alpha}$
12.	$\frac{r}{\alpha}$	$\frac{r}{\beta}$	$\frac{r}{\gamma}$
13.	$2\ln\frac{r}{\alpha}$	$k\ln\frac{r}{\beta}$	$l\ln\frac{r}{\gamma}$
14.	$k\ln\frac{r}{\alpha}$	$2\ln\frac{r}{\beta}$	$l\ln\frac{r}{\gamma}$
15.	$k\ln\frac{r}{\alpha}$	$l\ln\frac{r}{\beta}$	$2\ln\frac{r}{\gamma}$
16.	$k\ln\frac{r}{\alpha}$	$l\ln\frac{r}{\beta}$	$h\ln\frac{r}{\gamma}$

For the values of functions $\mu(r)$, $\nu(r)$ or $\lambda(r)$ given in Table 2 the system (13) give us five Noether symmetries, four Noether symmetries are given in Equation (14) and fifth Noether symmetry along with conservation law and gauge function are given in Table 3 correspondingly.

Table 3. Fifth symmetry with gauge term and respective first integral.

No.	Fifth Symmetry	Gauge Functions	First Integral
1.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
2.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
3.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
4.	$\mathbf{Y_1} = s\frac{\partial}{\partial z}$	$A = -2z$	$\phi_4 = 2[s\dot{z} - z]$
5.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2b^2\theta$	$\phi_4 = 2b^2[s\dot{\theta} - \theta]$
6.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2b^2\theta$	$\phi_4 = 2b^2[s\dot{\theta} - \theta]$
7.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2b^2\theta$	$\phi_4 = 2b^2[s\dot{\theta} - \theta]$
8.	$\mathbf{Y_1} = s\frac{\partial}{\partial \theta}$	$A = -2b^2\theta$	$\phi_4 = 2b^2[s\dot{\theta} - \theta]$
9.	$\mathbf{Y_1} = s\frac{\partial}{\partial t}$	$A = 2t$	$\phi_4 = 2[-s\dot{t} + t]$
10.	$\mathbf{Y_1} = s\frac{\partial}{\partial t}$	$A = 2t$	$\phi_4 = 2[-s\dot{t} + t]$
11.	$\mathbf{Y_1} = s\frac{\partial}{\partial t}$	$A = 2t$	$\phi_4 = 2[-s\dot{t} + t]$

Table 3. *Cont.*

No.	Fifth Symmetry	Gauge Functions	First Integral
12.	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\beta}\frac{\partial}{\partial \theta} - \frac{z}{2\gamma}\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = 2\dot{r} + \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} - \frac{b^2\theta\dot{\theta}e^{\frac{r}{\beta}}}{\beta} - \frac{z\dot{z}e^{\frac{r}{\gamma}}}{\gamma}$
13.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-l}{4}z\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = sL + \frac{(2-k)\theta\dot{\theta}b^2}{2}\frac{r^k}{\alpha^k} + \frac{(2-l)z\dot{z}}{2}\frac{r^l}{\alpha^l}$
14.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{2-l}{4}z\frac{\partial}{\partial z}$	$A = 0$	$\phi_4 = sL - \frac{(2-k)t\dot{t}b^2}{2}\frac{r^k}{\alpha^k} + \frac{(2-l)z\dot{z}}{2}\frac{r^l}{\alpha^l}$
15.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{2-l}{4}\theta\frac{\partial}{\partial \theta}$	$A = 0$	$\phi_4 = sL - \frac{(2-k)t\dot{t}}{2}\frac{r^k}{\alpha^k} + \frac{(2-l)b^2\theta\dot{\theta}}{2}\frac{r^l}{\alpha^l}$
16.	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{2-l}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-h}{4}z\frac{\partial}{\partial z}$	$A = 0$	$sL - \frac{(2-k)t\dot{t}}{2}\frac{r^k}{\alpha^k} + \frac{(2-l)\theta\dot{\theta}b^2}{2}\frac{r^l}{\alpha^l} + \frac{(2-h)z\dot{z}}{2}\frac{r^h}{\alpha^h}$

4. Six Symmetries

In Table 4, a list of the metric coefficients, additional symmetries and gauge functions are given. There are 24 different classes given in Table 4 which are the only candidates for six Noether symmetries. The four Noether symmetries are given in Equation (14) and the corresponding two additional Noether symmetries along with metric coefficients $\mu(r)$, $\nu(r)$ and $\lambda(r)$ and gauge function are given in Table 4. There are 24 different solutions of the system given in Equation (13). We see from Table 4 that the 24 classes of six Noether symmetries split in four different classes according to different types of Noehter symmetries as

(i) in classes 1-3 the first symmetry is a mixed Noether symmetry of translation and scaling, and the second Noether symmetry is a galilean transformation,
(ii) in classes 4-6 the first symmetry is the mixed one and the second is either boast or rotation,
(iii) in classes 7-15 the first symmetry is scaling while the second is galilean transformation and,
(iv) in classes 16-24 the first symmetry is scaling and the second symmetry is either boast or rotation.

Table 4. Metric coefficients, additional symmetries and gauge functions.

No.	Metrics	Symmetries, Gauge Functions
1.	$\nu(r) = \frac{r}{\alpha}, \mu(r) = const, \lambda(r) = \frac{r}{\beta}$	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{z}{2\beta}\frac{\partial}{\partial z}, \mathbf{Y_1} = s\frac{\partial}{\partial \theta}, A_1 = -2b^2\theta$
2.	$\nu(r) = \frac{r}{\alpha}, \mu(r) = \frac{r}{\beta}, \lambda(r) = const$	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\beta}\frac{\partial}{\partial \theta}, \mathbf{Y_1} = s\frac{\partial}{\partial z}, A_1 = -2z$
3.	$\nu(r) = const, \mu(r) = \frac{r}{\alpha}, \lambda(r) = \frac{r}{\beta}$	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \mathbf{Y_1} = s\frac{\partial}{\partial t}, A_1 = -2t$
4.	$\nu(r) = \frac{r}{\alpha}, \mu(r) = \frac{r}{\beta}, \lambda(r) = \frac{r}{\alpha}$	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\beta}\frac{\partial}{\partial \theta} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \mathbf{X_4} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$
5.	$\nu(r) = \frac{r}{\alpha}, \mu(r) = \frac{r}{\beta}, \lambda(r) = \frac{r}{\beta}$	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\beta}\frac{\partial}{\partial \theta} - \frac{z}{2\beta}\frac{\partial}{\partial z}, \mathbf{X_4} = z\frac{\partial}{\partial \theta} - b^2\theta\frac{\partial}{\partial z}$
6.	$\nu(r) = \frac{r}{\alpha}, \mu(r) = \frac{r}{\alpha}, \lambda(r) = \frac{r}{\beta}$	$\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{z}{2\beta}\frac{\partial}{\partial z}, \mathbf{X_4} = \frac{t}{b^2}\frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial t}$
7.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = 2\ln\frac{r}{\alpha}, \lambda(r) = const$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial z}, A_2 = -2z$
8.	$\nu(r) = 2\ln\frac{r}{\alpha}, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = const$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{z}{2}\frac{\partial}{\partial z}, \mathbf{Y_2} = s\frac{\partial}{\partial z}, \quad A_2 = -2z$
9.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = l\ln\frac{r}{\alpha}, \lambda(r) = const$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-l}{4}\theta\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z}, \mathbf{Y_2} = s\frac{\partial}{\partial z}, A_2 = -2z$
10.	$\nu(r) = const, \mu(r) = 2\ln\frac{r}{\alpha}, \lambda(r) = k\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2} = s\frac{\partial}{\partial t}, A_2 = 2t$
11.	$\nu(r) = const, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = 2\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{t}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta}, \mathbf{Y_2} = s\frac{\partial}{\partial}, A_2 = 2t$
12.	$\nu(r) = const, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = l\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-l}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2} = s\frac{\partial}{\partial z}, A_2 = 2t$
13.	$\nu(r) = 2\ln\frac{r}{\alpha}, \mu(r) = const, \lambda(r) = k\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, A_2 = -2b^2\theta$
14.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = const, \lambda(r) = 2\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta}, \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, A_2 = -2b^2\theta$
15.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = const, \lambda(r) = l\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{2-l}{4}z\frac{\partial}{\partial z}, \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, A_2 = -2b^2\theta$
16.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = 2\ln\frac{r}{\alpha}, \lambda(r) = 2\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \quad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$
17.	$\nu(r) = 2\ln\frac{r}{\alpha}, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = k\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \mathbf{X_3} = z\frac{\partial}{\partial \theta} - b^2\theta\frac{\partial}{\partial z}$
18.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = l\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta}, \mathbf{X_3} = \theta\frac{\partial}{\partial \theta} + \frac{t}{b^2}\frac{\partial}{\partial \theta}$
19.	$\nu(r) = 2\ln\frac{r}{\alpha}, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = 2\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta}, \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$
20.	$\nu(r) = 2\ln\frac{r}{\alpha}, \mu(r) = 2\ln\frac{r}{\alpha}, \lambda(r) = k\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \mathbf{X_3} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}$
21.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = 2\ln\frac{r}{\alpha}, \lambda(r) = 2\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}t\frac{\partial}{\partial t}, \mathbf{X_3} = z\frac{\partial}{\partial \theta} - b^2\theta\frac{\partial}{\partial z}$
22.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = l\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-l}{4}z\frac{\partial}{\partial z}, \mathbf{X_3} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}$
23.	$\nu(r) = k\ln\frac{r}{\alpha}, \mu(r) = l\ln\frac{r}{\alpha}, \lambda(r) = k\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-l}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$
24.	$\nu(r) = l\ln\frac{r}{\alpha}, \mu(r) = k\ln\frac{r}{\alpha}, \lambda(r) = k\ln\frac{r}{\alpha}$	$\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-l}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \mathbf{X_3} = z\frac{\partial}{\partial \theta} - \theta b^2\frac{\partial}{\partial z}$

Table 5 contains the first integrals (Conservation laws) corresponding to the Noether symmetries given in Table 4.

Table 5. Symmetry generators and first integrals for Table 4.

No.	Generators	First Integrals
1.	$\mathbf{X_3}$, $\mathbf{Y_1}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\alpha}} t\dot{t}}{\alpha} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}, \quad \phi_5 = 2b^2[s\dot{\theta} - \theta]$
2.	$\mathbf{X_3}$, $\mathbf{Y_1}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\alpha}} t\dot{t}}{\alpha} - \frac{e^{\frac{r}{\beta}} \theta\dot{\theta}}{\beta}, \quad \phi_5 = 2b^2[s\dot{z} - z]$
3.	$\mathbf{X_3}$, $\mathbf{Y_1}$	$\phi_4 = 2\dot{r} - \frac{e^{\frac{r}{\alpha}} \theta\dot{\theta}}{\alpha} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}, \quad \phi_5 = 2b^2[-s\dot{t} + t]$
4.	$\mathbf{X_3}$, $\mathbf{X_4}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\beta}} t\dot{t}}{\beta} - \frac{e^{\frac{r}{\beta}} \theta\dot{\theta}}{\beta} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}, \quad \phi_5 = 2e^{\frac{r}{\alpha}}[t\dot{z} - z\dot{t}]$
5.	$\mathbf{X_3}$, $\mathbf{X_4}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\beta}} t\dot{t}}{\beta} - \frac{e^{\frac{r}{\beta}} \theta\dot{\theta}}{\beta} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}, \quad \phi_5 = 2b^2 e^{\frac{r}{\beta}}[z\dot{\theta} - \theta\dot{z}]$
6.	$\mathbf{X_3}$, $\mathbf{X_4}$	$\phi_4 = 2\dot{r} + \frac{e^{\frac{r}{\beta}} t\dot{t}}{\beta} - \frac{e^{\frac{r}{\alpha}} \theta\dot{\theta}}{\alpha} - \frac{e^{\frac{r}{\beta}} z\dot{z}}{\beta}, \quad \phi_5 = 2e^{\frac{r}{\alpha}}[t\dot{\theta} - \theta\dot{t}]$
7.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + r\dot{r} + z\dot{z}, \quad \phi_5 = 2[s\dot{z} - z]$
8.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL + \frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta} + r\dot{r} + z\dot{z}, \quad \phi_5 = 2[s\dot{z} - z]$
9.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + \frac{2-l}{2}\frac{r^l}{\alpha^l} \theta\dot{\theta} + r\dot{r} + z\dot{z}, \quad \phi_5 = 2[s\dot{z} - z]$
10.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - t\dot{t} + r\dot{r} + \frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2[t - s\dot{t}]$
11.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - t\dot{t} + r\dot{r} + \frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta}, \quad \phi_5 = 2[t - s\dot{t}]$
12.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - t\dot{t} + \frac{2-l}{2}{\frac{r^l}{\alpha^l}}^2 \theta\dot{\theta} + r\dot{r} + \frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2[-s\dot{t} + t]$
13.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL + r\dot{r} + b^2\theta\dot{\theta} + \frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2b^2[s\dot{\theta} - \theta]$
14.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + r\dot{r} + b^2\theta\dot{\theta}, \quad \phi_5 = 2b^2[s\dot{\theta} - \theta]$
15.	$\mathbf{Y_1}$, $\mathbf{Y_2}$	$\phi_4 = sL - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + r\dot{r} + b^2\theta\dot{\theta} + \frac{2-l}{2}\frac{r^l}{\alpha^l} z\dot{z}, \quad \phi_5 = 2b^2[s\dot{\theta} - \theta]$
16.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + r\dot{r}, \quad \phi_5 = 2[t\dot{z} - z\dot{t}]$
17.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} + b^2\theta\dot{\theta} + \frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta} + \frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2b^2[z\dot{\theta} - \theta\dot{z}]$
18.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + r\dot{r} + b^2\frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta} + \frac{2-l}{2}\frac{r^l}{\alpha^l} z\dot{z}, \quad \phi_5 = 2b^2[t\dot{\theta} - \theta\dot{t}]$
19.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} + b^2\frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta}, \quad \phi_5 = 2[t\dot{z} - z\dot{t}]$
20.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} + b^2\frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2[t\dot{\theta} - \theta\dot{t}]$
21.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} + b^2\frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t}, \quad \phi_5 = 2b^2[z\dot{\theta} - \theta\dot{z}]$
22.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + b^2\frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta} + \frac{2-l}{2}\frac{r^l}{\alpha^l} z\dot{z}, \quad \phi_5 = 2[t\dot{\theta} - \theta\dot{t}]$
23.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} - \frac{2-k}{2}\frac{r^k}{\alpha^k} t\dot{t} + b^2\frac{2-l}{2}\frac{r^l}{\alpha^l} \theta\dot{\theta} + \frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2[t\dot{z} - z\dot{t}]$
24.	$\mathbf{Y_1}$, $\mathbf{X_3}$	$\phi_4 = sL + r\dot{r} - \frac{2-l}{2}\frac{r^l}{\alpha^l} t\dot{t} + b^2\frac{2-k}{2}\frac{r^k}{\alpha^k} \theta\dot{\theta} + \frac{2-k}{2}\frac{r^k}{\alpha^k} z\dot{z}, \quad \phi_5 = 2b^2[z\dot{\theta} - \theta\dot{z}]$

5. Seven Symmetries

The classes for seven Noether symmetries are given in this section. There are only three classes of cylindrically symmetric static spacetime that admit seven Noether symmetries.

Class 1 of seven Noether Symmetries:

$$ds^2 = dt^2 - dr^2 - b^2(\frac{r}{\alpha})^k d\theta^2 - (\frac{r}{\alpha})^k dz^2 \tag{16}$$

The Lagrangian corresponding to the metric (16) admits seven Noether symmetries, four of which are given in Equation (14) and the remaining are listed below.

$$\begin{aligned}
\mathbf{Y_1} &= s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z} \\
\mathbf{Y_2} &= s\frac{\partial}{\partial t}, \quad A_2 = 2t, \qquad \mathbf{X_3} = -\frac{z}{b^2}\frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial z}
\end{aligned} \tag{17}$$

The first integrals corresponding to the Noether symmetries are given in Equation (17) are given in Table 6.

Table 6. First Integrals corresponding to symmetries given in (17).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^k}{\alpha^k}[\theta\dot{z} - z\dot{\theta}]$
$\mathbf{Y_1}$	$\phi_5 = sL - t\dot{t} + r\dot{r} + b^2\frac{r^k}{\alpha^k}\frac{2-k}{2}\theta\dot{\theta} + \frac{r^k}{\alpha^k}\frac{2-k}{2}z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2[t - s\dot{t}]$

Class 2 of seven Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^k dt^2 - dr^2 - b^2(\frac{r}{\alpha})^k d\theta^2 - dz^2 \tag{18}$$

We have seven Noether symmetries for the Lagrangian corresponding to the spacetime given in Equation (18), four Noether syymmetries are given in Equation (14) and the additional three symmetries are,

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z} \\ &\mathbf{Y_2} = s\frac{\partial}{\partial z}, \qquad A_2 = -2z, \qquad \mathbf{X_3} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta} \end{aligned} \tag{19}$$

The corresponding first integrals are given in Table 7.

Table 7. First Integrals corresponding to symmetries given in (19).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^k}{\alpha^k}[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = sL - \frac{r^k}{\alpha^k}\frac{2-k}{2}t\dot{t} + r\dot{r} + b^2\frac{r^k}{\alpha^k}\frac{2-k}{2}\theta\dot{\theta} + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2b^2[-z + s\dot{z}]$

Class 3 of seven Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^k dt^2 - dr^2 - b^2 d\theta^2 - (\frac{r}{\alpha})^k dz^2 \tag{20}$$

The following three are the additional Noether symmetries corresponding to the Lagrangian for spacetime (20).

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z} \\ &\mathbf{Y_2} = s\frac{\partial}{\partial \theta}, \qquad A_2 = -2b^2\theta, \qquad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} \end{aligned} \tag{21}$$

The first integrals conservation laws corresponding to the Noether symmetries given in Equation (21) are given in Table 8.

Table 8. First Integrals corresponding to symmetries given in (21).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^k}{\alpha^k}[t\dot{z} - z\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = sL - \frac{r^k}{\alpha^k}\frac{2-k}{2}t\dot{t} + r\dot{r} + b^2\theta\dot{\theta} + \frac{r^k}{\alpha^k}\frac{2-k}{2}z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2b^2[-\theta + s\dot{\theta}]$

6. Eight Symmetries

In this section we list all those classes of cylindrically symmetric static spacetimes which admit eight Noether symmetries. There are seven different classes of cylindrically symmetric static spacetime that admit eight Noether symmetries, the detail of which are given below.

Class 1 of eight Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^k dt^2 - dr^2 - b^2 d\theta^2 - dz^2 \tag{22}$$

The Lagrangian of the spacetime (22) admits eight Noether symmetries, four of which are given in Equation (14) and the remaining four are listed here in Equation (23)

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4} t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, \quad A_2 = -2b^2\theta \\ &\mathbf{Y_3} = s\frac{\partial}{\partial z}, \qquad A_3 = -2z, \qquad \mathbf{X_3} = z\frac{\partial}{\partial \theta} - b^2\theta\frac{\partial}{\partial z} \end{aligned} \tag{23}$$

The corresponding conservation laws are given in Table 9.

Table 9. First Integrals corresponding to symmetries given in (23).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2b^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{Y_1}$	$\phi_5 = sL - \frac{r^k}{\alpha^k}\frac{2-k}{2}t\dot{t} + r\dot{r} + b^2\theta\dot{\theta} + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2b^2[-\theta + s\dot{\theta}]$
$\mathbf{Y_3}$	$\phi_7 = 2[s\dot{z} - z]$

Class 2 of eight Noether Symmetries:

$$ds^2 = dt^2 - dr^2 - b^2\frac{r^k}{\alpha^k}d\theta^2 - dz^2 \tag{24}$$

We obtain eight Noether symmetries for the Lagrangian corresponding to the spacetime (24). The additional four Noether symmetries are given below,

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial t}, \quad A_2 = 2t \\ &\mathbf{Y_3} = s\frac{\partial}{\partial z}, \quad A_3 = -2z, \quad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} \end{aligned} \tag{25}$$

The corresponding first integrals are given below in Table 10.

Table 10. First Integrals corresponding to symmetries given in (25).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2[t\dot{z} - z\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = sL - t\dot{t} + r\dot{r} + b^2\frac{r^k}{\alpha^k}\frac{2-k}{2}\theta\dot{\theta} + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = 2[t - s\dot{t}]$
$\mathbf{Y_3}$	$\phi_7 = 2[s\dot{z} - z]$

Class 3 of eight Noether Symmetries:

$$ds^2 = dt^2 - dr^2 - b^2 d\theta^2 - (\frac{r}{\alpha})^k dz^2 \tag{26}$$

The additional Noether symmetries of the Lagrangian of spacetime (26) are given here,

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s\frac{\partial}{\partial t}, \quad A_2 = 2t \\ &\mathbf{Y_3} = s\frac{\partial}{\partial \theta}, \quad A_3 = -2b^2\theta, \quad \mathbf{X_3} = b^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta} \end{aligned} \tag{27}$$

The first integral for these symmetries are given in Table 11.

Table 11. First Integrals corresponding to symmetries given in (27).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2b^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = sL - t\dot{t} + r\dot{r} + b^2\theta\dot{\theta} + \frac{2-k}{2}z\dot{z}\frac{r^k}{\alpha^k}$
$\mathbf{Y_2}$	$\phi_6 = 2[t - s\dot{t}]$
$\mathbf{Y_3}$	$\phi_7 = 2b^2[s\dot{\theta} - \theta]$

Class 4 of eight Noether Symmetries:

$$ds^2 = dt^2 - dr^2 - b^2(\frac{r}{\alpha})^2 d\theta^2 - (\frac{r}{\alpha})^2 dz^2 \tag{28}$$

This spacetime admit the following additional Noether symmetries

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{t}{2}\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \quad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + st\frac{\partial}{\partial t} + sr\frac{\partial}{\partial r}, \quad A_2 = t^2 - r^2 \\ &\mathbf{Y_3} = s\frac{\partial}{\partial t}, \quad A_3 = 2t, \quad \mathbf{X_3} = -\frac{z}{b^2}\frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial z} \end{aligned} \tag{29}$$

The corresponding first integrals are given in Table 12.

Table 12. First Integrals corresponding to symmetries given in (29).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^2}{\alpha^2}[\theta\dot{z} - z\dot{\theta}]$
$\mathbf{Y_1}$	$\phi_5 = sL + r\dot{r} - t\dot{t}$
$\mathbf{Y_2}$	$\phi_6 = s^2L + 2sr\dot{r} - 2st\dot{t} + t^2 - r^2$
$\mathbf{Y_3}$	$\phi_7 = 2[-s\dot{t} + t]$

Class 5 of eight Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^2 dt^2 - dr^2 - b^2 d\theta^2 - (\frac{r}{\alpha})^2 dz^2 \tag{30}$$

Following four are the additional Noether symmetries obtain for the Lagrangian corresponding to spacetime (30).

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{\theta}{2}\frac{\partial}{\partial \theta}, \quad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sr\frac{\partial}{\partial t} + s\theta\frac{\partial}{\partial \theta}, \quad A_2 = -r^2 - b^2\theta^2 \\ &\mathbf{Y_3} = s\frac{\partial}{\partial \theta}, \quad A_3 = -2b^2\theta, \quad \mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} \end{aligned} \tag{31}$$

Table 13 contains the first integrals for Noether symmetries given in Equation (31).

Table 13. First Integrals corresponding to symmetries given in (31).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^2}{\alpha^2}[t\dot{z} - z\dot{t}]$
$\mathbf{Y_1}$	$\phi_5 = sL + r\dot{r} + b^2\theta\dot{\theta}$
$\mathbf{Y_2}$	$\phi_6 = s^2L + 2sr\dot{r} + 2sb^2\theta\dot{\theta} - r^2 - b^2\theta^2$
$\mathbf{Y_3}$	$\phi_7 = 2b^2[s\dot{\theta} - \theta]$

Class 6 of eight Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^2 dt^2 - dr^2 - b^2(\frac{r}{\alpha})^2 d\theta^2 - dz^2 \tag{32}$$

Additional symmetries are are

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{z}{2}\frac{\partial}{\partial z}, \quad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sr\frac{\partial}{\partial t} + sz\frac{\partial}{\partial z}, \quad A_2 = -r^2 - z^2 \\ &\mathbf{Y_3} = s\frac{\partial}{\partial z}, \quad A_3 = -2z, \quad \mathbf{X_3} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}. \end{aligned} \tag{33}$$

The first integrals are given in Table 14.

Table 14. First Integrals corresponding to symmetries given in (33).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^2}{\alpha^2}[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_6 = sL + r\dot{r} + z\dot{z}$
$\mathbf{Y_2}$	$\phi_6 = s^2L + 2sr\dot{r} + 2sz\dot{z} - r^2 - z^2$
$\mathbf{Y_3}$	$\phi_7 = 2[s\dot{z} - z]$

Class 7 of eight Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^k dt^2 - dr^2 - b^2(\frac{r}{\alpha})^k d\theta^2 - (\frac{r}{\alpha})^k dz^2 \tag{34}$$

The Lagrangian of the spacetime (34) admits eight Noether symmetries, list of four symmetries are given in Equation (14) and the remaining four additional Noether symmetries are given in Equation (35)

$$\begin{aligned} &\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{2-k}{4}t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r} + \frac{2-k}{4}\theta\frac{\partial}{\partial \theta} + \frac{2-k}{4}z\frac{\partial}{\partial z} \\ &\mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \quad \mathbf{X_4} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}, \quad \mathbf{X_5} = z\frac{\partial}{\partial \theta} - b^2\theta\frac{\partial}{\partial z} \end{aligned} \tag{35}$$

The first integrals corresponding to the symmetries given in Equation (35) are given in Table 15.

Table 15. First Integrals corresponding to symmetries given in (35).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\frac{r^k}{\alpha^k}[t\dot{z} - z\dot{t}]$
$\mathbf{X_4}$	$\phi_5 = 2\frac{r^k}{\alpha^k}[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = 2b^2\frac{r^k}{\alpha^k}[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{Y_1}$	$\phi_7 = sL + \frac{r^k}{\alpha^k}\frac{(2-k)}{2}[-t\dot{t} + b^2\theta\dot{\theta} + z\dot{z}] + r\dot{r}$

7. Nine Symmetries

In this section, all the cases are given where the Lagrangian admits nine Noether symmetries. There are four classes of cylindrically symmetric static spacetime which admit nine Noether symmetries. We discuss them in detail in this section.

Class 1 of nine Noether Symmetries:

$$ds^2 = dt^2 - dr^2 - b^2 e^{\frac{r}{\alpha}} d\theta^2 - e^{\frac{r}{\alpha}} dz^2 \tag{36}$$

The Lagrangian corresponding to the metric (36) admits the following five additional Noether symmetries along with the minimal set which given in Equation (14).

$$\begin{aligned}
\mathbf{X_3} &= \frac{\partial}{\partial r} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \mathbf{X_4} = -\frac{z}{b^2}\frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial z} \\
\mathbf{X_5} &= z\frac{\partial}{\partial r} - \frac{z\theta}{2\alpha}\frac{\partial}{\partial \theta} + [\frac{b^2\theta^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial z} \\
\mathbf{X_6} &= \theta\frac{\partial}{\partial r} + [\frac{-b^2\theta^2 + z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha b^2}]\frac{\partial}{\partial \theta} - \frac{z\theta}{2\alpha}\frac{\partial}{\partial z} \\
\mathbf{Y_1} &= s\frac{\partial}{\partial t}, \qquad A = 2t
\end{aligned} \tag{37}$$

The corresponding first integrals are given in Table 16.

Table 16. First Integrals corresponding to symmetries given in (37).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = \frac{\theta\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha} + 2\dot{r} - \frac{z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}[\theta\dot{z} - z\dot{\theta}]$
$\mathbf{X_5}$	$\phi_6 = \frac{[(b^2\theta^2 - z^2)e^{\frac{r}{\alpha}} + 4\alpha^2]\dot{\theta}}{2\alpha} + 2t\dot{r} - \frac{\theta z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_6}$	$\phi_7 = -\frac{[(b^2\theta^2 + z^2)e^{\frac{r}{\alpha}} - 4\alpha^2]\dot{\theta}}{2\alpha} + 2z\dot{r} - \frac{\theta z\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{Y_1}$	$\phi_8 = 2b^2[-s\dot{t} + t]$

Class 2 of nine Noether Symmetries:

$$ds^2 = e^{\frac{r}{\alpha}} dt^2 - dr^2 - b^2 d\theta^2 - e^{\frac{r}{\alpha}} dz^2 \tag{38}$$

This Lagrangian for this metric gives the following five additional Noether symmetries along with the minimal set (14).

$$\begin{aligned}
\mathbf{X_3} &= \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \quad \mathbf{X_4} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} \\
\mathbf{X_5} &= t\frac{\partial}{\partial r} - [\frac{t^2 - z^2 + 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial t} - \frac{zt}{2\alpha}\frac{\partial}{\partial z} \\
\mathbf{X_6} &= z\frac{\partial}{\partial r} - \frac{zt}{2\alpha}\frac{\partial}{\partial t} - [\frac{t^2 + z^2 - 4\alpha^2 e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial z} \\
\mathbf{Y_1} &= s\frac{\partial}{\partial \theta}, \qquad A = -2b^2\theta
\end{aligned} \tag{39}$$

The first integrals or conservation laws are given in Table 17.

Table 17. First Integrals corresponding to symmetries given in (39).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} + 2\dot{r} - \frac{z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}[t\dot{z} - z\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = \frac{[(t^2-z^2)e^{\frac{r}{\alpha}}+4\alpha^2]\dot{t}}{2\alpha} + 2t\dot{r} - \frac{tz\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_6}$	$\phi_7 = -\frac{[(t^2+z^2)e^{\frac{r}{\alpha}}-4\alpha^2]\dot{\theta}}{2\alpha} + 2z\dot{r} - \frac{tz\dot{t}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{Y_1}$	$\phi_8 = 2b^2[s\dot{\theta} - \theta]$

Class 3 of nine Noether Symmetries:

$$ds^2 = e^{\frac{r}{\alpha}}dt^2 - dr^2 - b^2e^{\frac{r}{\alpha}}d\theta^2 - dz^2 \tag{40}$$

For spacetime (40) the Lagrangian admits nine Neother symmetries , four of which are given in Equation (14) and the remaining five additional Noether symmetries are gievn over here,

$$\begin{aligned}
&\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta}, \mathbf{X_4} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}\\
&\mathbf{X_5} = t\frac{\partial}{\partial r} - [\frac{t^2 - b^2\theta^2 + 4\alpha^2e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial t} - \frac{t\theta}{2\alpha}\frac{\partial}{\partial \theta}\\
&\mathbf{X_6} = \theta\frac{\partial}{\partial r} - \frac{t\theta}{2\alpha}\frac{\partial}{\partial t} - [\frac{t^2 - b^2\theta^2 - 4\alpha^2e^{\frac{-r}{\alpha}}}{4\alpha b^2}]\frac{\partial}{\partial \theta}\\
&\mathbf{Y_1} = s\frac{\partial}{\partial z}, \qquad A = -2z.
\end{aligned} \tag{41}$$

Table 18 contains the conservation laws corresponding to the additional Noether symmetries given in Equation (41).

Table 18. First Integrals corresponding to symmetries given in (41).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} + 2\dot{r} - \frac{\theta\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2e^{\frac{r}{\alpha}}[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = \frac{[(t^2-b^2\theta^2)e^{\frac{r}{\alpha}}+4\alpha^2]\dot{t}}{2\alpha} + 2t\dot{r} - \frac{t\theta\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_6}$	$\phi_7 = -\frac{[(t^2-b^2\theta^2)e^{\frac{r}{\alpha}}-4\alpha^2]\dot{\theta}}{2\alpha} + 2\theta\dot{r} + \frac{t\theta\dot{t}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{Y_1}$	$\phi_8 = 2[s\dot{z} - z]$

Class 4 of nine Noether Symmetries:

$$ds^2 = (\frac{r}{\alpha})^2dt^2 - dr^2 - b^2(\frac{r}{\alpha})^2d\theta^2 - (\frac{r}{\alpha})^2dz^2 \tag{42}$$

The following are the additional five Noether symmetries for the Lagrangian of spacetime (42).

$$\begin{aligned}
&\mathbf{Y_1} = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r}, \qquad \mathbf{Y_2} = s^2\frac{\partial}{\partial s} + sr\frac{\partial}{\partial t}, \qquad A_2 = -r^2\\
&\mathbf{X_3} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad \mathbf{X_4} = z\frac{\partial}{\partial \theta} - b^2t\frac{\partial}{\partial z}, \qquad \mathbf{X_5} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}.
\end{aligned} \tag{43}$$

The Corresponding first integrals are given in Table 19.

Table 19. First Integrals corresponding to symmetries given in (43).

Gen	First Integrals
$\mathbf{Y_1}$	$\phi_4 = sL + r\dot{r}$
$\mathbf{Y_2}$	$\phi_5 = s^2L - r^2 - \frac{2sr^3\dot{t}}{\alpha^2}$
$\mathbf{X_3}$	$\phi_6 = 2\frac{r^2}{\alpha^2}[t\dot{z} - z\dot{t}]$
$\mathbf{X_4}$	$\phi_7 = 2\frac{r^2}{\alpha^2}[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_5}$	$\phi_8 = 2\frac{r^2}{\alpha^2}[t\dot{\theta} - \theta\dot{t}]$

8. Eleven Symmetries

There is only one case for 11 Noether symmetries.

Class of 11 Noether Symmetries:

$$ds^2 = e^{\frac{r}{\alpha}}dt^2 - dr^2 - b^2e^{\frac{r}{\alpha}}d\theta^2 - e^{\frac{r}{\alpha}}dz^2 \tag{44}$$

We obtain 11 Noether symmetries for the Lagrangian of the spacetime (44). Four Noethter symmetries are given in Equation (14) and the remaining additional seven Noether symmetries are given below.

$$\begin{aligned}
&\mathbf{X_3} = \frac{\partial}{\partial r} - \frac{t}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{z}{2\alpha}\frac{\partial}{\partial z}, \qquad \mathbf{X_4} = \theta\frac{\partial}{\partial t} + \frac{t}{b^2}\frac{\partial}{\partial \theta}\\
&\mathbf{X_5} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, \qquad \mathbf{X_6} = z\frac{\partial}{\partial \theta} - \theta b^2\frac{\partial}{\partial z}\\
&\mathbf{X_7} = t\frac{\partial}{\partial r} - [\frac{t^2 + b^2\theta^2 + z^2 + 4\alpha^2e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial t} - \frac{t\theta}{2\alpha}\frac{\partial}{\partial \theta} - \frac{tz}{2\alpha}\frac{\partial}{\partial z}\\
&\mathbf{X_8} = z\frac{\partial}{\partial r} + \frac{tz}{2\alpha}\frac{\partial}{\partial t} - \frac{\theta z}{2\alpha}\frac{\partial}{\partial \theta} + [\frac{-t^2 + b^2\theta^2 - z^2 + 4\alpha^2e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial z}\\
&\mathbf{X_9} = \theta\frac{\partial}{\partial r} + \frac{t\theta}{2\alpha}\frac{\partial}{\partial t} + [\frac{-t^2 - b^2\theta^2 + z^2 + 4\alpha^2e^{\frac{-r}{\alpha}}}{4\alpha}]\frac{\partial}{\partial \theta} - \frac{\theta z}{2\alpha}\frac{\partial}{\partial z}.
\end{aligned} \tag{45}$$

The symmetries and first integrals are given in Table 20.

Table 20. First Integrals corresponding to symmetries given in (45).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2\dot{r} + \frac{t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} - \frac{\theta\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha} - \frac{z\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_4}$	$\phi_5 = 2b^2e^{\frac{r}{\alpha}}[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = 2e^{\frac{r}{\alpha}}[t\dot{z} - z\dot{t}]$
$\mathbf{X_6}$	$\phi_7 = 2b^2e^{\frac{r}{\alpha}}[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_7}$	$\phi_8 = \frac{(t^2+b^2\theta^2+z^2)e^{\frac{r}{\alpha}}+4\alpha^2}{2\alpha}\dot{t} + 2t\dot{r} - \frac{b^2\theta t\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha} - \frac{zt\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_8}$	$\phi_9 = \frac{(-t^2+b^2\theta^2-z^2)e^{\frac{r}{\alpha}}+4\alpha^2}{2\alpha}\dot{z} + 2z\dot{r} - \frac{b^2\theta z\dot{\theta}e^{\frac{r}{\alpha}}}{\alpha} - \frac{zt\dot{t}e^{\frac{r}{\alpha}}}{\alpha}$
$\mathbf{X_9}$	$\phi_{10} = \frac{(-t^2-b^2\theta^2+z^2)e^{\frac{r}{\alpha}}+4\alpha^2}{2\alpha}b^2\dot{\theta} + 2\theta\dot{r} - \frac{\theta t\dot{t}e^{\frac{r}{\alpha}}}{\alpha} - \frac{z\theta\dot{z}e^{\frac{r}{\alpha}}}{\alpha}$

9. Seventeen Symmetries

Class of 17 Noether Symmetries:

The Minkowski spacetime is the only case having 17 Noether symmetries.

$$ds^2 = dt^2 - dr^2 - b^2d\theta^2 - dz^2 \tag{46}$$

The additional symmetries are

$$
\begin{aligned}
&\mathbf{X_3} = \frac{\partial}{\partial r}, \mathbf{X_4} = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}, \qquad \mathbf{X_5} = -b^2\theta\frac{\partial}{\partial r} + r\frac{\partial}{\partial \theta} \\
&\mathbf{X_6} = z\frac{\partial}{\partial \theta} - b^2\theta\frac{\partial}{\partial z}, \qquad \mathbf{X_7} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} \\
&\mathbf{X_8} = z\frac{\partial}{\partial r} - r\frac{\partial}{\partial z}, \qquad \mathbf{X_9} = b^2\theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta} \\
&\mathbf{Y_1} = 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}, \qquad \mathbf{Y_2} = s\frac{\partial}{\partial \theta}, A_2 = -2b^2\theta \\
&\mathbf{Y_3} = s[s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + \theta\frac{\partial}{\partial \theta} + z\frac{\partial}{\partial z}], \qquad A_3 = t^2 - r^2 - b^2\theta^2 - z^2 \\
&\mathbf{Y_4} = s\frac{\partial}{\partial t}, \qquad A_4 = 2t, \mathbf{Y_5} = s\frac{\partial}{\partial r}, \qquad A_5 = -2r, \qquad \mathbf{Y_6} = s\frac{\partial}{\partial z}, \qquad A_6 = -2z
\end{aligned}
\tag{47}
$$

The first integrals corresponding to the Noether symmetries given in Equation (47) are listed in Table 21.

Table 21. First Integrals corresponding to symmetries given in (47).

Gen	First Integrals
$\mathbf{X_3}$	$\phi_4 = 2[\dot{r} - r]$
$\mathbf{X_4}$	$\phi_5 = 2[t\dot{r} - r\dot{t}]$
$\mathbf{X_5}$	$\phi_6 = 2b^2[r\dot{\theta} - \theta\dot{r}]$
$\mathbf{X_6}$	$\phi_7 = 2b^2[z\dot{\theta} - \theta\dot{z}]$
$\mathbf{X_7}$	$\phi_8 = 2[t\dot{z} - z\dot{t}]$
$\mathbf{X_8}$	$\phi_9 = 2[z\dot{r} - r\dot{z}]$
$\mathbf{X_9}$	$\phi_{10} = 2b^2[t\dot{\theta} - \theta\dot{t}]$
$\mathbf{Y_1}$	$\phi_{11} = 2[sL - t\dot{t} + r\dot{r} + b^2\theta\dot{\theta} + z\dot{z}]$
$\mathbf{Y_2}$	$\phi_{12} = 2b^2[s\dot{\theta} - \theta]$
$\mathbf{Y_3}$	$\phi_{13} = s^2L + 2s[-t\dot{t} + r\dot{r} + b^2\theta\dot{\theta} + z\dot{z}] + t^2 - r^2 - b^2\theta^2 - z^2$
$\mathbf{Y_4}$	$\phi_{14} = 2[t - s\dot{t}]$
$\mathbf{Y_5}$	$\phi_{15} = 2[s\dot{r} - r]$
$\mathbf{Y_6}$	$\phi_{16} = 2[s\dot{z} - z]$

10. Conclusions

In this paper we classify the cylindrically symmetric static space-times according to Noether symmetries from their Lagrangian. To get all possible metrics, the usual Lagrangian for the general cylindrically symmetric static metric has been considered. It has been observed that

- there may be 5, 6, 7, 8, 9, 11, and 17 Noether symmetries for the Lagrangian of cylindrically symmetric static space-times.
- There may be infinite metrics whose Lagrangian admits five Noether symmetries.
- There are 24 classes for six Noether symmetries, three classes of seven Noether symmetries, seven for eight Noether symmetries, four for nine Noether symmetries and one class for 11 Noether symmetries.
- The maximum number of Noether symmetries, i.e., 17, appears for the Minkowski spacetime.

The first integrals in each case are also given correspondingly in tabulated form. It is important to note that in this classification, all the metrics given in [8] have been recovered.

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