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Skew Continuous Morphisms of Ordered Lattice Ringoids

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Abstract: Skew continuous morphisms of ordered lattice semirings and ringoids are studied. Different associative semirings and non-associative ringoids are considered. Theorems about properties of skew morphisms are proved. Examples are given. One of the main similarities between them is related to cones in algebras of non locally compact groups.

Keywords: non-associative; algebra; morphism; idempotent; skew; semiring; ringoid

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1. Introduction

Semirings, ringoids, algebroids and non-associative algebras play important role in algebra and among them ordered semirings and lattices as well [1–8]. This is also motivated by idempotent mathematical physics naturally appearing in quantum mechanics and quantum field theory (see, for example, [9] and references therein). They also arise from the consideration of algebroids and ringoids associated with non locally compact groups. Namely, this appears while the studies of representations of non locally compact groups, quasi-invariant measures on them and convolution algebras of functions and measures on them [10–13]. The background for this is A. Weil's theorem (see [14]) asserting that if a topological group has a quasi-invariant σ-additive non trivial measure relative to the entire group, then it is locally compact. Therefore, it appears natural to study inverse mapping systems of non locally compact groups and their dense subgroups. Such spectra lead to structures of algebroids and ringoids. Investigations of such objects are also important for making advances in representation theory of non locally compact groups.

In this paper methods of categorial topology are used (see [15–18] and references therein).

This article is devoted to ordered ringoids and semirings with an additional lattice structure. Their continuous morphisms are investigated in Section 3. Preliminaries are given in Section 2. Necessary definitions 2.1–2.4 are recalled. For a topological ringoid K and a completely regular topological space X new ringoids $C(X,K)$ are studied, where $C(X,K)$ consists of all continuous mappings $f : X \to K$ with point-wise algebraic operations. Their ideals, topological directed structures and idempotent operations are considered in Lemmas 2.6, 2.8, 2.9, 2.12 and Corollary 2.7. There are also given several examples 2.13–2.18 of objects. One of the main examples between them is related to cones in algebras of non locally compact groups. Another example is based on ordinals. Construction of ringoids with the help of inductive limits is also considered.

Structure and properties of these objects are described in Section 3. Definitions of morphisms of ordered semirings and some their preliminaries are described in Subsection 3.1. An existence of idempotent K-homogeneous morphisms under definite conditions is proved in Lemma 3.4. A relation between order preserving weakly additive morphisms and non-expanding morphisms is given in

Lemma 3.7. An extension of an order preserving weakly additive morphism is considered in Lemma 3.9.

Then a weak* topology on a family $\mathcal{O}(X, K)$ of all order preserving weakly additive morphisms on a Hausdorff topological space X with values in K is taken. The weak* compactness of $\mathcal{O}(X, K)$ under definite conditions is proved in Theorem 3.10. Further in Proposition 3.11 there is proved that $I(X, K)$ and $I_h(X, K)$ are closed in $\mathcal{O}(X, K)$, where $I(X, K)$ denotes the set of all idempotent K-valued morphisms, also $I_h(X, K)$ denotes its subset of idempotent homogeneous morphisms.

Categories related to morphisms and ringoids are presented in Subsection 3.2. An existence of covariant functors, their ranges and continuity of morphisms are studied in Lemmas 3.14, 3.16, 3.21, 3.34 and Propositions 3.15, 3.22. In Propositions 3.24, 3.26 and 3.29 such properties of functors as being monomorphic and epimorphic are investigated. Supports of functors are studied in Proposition 3.31. Moreover, in Proposition 3.32 it is proved that definite functors preserve intersections of closed subsets. Then functors for inverse systems are described in Proposition 3.33. Bi-functors preserving pre-images are considered in Proposition 3.35. Monads in certain categories are investigated in Theorem 3.38. Exact sequences in categories are considered in Proposition 3.39.

Lattices associated with actions of groupoids on topological spaces are investigated in subsection 3.3. Supports of (T, G)-invariant semi-idempotent continuous morphisms are estimated in Proposition 3.42, where G is a topological groupoid and T is its representation described in Lemma 3.40. Structures of families of all semi-idempotent continuous morphisms associated with a groupoid G and a ringoid K are investigated in Proposition 3.43 and Theorems 3.44, 3.45.

The main results are Propositions 3.22, 3.24, 3.29, 3.32, 3.33, 3.35, 3.39, 3.43, Theorems 3.38, 3.44 and 3.45. All main results of this paper are obtained for the first time. The obtained results can be used for further studies of such objects, their classes and classification. They can be applied to investigations of non locally compact group algebras also.

2. Ringoids and Lattice Structure

2.1. Preliminaries

To avoid misunderstandings we first present our definitions.

1. Definitions. Let K be a set and let two operations $+ : K^2 \to K$ the addition and $\times : K^2 \to K$ the multiplication be given so that K is a semigroup (with associative binary operations) or a quasigroup (with may be non-associative binary operations) relative to $+$ and $\times$ with neutral elements $e_+ =: 0$ and $e_\times =: 1$ so that $a \times 0 = 0 \times a = 0$ for each $a \in K$ and either the right distributivity $a(b + c) = ab + ac$ for every $a, b, c \in K$ or the left distributivity $(b + c)a = ba + ca$ for every $a, b, c \in K$ is accomplished, then K is called a semiring or a ringoid respectively with either right or left distributivity correspondingly. If it is simultaneously right and left distributive, then it is called simply a semiring or a ringoid respectively.

A semiring K (or a ringoid, or a ring, or a non-associative ring) having also a structure of a linear space over a field $\mathbf{F}$ and such that $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$ and $(\alpha\beta)a = \alpha(\beta a)$ for each $\alpha, \beta \in \mathbf{F}$ and $a, b \in K$ is called a semialgebra (or an algebroid, or an algebra or a non-associative algebra correspondingly).

A semiring K (or a semialgebra and so on) supplied with a topology on K (or on K and $\mathbf{F}$ correspondingly) relative to which algebraic operations are continuous is called a topological semiring (or a topological semialgebra and so forth correspondingly).

A set K with binary operations $\mu_1, ..., \mu_n$ will also be called an algebraic object. An algebraic object is commutative relative to an operation μ_p if $\mu_p(a, b) = \mu_p(b, a)$ for each $a, b \in K$.

An algebraic object K with binary operations $\mu_1, ..., \mu_n$ is called either directed or linearly ordered or well-ordered if it is such as a set correspondingly and its binary operations preserve an ordering: $\mu_p(a, b) \le \mu_p(c, d)$ for each $p = 1, ..., n$ and for every $a, b, c, d \in K$ so that $a \le c$ and $b \le d$ when a, b, c, d belong to the same linearly ordered set Z in K.

Henceforward, we suppose that the minimal element in an ordered K is zero.

Henceforth, for semialgebras, non-associative algebras or algebroids A speaking about ordering on them we mean that only their non-negative cones $K = \{y : y \in A, 0 \leq y\}$ are considered. For non-negative cones K in semialgebras, non-associative algebras or algebroids only the case over the real field will be considered.

2. Definition. A (non-associative) topological algebra or a topological ringoid, etc., we call topologically simple if it does not contain closed ideals different from $\{0\}$ and K, where $K \neq \{0\}$.

3. Definition. We consider a directed set K which satisfies the condition:

(DW) for each linearly ordered subset A in K there exists a well-ordered subset B in K such that $A \subset B$.

4. Definitions. Let K be a well-ordered (or directed satisfying condition $3(DW)$) either semiring or ringoid (or a non-negative cone in a algebroid over the real field $\mathbf{R}$) such that

(1) $\sup E \in K$ for each $E \in T$, where T is a family of subsets of K.

If K is a directed topological either semiring or ringoid, we shall suppose that it is supplied with a topology

(2) $\tau = \tau_K$ so that every set

$L_b := \{y : y \in K, y < b$ or y is not comparable with $b\}$ and
$G_b := \{y : y \in K, b < y$ or y is not comparable with $b\}$
is open relative to it.

That is, if a set Z is linearly ordered in K this topology τ_K provides the hereditary topology on Z which is not weaker than the interval topology on Z generated by the base $\{(a,b)_Z : a < b \in Z\}$, where $(a,b)_Z := \{c : c \in Z, a < c < b\}$.

For a completely regular topological space X and a topological semiring (or ringoid) K let $C(X,K)$ denote a semiring (or a ringoid respectively) of all continuous mappings $f : X \to K$ with the element-wise addition $(f+g)(x) = f(x) + g(x)$ and the element-wise multiplication $(fg)(x) = f(x)g(x)$ operations for every $f, g \in C(X,K)$ and $x \in X$.

If K is a directed semiring (or a directed ringoid) and X is a linearly ordered set, $C_+(X,K)$ (or $C_-(X,K)$) will denote the set of all monotone non-decreasing (or non-increasing correspondingly) maps $f \in C(X,K)$.

For the space $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$) we suppose that

(3) a family T of subsets of K contains the family $\{f(X) : f \in C(X,K)\}$ (or $\{f(X) : f \in C_+(X,K)\}$ or $\{f(X) : f \in C_-(X,K)\}$ correspondingly) and K satisfies Condition (1).

5. Remark. For example, the class On of all ordinals has the addition $\mu_1 = +_o$ and the multiplication $\mu_2 = \times_o$ operations which are generally non-commutative, associative, with unit elements 0 and 1 respectively, on On the right distributivity is satisfied (see Propositions 4.29–4.31 and Examples 1–3 in [19]). Relative to the interval topology generated by the base $\{(a,b) : a < b \in On\}$ the class On is the topological well-ordered semiring, where $(a,b) = \{c : c \in On, a < c < b\}$. For each non-void set A in On there exists $\sup A \in On$ (see [20]).

If K is a linearly ordered non-commutative relative to the addition semiring (or a ringoid), then the new operation $(a,b) \mapsto \max(a,b) =: a +_2 b$ defines the commutative addition. Then $c(a +_2 b) = \max(ca, cb) = ca +_2 cb$ and $(a +_2 b)c = \max(ac, bc) = ac +_2 bc$ for every $a, b, c \in K$, that is $(T, +_2, \times)$ is left and right distributive.

As an example of a semiring (or a ringoid) K in Definitions 4 one can take $K = On$ or $K = \{A : A \in On, |A| \leq b\}$, where b is a cardinal number such that $\aleph_0 \leq b$. Each segment $[a,b] := \{c : c \in On, a \leq c \leq b\}$ is compact in On, where $a < b \in On$. Evidently, $K = On$ satisfies Condition $4(1)$, since $\sup E$ exists for each set E in On (see [20]).

Particularly, if a topological space X is compact and $C(X,K)$ is a semiring (or a ringoid) of all continuous mappings $f : X \to K$, then a family T contains the family of compact subsets

$\{f(X) : f \in C(X,K)\}$, since a continuous image of a compact space is compact (see Theorem 3.1.10 [21]).

It is possible to modify Definition 4 in the following manner. For a well-ordered K without Condition $4(1)$ one can take the family of all continuous bounded functions $f : X \to K$ and denote this family of functions by $C(X,K)$ for the uniformity of the notation.

For a directed K satisfying Condition $3(DW)$ without Condition $4(1)$ it is possible to take the family of all monotone non-decreasing (or non-increasing) bounded functions $f : X \to K$ for a linearly ordered set X and denote this family by $C_+(X,K)$ ($C_-(X,K)$ correspondingly) also.

Naturally, $C(X,K)$ has also the structure of the left and right module over the semiring (or the ringoid correspondingly) K, *i.e.*, af and fa belong to $C(X,K)$ for each $a \in K$ and $f \in C(X,K)$. To any element $a \in K$ the constant mapping $g^a \in C(X,K)$ corresponds such that $g^a(x) = a$ for each $x \in X$. If K is right (or left) distributive, then $q(f+h) = qf + qh$ (or $(f+h)q = fq + hq$ correspondingly) for every $q, f, h \in C(X,K)$.

The semiring (or the ringoid) $C(X,K)$ will be considered directed:

(1) $f \le g$ if and only if $f(x) \le g(x)$ for each $x \in X$.

Indeed, if $f,h \in C(X,K)$, then $a = \sup(f(X)) \in K$ and $b = \sup(h(X)) \in K$ according to Condition $4(1)$. Then there exists $c \in K$ so that $a \le c$ and $b \le c$, consequently, $f \le g^c$ and $h \le g^c$. Thus for each $f,h \in C(X,K)$ there exists $q \in C(X,K)$ so that $f \le q$ and $h \le q$. From $a+b \le c+d$ and $ab \le cd$ for each $a \le c$ and $b \le d$ in K it follows that $f+q \le g+h$ and $fq \le gh$ for each $f \le g$ and $q \le h$ in $C(X,K)$.

If $f \le g$ and $f \ne g$ (i.e. $\exists x \in X\ f(x) \ne g(x)$), then we put $f < g$.

For a mapping $f \in C(X,K)$ its support $supp(f)$ is defined as usually

(2) $supp(f) := cl_X\{x : \ x \in X, f(x) \ne 0\}$, where $cl_X A$ denotes the closure of A in X when $A \subset X$.

Henceforth, we consider cases, when

(3) a topology on X is sufficiently fine so that functions separate points in X, *i.e.*, for each $x \ne z$ in X there exists f in $C(X,K)$(or $C_-(X,K)$ or $C_+(X,K)$ correspondingly) such that $f(x) \ne f(z)$.

The latter is always accomplished in the purely algebraic discrete case.

2.2. Directed Ringoids $C(X,K)$ of Mappings

6. Lemma. *If E is a closed subspace in a topological space X, then $C(X,K|E) := \{f : f \in C(X,K), supp(f) \subset E\}$ is an ideal in $C(X,K)$.*

Proof. If $f \in C(X,K|E)$ and $g \in C(X,K)$, then $f(x)g(x) = 0$ and $g(x)f(x) = 0$ when $f(x) = 0$, consequently, $supp(fg)$ and $supp(gf)$ are contained in E. Moreover, if $f,h \in C(X,K|E)$, then $supp(f+h)$ and $supp(h+f)$ are contained in E, since $f(x)+h(x) = 0$ and $h(x)+f(x) = 0$ for each $x \in X \setminus E$, while $X \setminus E$ is open in X. Thus $C(X,K|E)$ is a semiring (or a ringoid respectively) and $C(X,K|E)C(X,K) \subseteq C(X,K|E)$ and $C(X,K)C(X,K|E) \subseteq C(X,K|E)$.

7. Corollary. *If E is clopen (i.e. closed and open simultaneously) in X, then $C(E,K)$ is an ideal in $C(X,K)$.*

Proof. For a clopen topological subspace E in X one gets $C(E,K)$ isomorphic with $C(X,K|E)$, since each $f \in C(E,K)$ has the zero extension on $X \setminus E$.

8. Lemma. *For a linearly ordered set X and a directed semiring (ringoid) K there are directed semirings (or ringoids correspondingly) $C_+(X,K)$ and $C_-(X,K)$.*

Proof. The sets $C_+(X,K)$ and $C_-(X,K)$ are directed according to Condition $5(1)$ with a partial ordering inherited from $C(X,K)$. Since $a+b \le c+d$ and $ab \le cd$ for each $a \le c$ and $b \le d$ in K, then $f+q \le g+h$ and $fq \le gh$ for each $f \le g$ and $q \le h$ all either in $C_+(X,K)$ or $C_-(X,K)$. On the other hand, for each $f,h \in C(X,K)$ there exists $g^c \in C(X,K)$ so that $f \le g^c$ and $h \le g^c$

(see § 5). If $f(x) \leq f(y)$ and $h(x) \leq h(y)$ for $f, h \in C_+(X, K)$ and each $x \leq y$ in K, then $f(x) + h(x) \leq f(y) + h(y)$ and $f(x)h(x) \leq f(y)h(y)$, consequently, $f + h$ and fh are in $C_+(X, K)$. Analogously, if $f, h \in C_-(X, K)$, then $f + h$ and fh are in $C_-(X, K)$. But a constant mapping g^c belongs to $C_+(X, K)$ and $C_-(X, K)$. Thus $C_+(X, K)$ and $C_-(X, K)$ are directed semirings (or ringoids correspondingly).

9. Lemma. *If $H = H_X$ is a covering of X and τ_K is a topology on K satisfying Conditions $3(DW)$ and $4(1-3)$, then a semiring (or ringoid or a non-negative cone in a algebroid over* **R***) $C(X, K)$ can be supplied with a topology relative to which it is a topological directed (TD) semiring (or a TD ringoid or a TD algebroid respectively).*

Proof. Take a topology τ_C on $C(X, K)$ with the base β_C formed by the following sets and their finite intersections:

(1) $B_1(g, A, V) := \{f : f \in C(X, K), f(A) \subset g + V\}$,
$B_2(g, A, V) := \{f : f \in C(X, K), f(A) \subset V + g\}$,
$B_3(g, A, V) := \{f : f \in C(X, K), f(A) \subset gV\}$,
$B_4(g, A, V) := \{f : f \in C(X, K), f(A) \subset Vg\}$,

where $g \in C(X, K)$, $A \in H$, $V \in \tau_K$. Evidently, the addition $+ = \mu_1$ and the multiplication $\times = \mu_2$ are continuous relative to this topology, since each $U \in \tau_C$ is the union of base sets $P \in \beta_C$. In view of Section 5 $C(X, K)$ is directed: $\mu_p(f, h) \leq \mu_p(g, u)$ for each $p = 1, 2$ and for every $f, g, h, u \in C(X, K)$ so that $f \leq g$ and $h \leq u$ when f, g, h, u belong to the same linearly ordered set in $C(X, K)$, since element-wise these inequalities are satisfied in K, i.e. for $f(x), g(x), h(x), u(x)$ with $x \in X$ (see §1).

10. Note. Henceforward, it will be supposed that $C(X, K)$ is supplied with the topology τ_C of Lemma 9, while $C_+(X, K)$ and $C_-(X, K)$ are considered relative to the topology inherited from $C(X, K)$. Particularly, if $X \in H_X$, then it provides the topology of the uniform convergence on $C(X, K)$.

11. Corollary. *If the conditions of Lemma 9 are satisfied and $H = 2^X$ is the family of all subsets in X and a topology τ_K on K is discrete, then τ_C is the discrete topology on $C(X, K)$.*

12. Lemma. *Suppose that the conditions of Lemma 9 are satisfied. Then the functions*

(1) $f \vee g(x) := \max(f(x), g(x))$ *and*
(2) $f \wedge g(x) := \min(f(x), g(x))$
are in $C(X, K)$ (or in $C_+(X, K)$ or in $C_-(X, K)$) for every pair of functions $f, g \in C(X, K)$ (or in $C_+(X, K)$ or in $C_-(X, K)$ correspondingly) satisfying the condition:
(3) *for each $x \in X$ either $f(x) < g(x)$ or $g(x) < f(x)$ or $f(x) = g(x)$.*

Proof. Let $f, g \in C(X, K)$ satisfy Condition (3). Then the sets $\{x : x \in X, f(x) \leq g(x)\}$ and $\{x : x \in X, f(x) \leq g(x)\}$ are closed in X, since f and g are continuous functions on X and the topology τ_K on K satisfies Condition 4(2). For each closed set E in K the sets
$(f \vee g)^{-1}(E) = [f^{-1}(E) \cap \{x : x \in X, g(x) \leq f(x)\}] \cup [g^{-1}(E) \cap \{x : x \in X, f(x) \leq g(x)\}]$ and
$(f \wedge g)^{-1}(E) = [f^{-1}(E) \cap \{x : x \in X, f(x) \leq g(x)\}] \cup [g^{-1}(E) \cap \{x : x \in X, g(x) \leq f(x)\}]$
are closed in X, consequently, the mappings $f \vee g$ and $f \wedge g$ are continuous. If $f, g \in C_+(X, K)$ and $x < y \in X$, then $f(x) \leq f(y)$ and $g(x) \leq g(y)$. If $f(x) \leq g(x)$ and $g(y) \leq f(y)$, then $(f \vee g)(x) = g(x) \leq g(y) \leq f(y) = (f \vee g)(y)$ and $(f \wedge g)(x) = f(x) \leq g(x) \leq g(y) = (f \wedge g)(y)$. If $f(x) \leq g(x)$ and $f(y) \leq g(y)$, then $(f \vee g)(x) = g(x) \leq g(y) = (f \vee g)(y)$ and $(f \wedge g)(x) = f(x) \leq f(y) = (f \wedge g)(y)$. Therefore, $(f \vee g)(x) \leq (f \vee g)(y)$ and $(f \wedge g)(x) \leq (f \wedge g)(y)$ for each $x < y \in X$. Thus $f \vee g$ and $f \wedge g \in C_+(X, K)$. Analogously if $f, g \in C_-(X, K)$, then $f \vee g$ and $f \wedge g \in C_-(X, K)$.

Relative to the topology of §9 on $C(X, K)$ operations $\vee$ and $\wedge$ are continuous on $C(X, K)$, $C_+(X, K)$ and $C_-(X, K)$.

2.3. Examples of Directed Ringoids

13. Example. Ringoids and ordinals. The class On of all ordinals has the addition $\mu_1 = +_o$ and the multiplication $\mu_2 = \times_o$ operations which are generally non-commutative, associative, with unit elements 0 and 1 respectively, on On the right distributivity is satisfied (see Propositions 4.29–4.31

and Examples 1–3 in [19,22]). Relative to the interval topology generated by the base $\{(a,b) : a < b \in On\}$ the class On is the topological well-ordered semiring, where $(a,b) = \{c : c \in On, a < c < b\}$. For each non-void set A in On there exists $\sup A \in On$ (see [20]).

14. Example. Construction of ringoids with the help of inductive limits. Let J be a directed set of the cardinality $card(J) \ge \aleph_0$ such that for each $l, k \in J$ there exists $j \in J$ with $l \le j$ and $k \le j$ (see also §I.3 [21]), and let $\phi : J \to J$ be a monotone decreasing map, $G_j \subseteq [0,\infty)$, let also $p^k_j : G_k \to G_j$ be an embedding for each $k \le j \in J$. There is considered G_j as a ringoid with the addition, the multiplication, with neutral elements $0_j = 0$ by addition and $1_j = 1$ by multiplication and the linear ordering $x_j < y_j$ inherited from $[0,\infty) = \{t : t \in \mathbf{R}, 0 \le t < \infty\}$ for each $j \in J$. Put $G_0 = \lim\{G_j, p^k_j, J\}$ to be the inductive limit of the direct mapping system so that G is the quotient $(\bigoplus_j G_j)/\Xi$ of the direct sum $\bigoplus_j G_j$ by the equivalence relation Ξ caused by mappings p^k_j. Then consider $G := \{x : x \in G_0, \sup_{j\in J} x_j < \infty\}$, where $x_j = p_j(x)$, $p_j : G \to G_j$ notates the projection.

Then we define $g +_1 h := \{v_j : v_j = g_j + h_j \forall j \in J\}$ and $g \times_1 h := \{w_j : w_j = g_j p^k_j(h_k) \forall j \in J$ with $k = \phi(j)\}$ for all $g, h \in G$, where $g_j = p_j(g)$ for each $j \in J$. Let also $x <_1 y$ in G if and only if $x_j < y_j$ for each $j \in J$. Certainly for each $x, y \in G$ there exists $z \in G$ so that $x \le_1 z$ and $y \le_1 z$, for example, $z_j = \max(x_j, y_j)$ for each $j \in J$. Therefore we get that if $x <_1 y$ and $u <_1 z$ in G, then $x +_1 u <_1 y +_1 z$ and $x \times_1 u <_1 y \times_1 z$. We supply G with a topology τ_b inherited from the inductive limit topology on G_0, where $[0,\infty)$ is supplied with the standard metric of $\mathbf{R}$ and G_j has the topology inherited from $[0,\infty)$. Then we deduce that $U(x_j, b, j) + U(z_j, b, j) \subset U(x_j + z_j, 2b, j)$ and $U(x_j, b, j)U(z_k, b, k) \subset U(x_j p^k_j(z_k), b(1 + x_j + z_k), j)$ for every $x, z \in G$ and $b > 0$ and $j \in J$ with $k = \phi(j)$, where $U(x_j, b, j) := \{y_j : y_j \in G_j, x_j - b < y_j < x_j + b\}$. Since $\sup_{j\in J} x_j < \infty$ for each $x \in G$, then the addition and the multiplication in G are continuous. Thus $(G, +_1, \times_1, <_1, \tau_b)$ is the topological directed ringoid with the left and the right distributivity in which the multiplication $\times_1$ is non-associative, since $\phi(j) < j$ for each $j \in J$. It is worth to note that each set of the form $S(x) := \{y : y \in G$, either $x < y$ or x is incomparable with $y\}$ is open in (G, τ_b), where $x \in G$.

15. Example. The case of $G_j \subseteq [0,\infty)^\omega$ for each $j \in J$, where ω is a directed set, can be considered analogously to Example 14, taking the lexicographic ordering on the Cartesian product $M := \omega \times J$ and considering M instead of J.

16. Example. On G from Example 14 one can take also $x +_2 y := \{v_j : v_j = \max(x_j, y_j) \forall j \in J\}$ and $x \times_2 y := \{w_j : w_j = \min(x_j, y_k) \forall j \in J$ with $k = \phi(j)\}$. Then $(G, +_2, \times_2, <_1, \tau_b)$ is a topological non-associative ringoid with the left and right distributivity.

17. Example. Ringoids associated with families of measures. Let G_j be a Boolean algebra on a set H_j and let $p^k_j : G_k \to G_j$ be an embedding for each $j \in J$ with $k = \phi(j)$ so that $m_j(p^k_j(C_k)) \le m_k(C_k)$ for each $C_k \in G_k$, where J and ϕ are as in subsection 14. Suppose that on each Boolean algebra G_j there is a probability (finitely additive) measure $m_j : G_j \to [0,1]$ so that G_j is metrizable by the metric $d_j(A_j, B_j) := m_j(A_j \triangle B_j)$, where $A_j \triangle B_j := (A_j \setminus B_j) \cup (B_j \setminus A_j)$. Otherwise it is possible to consider the quotient algebra G_j/Ξ_j, where $A_j \Xi_j B_j$ if and only if $d_j(A_j, B_j) = 0$. Put $A + B := \{C : C_j = A_j \cup B_j \forall j \in J\}$ and $A \times B := \{C : C_j = A_j \cap p^k_j(B_k) \forall j \in J$ with $k = \phi(j)\}$, where $A, B \in G$, $A_j, B_j \in G_j$, $G = \lim\{G_j, p^k_j, J\}$ is the inductive limit of Boolean algebras, $A_j = p_j(A)$, $p_j : G \to G_j$ denotes the projection.

Consider on G the inductive limit topology τ_b, where G_j is supplied with the metric d_j for each $j \in J$. Naturally it is possible to put $A \le B$ in G if and only if $A_j \subseteq B_j$ for each $j \in J$. Then the inequalities

$m_j((A_j \cup C_j) \triangle (B_j \cup D_j)) \le m_j((A_j \triangle B_j) \cup (C_j \triangle D_j)) \le m_j(A_j \triangle B_j) + m_j(C_j \triangle D_j)$ and

$m_j((A_j \cap p^k_j(C_k)) \triangle (B_j \cap p^k_j(D_k))) \le m_j(A_j \triangle B_j) + m_j(p^k_j(C_k) \triangle p^k_j(D_k)) \le m_j(A_j \triangle B_j) + m_k(C_k \triangle D_k)$

are fulfilled for each $A_j, B_j \in G_j$ and $C_k, D_k \in G_k$. Therefore $(G, +, \times, <, \tau_b)$ is the topological ringoid with the left and right distributivity and the non-associative multiplication.

Instead of measures it is possible more generally to consider submeasures m_j, that is possessing the subadditivity property: $m_j(C_j) \le m_j(A_j) + m_j(B_j)$ for each $A_j, B_j, C_j \in G_j$ satisfying the inclusion $C_j \subset A_j \cup B_j$.

18. Example. Ringoids induced by spectra of non locally compact groups. Let $\{G_j, p^k_j, J\}$ be a family of topological non locally compact groups G_j, where J is a directed set, $p^k_j : G_k \to G_j$ is a continuous injective homomorphism for each $j < k$ in J. Let also $\phi : J \to J$ be an increasing map and let $m_j : \mathcal{B}_j \to [0,1]$ be a Radon probability σ-additive measure on the Borel σ-algebra $\mathcal{B}_j$ of G_j such that m_j is left quasi-invariant relative to $p^k_j(G_k)$ for each $j \in J$ with $k = \phi(j)$. That is there exists the Radon-Nikodym derivative (i.e., the left quasi-invariance factor) $d_m(v,g) := m^v(dg)/m(dg)$ for each $m = m_j$, where $v \in G_k$, $g \in G_j$, $m^v(A) := m((p^k_j(v))^{-1}A)$ for each $A \in Af(G_j, m_j)$, where $Af(G_j, m_j)$ denotes a σ-algebra which is the completion of $\mathcal{B}_j$ by m_j-null sets.

It is assumed that a uniformity τ_{G_j} on G_j is such that $\tau_{G_j}|G_k \subset \tau_{G_k}$ and (G_j, τ_{G_j}) is complete for each $j \in J$ with $k = \phi(j)$. Suppose also that there exists an open base of neighborhoods of $e_k \in G_k$ such that their closures in G_j are compact.

It is known that such systems exist for loop groups and groups of diffeomorphisms and Banach-Lie groups.

Then $L^p_{G_k}(G_j, m_j, \mathbf{R})$ for $1 \le p \le \infty$ denotes the Banach space of all m_j-measurable functions $f : G_j \to \mathbf{R}$ such that $f^h(g) \in L^p(G_j, m_j, \mathbf{R})$ for each $h \in G_k$ and

$$\|f\|_{L^p_{G_k}(G_j,m_j,\mathbf{R})} := \sup_{h\in G_k} \|f^h\|_{L^p(G_j,m_j,\mathbf{R})} < \infty,$$

where $f^h(g) := f((p^k_j(h))^{-1}g)$ for each $g \in G_j$ and $h \in G_k$, $j \in J$ with $k = \phi(j)$. Next we consider the space

$L^\infty(L^1_{G_k}(G_j, m_j, \mathbf{R}) : j \in J, k = \phi(j)) := \{f = (f_j : j \in J);\ f_j \in L^1_{G_k}(G_j, m_j, \mathbf{R})$ for each $j \in J$; $\|f\|_\infty := \sup_{j\in J} \|f_j\|_{L^1_{G_k}(G_j)} < \infty$, where $k = \phi(j)\}$.

There exists the non-associative normed algebra $\mathcal{E} := L^\infty(L^1_{G_k}(G_j, m_j, \mathbf{R}) : j \in J, k = \phi(j))$ supplied with the multiplication

$f \tilde{\star} u = w$ such that

$$w_j(g_j) = (f_k \tilde{\star} u_j)(g_j) = \int_{G_k} f_k(t_k) u_j(p^k_j(t_k) g_j) m_k(dt_k)$$

for every $f, u \in \mathcal{E}$ and $g \in G = \prod_{\alpha\in J} G_\alpha$, where $k = \phi(j),\ j \in J,\ g_j \in G_j$ (see [11–13,23]).

Now we take the positive cone

$F := \{f : f \in \mathcal{E},\ \forall j \in J\ f_j(g_j) \ge 0$ for $m_j -$ almost all $g_j \in G_j\}$ in $\mathcal{E}$ and put

$f + h = \{(f+h)_j(g_j) = f_j(g_j) + h_j(g_j)\ \forall j \in J\ \forall g_j \in G_j\}$,

$f \times h = f \tilde{\star} h$ for each $f, h \in F$ and define

$u \le f$ in F if and only if $u_j(g_j) \le f_j(g_j)$ for each $j \in J$ and m_j-almost all $g_j \in G_j$. Therefore, $w_j(g_j) \ge 0$ for every $f, u \in F,\ j \in J$ and m_j-almost all $g_j \in G_j$, where $w = f \times u$, since m_k is the probability measure, $f_k(t_k) \ge 0$ and $u_j(p^k_j(t_k)g_j) \ge 0$ for m_k-almost all $t_k \in G_k$ and m_j-almost all $g_j \in G_j$ correspondingly. Thus $f \times u \in F$ for each $f, u \in F$.

If $f, h, q, u \in F$ and $f \le q$, $h \le u$, then $f_j(g_j) + h_j(g_j) \le q_j(g_j) + u_j(g_j)$ and

$$(f_k \tilde{\star} h_j)(g_j) = \int_{G_k} f_k(t_k) h_j(p^k_j(t_k) g_j) m_k(dt_k)$$

$$\le \int_{G_k} q_k(t_k) u_j(p^k_j(t_k) g_j) m_k(dt_k) = (q_k \tilde{\star} u_j)(g_j)$$

for m_j-almost all $g_j \in G_j$ and hence $f + h \le q + u$ and $f \times h \le q \times u$. Then we infer that

$$((f_k + h_k)\tilde{\star} u_j)(g_j) = \int_{G_k} (f_k(t_k) + h_k(t_k))u_j(p_j^k(t_k)g_j)m_k(dt_k)$$

$$= \int_{G_k} f_k(t_k)u_j(p_j^k(t_k)g_j)m_k(dt_k) + \int_{G_k} h_k(t_k)u_j(p_j^k(t_k)g_j)m_k(dt_k)$$

$$= (f_k\tilde{\star} u_j)(g_j) + (h_k\tilde{\star} u_j)(g_j)$$

for every $f, h, u \in \mathcal{E}$ and $g \in G = \prod_{i\in J} G_i$, where $k = \phi(j)$, $j \in J$, $\pi_j(g) = g_j \in G_j$, $\pi_j : G \to G_j$ is the projection, consequently, $(f + h) \times u = (f \times u) + (h \times u)$. Analogously it can be verified that $u \times (f + h) = (u \times f) + (u \times h)$ for every $f, h, u \in \mathcal{E}$.

For each $f, h \in F$ there exists an element $u \in F$ so that $f \le u$ and $h \le u$, for example, either $u = f + h$ or u given by the formula $u_j(g_j) = \max(f_j(g_j), h_j(g_j))$ for each $j \in J$ and m_j-almost all $g_j \in G_j$.

Take on F the topology τ_n inherited from the norm topology on $\mathcal{E}$. This implies that $(F, +, \times, <, \tau_n)$ is the directed topological non-associative ringoid with the left and right distributivity.

There is the decomposition $f = f^+ - f^-$ for each $f \in \mathcal{E}$, where $f^+ \in F$ and $f^- \in F$, $f_j^+(g_j) := \max(f_j(g_j), 0)$ for each $j \in J$ and $g_j \in G_j$.

If f and h in F are incomparable, there exist $j, l \in J$ (may be either $j = l$ or $j \ne l$) such that $m_j(A_j^+) > 0$ and $m_l(A_l^-) > 0$, where
$A_j^+ = A_j^+(f, h) := \{g_j : g_j \in G_j,\ f_j(g_j) > h_j(g_j)\}$ and
$A_l^- = A_l^-(f, h) := \{g_l : g_l \in G_l,\ f_l(g_l) < h_l(g_l)\}$. Then for
$0 < b < \min(m_j(A_j^+), m_l(A_l^-)) \min(1, \|(f_j - h_j)|_{A_j^+}\|_{L^1(A_j^+)}, \|(f_l - h_l)|_{A_l^-}\|_{L^1(A_l^-)})/4$
each element v in the ball $B(F, h, b) := \{q : q \in F, \|q - h\|_{\mathcal{E}} < b\}$ is incomparable with f, since $\|q_j\|_{L^1(G_j)} \le \|q_j\|_{L^1_{G_k}(G_j)}$ for each $q \in \mathcal{E}$ and $j \in J$, while m_j is the probability measure for each $j \in J$. On the other hand, if $v < u$ in F, there exists $l \in J$ so that $m_l(A_l^-(v, u)) > 0$ and $v_j(g_j) \le u_j(g_j)$ for m_j-almost all $g_j \in G_j$ for each $j \in J$. Therefore, for $0 < b$ prescribed by the inequality given above and each $q \in B(F, h, b)$ the inequality $q \le f$ is impossible, consequently, either q is incomparable with f or $f < q$. Thus each set of the form $S(f) := \{h : h \in F,$ either $f < h$ or f is incomparable with $h\}$ is open in (F, τ_n), where $f \in F$.

19. Note. Certainly relative to the discrete topology the aforementioned ringoids are also topological ringoids. Other examples can be constructed from these using the theorems and the propositions presented above.

3. Skew Morphisms of Ordered Semirings and Ringoids

3.1. Morphisms and Their Properties

1. Notation. Let $\times_2$ denote the mapping on $[K \times C(X, K)] \cup [C(X, K) \times K]$ with values in $C(X, K)$ such that

(1) $c \times_2 f := g^c + f =: g^c \times_2 f$ and $f \times_2 c := f + g^c =: f \times_2 g^c$ for each $c \in K$ and $f \in C(X, K)$, where $g^c(x) := c$ for each $x \in X$, whilst the sum is taken element-wise $(f + g)(x) = f(x) + g(x)$ for every $f, g \in C(X, K)$ and $x \in X$.

2. Definition. We call a mapping ν on $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) with values in K an idempotent (K-valued) morphism if it satisfies for each $f, g, g^c \in C(X, K)$ (or in $C_+(X, K)$ or $C_-(X, K)$ correspondingly) the following five conditions

(1) $\nu(g^c) = c$;
(2) $\nu(c \times_2 f) = c + \nu(f) =: c \times_2 \nu(f)$ and
(3) $\nu(f \times_2 c) = \nu(f) + c =: \nu(f) \times_2 c$;
(4) $\nu(f \vee g) = \nu(f) \vee \nu(g)$ when f, g satisfy Condition 2.12(3) and

(5) $\nu(f \wedge g) = \nu(f) \wedge \nu(g)$ if f, g satisfy Condition 2.12(3),
where $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$ for each $a, b \in K$ when either $a < b$ or $a = b$ or $b < a$.

A mapping (morphism) ν on $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) with values in K we call order preserving (non-decreasing), if

(6) $\nu(f) \leq \nu(g)$ for each $f \leq g$
in $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$ respectively), *i.e.*, when $f(x) \leq g(x)$ for each $x \in X$.

A morphism ν will be called K-homogeneous on $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) if

(7) $\nu(bf) = b\nu(f)$ and

(8) $\nu(fb) = \nu(f)b$

for each f in $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$ correspondingly) and $b \in K$.

3. Remark. If a morphism satisfies Condition 2(4), then it is order preserving.

The evaluation at a point morphism δ_x defined by the formula:

(1) $\delta_x f = f(x)$

is the idempotent K-homogeneous morphism on $C(X, K)$, where x is a marked point in X.

If morphisms $\nu_1, ..., \nu_n$ are idempotent and the multiplication in K is distributive, then for each constants

(2) $c_1 > 0, ..., c_n > 0$ in K with

(3) $c_1 + ... + c_n = 1$ morphisms

(4) $c_1\nu_1 + ... + c_n\nu_n$ and

(5) $\nu_1 c_1 + ... + \nu_n c_n$

are idempotent. Moreover, if the multiplication in K is commutative, associative and distributive and constants satisfy Conditions $(2, 3)$ and morphisms $\nu_1, ..., \nu_n$ are K-homogeneous, then morphisms of the form $(4, 5)$ are also K-homogeneous.

The considered here theory is different from the usual real field **R**, since **R** has neither an infimum nor a supremum, i.e. it is not well-ordered and satisfy neither $2.3(DW)$ nor $2.4(1)$.

4. Lemma. *Suppose that either*

(1) *K is well-ordered and satisfies Conditions $2.4(1-3)$ or*

(2) *X is linearly ordered and K is directed and satisfies Conditions $2.3(DW)$ and $2.4(1-3)$. Then there exists an idempotent K-homogeneous morphism ν on $C(X, K)$ in case (1), on $C_+(X, K)$ and $C_-(X, K)$ in case (2). Moreover, if $K \subset On$ and K is infinite, X is not a singleton, $\aleph_0 \leq |K|$, $|X| > 1$, then ν has not the form either $3(4)$ or $3(5)$ with the evaluation at a point morphisms $\nu_1, ..., \nu_n$ relative to the standard addition in On.*

Proof. Suppose that ν is an order preserving morphism on $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$). If f, g in $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$ respectively) satisfy Condition 2.12(3), then in accordance with Lemma 2.12 there exists $f \vee g$ and $f \wedge g$ in the corresponding $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$). Since $f \vee g \geq f$ and $f \vee g \geq g$ and $f \wedge g \leq f$ and $f \wedge g \leq g$ and the morphism ν is order preserving, then $\nu(f) \vee \nu(g) \leq \nu(f \vee g)$ and $\nu(f \wedge g) \leq \nu(f) \wedge \nu(g)$.

Let also E be a subset in X, we put

(3) $\nu(f) = \nu_E(f) = \sup_{x \in E} f(x)$.

This morphism exists due Conditions $2.4(1, 3)$, since in both cases (1) and (2) of this lemma, the image $f(E)$ is linearly ordered and is contained in K.

From the fact that the addition preserves ordering on K (see §2.1) it follows that Properties $(1-3, 7, 8)$ are satisfied for the morphism ν given by Formula (3). If $f \leq g$ on X, then for each $a \in f(E)$ there exists $b \in g(E)$ so that $a \leq b$, consequently, $\nu(f) \leq \nu(g)$, *i.e.*, 2(6) is fulfilled.

We consider any pair of functions f, g in $C(X, K)$ (or $C_+(X, K)$ or $C_-(X, K)$) satisfying Condition 8(3). In case (2) a topological space X is linearly ordered, in case (1) K is well-ordered, hence $f(X)$, $g(X)$, $f(E)$ and $g(E)$ are linearly ordered in K. Then for each $a \in f(E) \cup g(E)$ there exist $b \in (f \vee g)(E)$ so that $a \leq b$, while for each $c \in (f \vee g)(E)$ there exists $d \in f(E) \cup g(E)$ so that $c \leq d$, hence $\nu(f \vee g) = \nu(f) \vee \nu(g)$. Moreover, for each $a \in f(E) \cup g(E)$ there exists $b \in (f \wedge g)(E)$ so that $b \leq a$ and for each

$c \in (f \wedge g)(E)$ there exists $d \in f(E) \cup g(E)$ so that $d \leq c$, consequently, $\nu(f \wedge g) = \nu(f) \wedge \nu(g)$. Thus Properties 2(4, 5) are satisfied as well.

If E is chosen such that there exists $U \in H_X$ with $E \subset U$, then this morphism ν is continuous on $C(X,K)$, $C_+(X,K)$ and $C_-(X,K)$ (see §§2.3, 2.4, 2.9 and 2.10 also).

If a set X is not a singleton, $|X| > 1$, and $K \subset On$ is infinite, $\aleph_0 \leq |K|$, then taking a set E in X different from a singleton, $|E| > 1$, we get that the morphism given by Formula (3) can not be presented with the help of evaluation at a point morphisms $\nu_1 = \delta_{x_1}, ..., \nu_n = \delta_{x_n}$ by Formula either 3(4) or 3(5) relative to the standard addition in On, since functions f in $C(X,K)$ (or $C_+(X,K)$ or $C_-(X,K)$) separate points in X (see Remark 2.5(3)).

5. Remark. Relative to the idempotent addition $x \vee y = \max(x,y)$ the morphism ν_E given by 4(3) has the form $\nu_E(f) = \vee_{x \in E} \delta_x(f)$.

Let $I(X,K)$ denote the set of all idempotent K-valued morphisms, while $I_h(X,K)$ denotes its subset of idempotent homogeneous morphisms.

A set F of all continuous K-valued morphisms on $C(X,K)$ is supplied with the weak* topology having the base consisting of the sets

(1) $< \mu; g_1, ..., g_n; b >_1 := \{\nu : \nu \in F, \ \forall j = 1, ..., n \ \nu(g_j) < \mu(g_j) + b\}$;

$< \mu; g_1, ..., g_n; b >_2 := \{\nu : \nu \in F, \ \forall j = 1, ..., n \ \mu(g_j) < \nu(g_j) + b\}$;

$< \mu; g_1, ..., g_n; b >_3 := \{\nu : \nu \in F, \ \forall j = 1, ..., n \ \nu(g_j) < b + \mu(g_j)\}$;

$< \mu; g_1, ..., g_n; b >_4 := \{\nu : \nu \in F, \ \forall j = 1, ..., n \ \mu(g_j) < b + \nu(g_j)\}$

and their finite intersections, where $0 < b \in K$, $g_1, ..., g_n \in C(X,K)$, $\mu \in F$.

6. Definitions. A morphism $\nu : C(X,K) \to K$ is called weakly additive, if it satisfies Conditions 2(2, 3);

normalized at $c \in K$, if Formula 2(1) is fulfilled;

(1) non-expanding if $\nu(f) \leq \nu(h) + c$ when $f \leq h + g^c$ and $\nu(f) \leq c + \nu(h)$ when $f \leq g^c + h$ for any $f, h \in C(X,K)$ and $c \in K$,

where ν may be non-linear or discontinuous as well.

The family of all order preserving weakly additive morphisms on a Hausdorff topological space X with values in K will be denoted by $\mathcal{O}(X,K)$.

If $E \subset C(X,K)$ satisfies the conditions: $g^0 \in E$, $g + b$ and $b + g \in E$ for each $g \in E$ and $b \in K$, then E is called an A-subset.

7. Lemma. *If $\nu : C(X,K) \to K$ is an order preserving weakly additive morphism, then it is non-expanding.*

Proof. Suppose that $f, h \in C(X,K)$ and $b \in K$ are such that $f(x) \leq (h(x) + c)$ or $f(x) \leq (c + h(x))$ for each $x \in X$, then 2(2, 3, 6) imply that $\nu(f) \leq (\nu(h) + c)$ or $\nu(f) \leq (c + \nu(h))$ respectively. Thus the morphism ν is non-expanding.

8. Corollary. *Suppose that a topological ringoid K is well-ordered, satisfies 1(1) and with the interval topology, $X \in H$, $C(X,K)$ is supplied with the topology of §2.9. Then any order preserving weakly additive morphism $\nu : C(X,K) \to K$ is continuous.*

Proof. This follows from Lemma 7 and §§2.3, 2.4, since each subset $\{f : f \leq g\}$ and $\{f : g \leq f\}$ is closed in $C(X,K)$ in the topology of §2.9, where $g \in C(X,K)$.

9. Lemma. *Suppose that A is an A-subset (a left or right submodule over K) in $C(X,K)$ and $\nu : A \to K$ is an order preserving weakly additive morphism (left or right K-homogeneous with left or right distributive ringoid K correspondingly). Then there exists an order preserving weakly additive morphism $\mu : C(X,K) \to K$ such that its restriction on A coincides with ν.*

Proof. One can consider the set $\mathcal{F}$ of all pairs (B,μ) so that B is an A-subset (a left or right submodule over K respectively), $A \subseteq B \subseteq C(X,K)$, μ is an order preserving weakly additive morphism on B the restriction of which on A coincides with ν. The set $\mathcal{F}$ is partially ordered: $(B_1,\mu_1) \leq (B_2,\mu_2)$ if $B_1 \subseteq B_2$ and μ_2 is an extension of μ_1. In accordance with Zorn's lemma a maximal element (E,μ) in $\mathcal{F}$ exists.

If $E \neq C(X,K)$, there exists $g \in C(X,K) \setminus E$. Let $E_- := \{f : f \in E, f \leq g\}$ and $E_+ := \{f : f \in E, g \leq f\}$, then $\mu(h) \leq \mu(q)$ for each $h \in E_-$ and $q \in E_+$, consequently, an element $b \in K$ exists such that $\mu(E_-) \leq b \leq \mu(E_+)$ due to Conditions 2.3(DW) and 2.4(1) imposed on K. Then we put $F = E \cup \{g + g^c, g^c + g : c \in K\}$ (F is a minimal left or right module over K containing E and g correspondingly). Then one can put $\mu(g+g^c) = b + c$ and $\mu(g^c + g) = c + b$. Moreover, one gets $\mu(d(g+g^c)) = d\mu(g) + dc$ or $\mu((g+g^c)d) = \mu(g)d + cd$ for each $d \in K$ correspondingly for each $c \in K$. Then μ is an order preserving weakly additive morphism (left or right homogeneous correspondingly) on F. This contradicts the maximality of A.

10. Theorem. *If a ringoid K is well-ordered and satisfies $1(1)$, with the interval topology and K is locally compact, $X \in H_X$. Then $\mathcal{O}(X,K)$ is compact relative to the weak* topology.*

Proof. In view of Lemma 8 each $\nu \in \mathcal{O}(X,K)$ is continuous. The set $\mathcal{O}(X,K)$ is supplied with the weak* topology (see §5).

For each $\nu \in \mathcal{O}(X,K)$ one has $\nu(g^c) = \nu(g^c + g^0) = c$, since $g^c + g^0 = g^c$ and $\nu(g^c + g^0) = c + 0$. On the other hand, for each $g \in C(X,K)$ due to Condition 2.4(1) a supremum exists, $\|g\| := \sup_{x \in X} g(x) \in K$. Each segment $[a,b]$ in K is closed, bounded and hence compact relative to the interval topology. Therefore, $\mathcal{O}(X,K)$ is contained in the Tychonoff product $S = \prod\{[0,\|g\|] : g \in C(X,K)\}$, since $g \leq h$ and hence $\nu(g) \leq \nu(h)$ when $h(x) = \|g\|$ for each $x \in X$. This product is compact as the Tychonoff product of compact topological spaces by Theorem 3.2.13 [21]. It remains to prove, that $\mathcal{O}(X,K)$ is closed in S, since a closed subspace of a compact topological space is compact (see Theorem 3.1.2 [21]).

Each compact Hausdorff space has a uniformity compatible with its topology (see Theorems 3.19 and 8.1.20 [21]). To each element $y \in S$ a morphism $y : C(X,K) \to K$ corresponds, since $[0,\|g\|] \subset K$ for each $g \in C(X,K)$. If $\nu_n \in \mathcal{O}(X,K)$ is a net converging to q in S, then Properties $2(2,3,6)$ for each ν_n imply Properties $2(2,3,6)$ for q, since each segment $[a,b]$ in K is compact and hence complete as the uniform space due to Theorem 8.3.15 [21], where $a < b \in K$. Therefore, $\lim_n = q \in \mathcal{O}(X,K)$ according to Lemma 7 and Corollary 8. Thus $\mathcal{O}(X,K)$ is complete as the uniform space by Theorem 8.3.20 [21] and hence closed in S in accordance with Theorem 8.3.6 [21].

11. Proposition. *In the topological space $\mathcal{O}(X,K)$ the subsets $I(X,K)$ and $I_h(X,K)$ are closed.*

Proof. From the definitions above it follows that $I_h(X,K) \subset I(X,K) \subset \mathcal{O}(X,K)$. If ν_k is a net in $I(X,K)$ (or in $I_h(X,K)$) converging to a morphism $\mu \in \mathcal{O}(X,K)$ relative to the weak* topology (see also §1.6 [21]), then μ satisfies Conditions $2(1-5)$ (or to $2(1-5,7,8)$ respectively). Thus $I(X,K)$ and $I_h(X,K)$ are closed in $\mathcal{O}(X,K)$.

12. Corollary. *If the conditions of Theorem 10 are satisfied, then the topological spaces $I(X,K)$ and $I_h(X,K)$ are compact.*

3.2. Categories of Semirings, Ringoids and Morphisms

13. Definition. If topological spaces X and Y are given and $f : X \to Y$ is a continuous mapping, then it induces the mapping $\mathcal{O}(f) : \mathcal{O}(X,K) \to \mathcal{O}(Y,K)$ according to the formula: $(\mathcal{O}(f)(\nu))(g) = \nu(g(f))$ for each $g \in C(Y,K)$ and $\nu \in \mathcal{O}(X,K)$.

By $I(f)$ will be denoted the restriction of $\mathcal{O}(f)$ onto $I(X,K)$.

A T_1 topological space will be called K-completely regular (or K Tychonoff space), if for each closed subset F in X and each point $x \in X \setminus F$ a continuous function $h : X \to K$ exists such that $h(x) = 0$ and $h(F) = \{c\}$, i.e. h is constant on F, where $c \neq 0$.

Let RK denote a category such that a family $Ob(RK)$ of its objects consists of all K-regular topological spaces, a set of morphisms $Mor(X,Y)$ consists of all continuous mappings $f : X \to Y$ for every $X, Y \in Ob(RK)$, i.e. RK is a subcategory in the category of topological spaces. We denote by $\mathcal{O}K$ a category with objects $Ob(\mathcal{O}K) = \{\mathcal{O}(X,K) : X \in Ob(RK)\}$ and families of morphisms $Mor(\mathcal{O}(X,K), \mathcal{O}(Y,K))$.

14. Lemma. (1). *There exists a covariant functor $\mathcal{O}$ in the category RK.* (2). *Moreover, if a topological ringoid K is well-ordered, satisfies 2.4(1) and with the interval topology, when $f \in Mor(X,Y)$, $X \in H_X$, $Y \in H_Y$, $X,Y \in Ob(RK)$, then $\mathcal{O}(f)$ is continuous.*

Proof. (1). If $X,Y \in Ob(RK)$ and $f \in Mor(X,Y)$, $g \le h$ in $C(Y,K)$, then $g \circ f \le h \circ f$ in $C(X,K)$, consequently, $(\mathcal{O}(f)(\nu))(g) = \nu(g \circ f) \le \nu(h \circ f) = (\mathcal{O}(f)(\nu))(h)$ for each $\nu \in \mathcal{O}(X,K)$. If $c \in K$, $g^c \in C(Y,K)$, then $g^c \circ f \in C(X,K)$, $(\mathcal{O}(f)(\nu))(g^c + h) = \nu(g^c \circ f + h \circ f) = c + \nu(h \circ f) = c + (\mathcal{O}(f)(\nu))(h)$ and $(\mathcal{O}(f)(\nu))(h + g^c) = \nu(h \circ f + g^c \circ f) = \nu(h \circ f) + c = (\mathcal{O}(f)(\nu))(h) + c$ for each $h \in C(Y,K)$. If $1_X \in Mor(X,X)$, $1_X(x) = x$ for each $x \in X$, then $1_X \circ q = q$ for each $q \in Mor(Y,X)$ and $t \circ 1_X = t$ for each $t \in Mor(X,Y)$. On the other hand, $(\mathcal{O}(1_X)(\nu))(g) = \nu(g \circ 1_X) = \nu(g)$ for each $g \in C(X,K)$, *i.e.*, $\mathcal{O}(1_X) = 1_{\mathcal{O}(X)}$. Evidently, $(\mathcal{O}(f \circ s)(\nu))(g) = \nu(g \circ f \circ s) = (\mathcal{O}(s)(\nu)(g \circ f) = ((\mathcal{O}(f) \circ \mathcal{O}(s))(\nu))(g)$.

(2). If ν_j is a net converging to ν in $\mathcal{O}(X,K)$ relative to the weak* topology, then $\lim_j (\mathcal{O}(f)(\nu_j))(g) = \lim_j \nu_j(g \circ f) = \nu(g \circ f) = (\mathcal{O}(f)(\nu))(g)$ for each $f \in Mor(X,Y)$ and $g \in C(Y,K)$, since $\mathcal{O}(X,K)$ and $\mathcal{O}(Y,K)$ are weakly* compact according to Theorem 10, consequently, $\mathcal{O}$ is continuous from $\mathcal{O}(X,K)$ to $\mathcal{O}(Y,K)$.

15. Proposition. *If $f \in Mor(X,Y)$ for $X,Y \in Ob(RK)$, then*
$\mathcal{O}(f)(I(X,K)) \subseteq I(Y,K)$.

Proof. If $g,h \in C(Y,K)$ are such that $g \vee h$ or $g \wedge h$ exists and $f : X \to Y$ is a continuous mapping, then
$(\mathcal{O}(f)(\nu))(g \vee h) = \nu(g \circ f \vee h \circ f) = \nu(g \circ f) \vee \nu(h \circ f) = (\mathcal{O}(f)(\nu))(g) \vee (\mathcal{O}(f)(\nu))(h)$ or
$(\mathcal{O}(f)(\nu))(g \wedge h) = \nu(g \circ f \wedge h \circ f) = \nu(g \circ f) \wedge \nu(h \circ f) = (\mathcal{O}(f)(\nu))(g) \wedge (\mathcal{O}(f)(\nu))(h)$.
Then for each element $c \in K$ one gets
$(\mathcal{O}(f)(\nu))(g^c \times_2 h) = \nu(g^c \circ f \times_2 h \circ f) = \nu(g^c \circ f) \times_2 \nu(h \circ f) = c \times_2 (\mathcal{O}(f)(\nu))(h)$ and
$(\mathcal{O}(f)(\nu))(h \times_2 g^c) = \nu(h \circ f \times_2 g^c \circ f) = \nu(h \circ f) \times_2 \nu(g^c \circ f) = (\mathcal{O}(f)(\nu))(h) \times_2 c$.

16. Definitions. A covariant functor $F : RK \to RK$ will be called epimorphic (monomorphic) if it preserves epimorphisms (monomorphisms). If $\phi : A \hookrightarrow X$ is an embedding, then $F(A)$ will be identified with $F(\phi)(F(A))$.

If for each $f \in Mor(X,Y)$ and each closed subset A in Y, the equality $(F(f)^{-1})(F(A)) = F(f^{-1}(A))$ is satisfied, then a covariant functor F is called preimage-preserving. When $F(\bigcap_{j \in J} X_j) = \bigcap_{j \in J} F(X_j)$ for each family $\{X_j : j \in J\}$ of closed subsets in $X \in Ob(RK)$ the monomorphic functor F is called intersection-preserving.

If a functor F preserves inverse mapping system limits, it is called continuous.

A functor F is said to be weight-preserving when $w(X) = w(F(X))$ for each $X \in Ob(RK)$, where $w(X)$ denotes the topological weight of $X \in Ob(RK)$.

A functor is said to be semi-normal when it is continuous, monomorphic, epimorphic, preserves weights, intersections, preimages and the empty space.

If a functor is continuous, monomorphic, epimorphic, preserves weights, intersections and the empty space, then it is called weakly semi-normal.

17. Lemma. *Let Y be a normal topological space, let also A and B be nonintersecting closed subsets in Y, where T is a well-ordered set supplied with the interval topology. Suppose also that $c_1 < c_2 \in T$ are such that for each $a,b \in T$ with $c_1 \le a < b \le c_2$ an element $d \in T$ exists such that $a < d < b$ (i.e. a segment $[c_1,c_2]$ is without gaps). Then a continuous function $f : Y \to T$ exists such that $f(A) = \{c_1\}$ and $f(B) = \{c_2\}$.*

Proof. Consider the segment $[c_1,c_2]$ in T. There exists a set E dense in $[c_1,c_2]$ such that

(1) $|E| = d([c_1,c_2])$, $\inf E = c_1$, $\sup E = c_2$,

where $d(X)$ denotes the density of a topological space X, $|E|$ denotes the cardinality of E. There exist open subsets U and V in X such that

(2) $A \subset U$, $B \subset V$, $U \cap V = \emptyset$.

We define open subsets V_t in X such that

(3) $cl_X V_t \subset V_s$ for each $t < s \in E$,

(4) $A \subset V_{c_1}$, $B \subset X \setminus V_{c_2}$,

where $cl_X G$ denotes the closure of a set G in X.

Sets V_t will be defined by the transfinite induction. For this one can put $V_{c_1} = U$ and $V_{c_2} = X \setminus B$. Therefore, $A \subset V_{c_1} \subset X \setminus V = cl_X(X \setminus V) \subset V_{c_2}$, consequently, $cl_X V_{c_1} \subset V_{c_2}$. In view of the Zermelo theorem there exists an ordinal P such that $|P| = |E|$, a bijective surjective mapping $\theta : P \to E$ exists such that $\inf P = 0$, $1 \in P$, $\theta(0) = c_1$ and $\theta(1) = c_2$. Suppose that V_{t_j} satisfying Condition (3) are constructed for $j = 1, ..., n$, $j \in P$. There exist elements $a_n = \inf\{t_j : j \leq n, t_j < t_{n+1}\}$ and $b_n = \sup\{t_j : j \leq n, t_j < t_{n+1}\}$. Therefore, $cl_X V_{a_n} \subset V_{b_n}$. From the normality of X it follows that open sets U and V exist such that $cl_X V_{a_n} \subset U$, $X \setminus V_{b_n} \subset V$ and $U \cap V = \emptyset$, consequently, $U \subset X \setminus V \subset V_{b_n}$ and hence $cl_X U \subset cl_X(X \setminus V) = X \setminus V \subset V_{b_n}$. Then one puts $V_{t_{n+1}} = U$. This means that there exists a countable infinite sequence V_{t_j} for $j \in \omega_0 \subseteq P$ satisfying Conditions $(3, 4)$. If $\{t_j : j \in \omega_0\}$ is not dense in $[c_1, c_2]$ the process continues. Suppose that α is an ordinal such that $\omega_0 \subseteq \alpha \subset P$, V_{t_j} is defined for each $j \in \alpha$. If the set $\{t_j : j \in \alpha\}$ is not dense in $[c_1, c_2]$, there exists a segment

(5) $[a, b] \subset [c_1, c_2]$ such that $[a, b] \cap \{t_j : j \in \alpha\} = \emptyset$. We put $L = \bigcup_{t_j < a; j \in \alpha} V_{t_j}$ and $M = \bigcap_{b < t_j; j \in \alpha} V_{t_j}$. From $(3, 4)$ it follows that the set L is open in X and $L \subset M$. On the other hand,

(6) $V_{t_l} \subset L \subset cl_X L \subset cl_X M \subset cl_X V_{t_j} \subset V_{t_k}$ for every $l, j, k \in \alpha$ such that $t_l < a$ and $b < t_j < t_k$. If

(7) $cl_X L$ is not contained in $Int_X M$ this segment $[a, b]$ is skipped, where $Int_X M$ is an interior of M in X. If $cl_X L \subset Int_X M$ one can put $V_a = L$ and $V_b = Int_X M$. Then the process continues for $[a, b]$.

The family $\mathcal{F} = \{(V_j : j \in \alpha) : \alpha \subset P\}$ is ordered by inclusion: $(V_j : j \in \alpha) \leq (W_k : k \in \beta)$ if and only if a bijective monotonously increasing mapping $\theta : \alpha \to \beta$ exists such that $V_j = W_{\theta(j)}$ for each $j \in \alpha$. If a subfamily $\{(V_j : j \in \alpha) : \alpha_k \subset P, k \in \Lambda\}$ is linearly ordered, then its union is in $\mathcal{F}$. In view of the Kuratowski-Zorn lemma there exists a maximal element $(V_j : j \in \alpha_1)$ in $\mathcal{F}$ for some ordinal $\alpha_1 \subset P$ such that conditions $(3, 4)$ are satisfied.

Put $f(x) = \inf\{t : x \in V_t\}$ for $x \in V_{c_2}$ and $f(x) = c_2$ when $x \in X \setminus V_{c_2}$. Therefore, $f(x) \in [c_1, c_2]$ for each $x \in X$, $f(A) \subset \{c_1\}$ and $f(B) \subset \{c_2\}$. Since $[c_1, c_2]$ is supplied with the interval topology it is sufficient to prove that $f^{-1}([c_1, a))$ and $f^{-1}((b, c_2])$ are open in $[c_1, c_2]$ for each $c_1 < a \leq c_2$ and $c_1 \leq b < c_2$. From $(3, 4)$, also from $(6, 7)$ when (5) is fulfilled, and the definition of f it follows that $f^{-1}([c_1, a)) = \bigcup\{V_{t_j} : t_j < a, j \in \alpha_1\}$ and $f^{-1}((b, c_2]) = \bigcup\{X \setminus cl_X V_{t_j} : b < t_j, j \in \alpha_1\}$ are open in $[c_1, c_2]$.

18. Lemma. *If X is well-ordered and E is a segment $[a, b]$ in X, while K satisfies Condition 2.3(DW), then each $f \in C_+(E, K)$ has a continuous extension $g \in C_+(X, \mathbf{K})$.*

Proof. Since $f(E) =: A$ is linearly ordered in K, then by 2.3(DW) there exists a well ordered subset B in K such that $A \subset B$. So putting $g(x) = \inf A$ for each $x < a$ in X, whilst $g(x) = \sup A$ for each $b < x$ in X one gets the continuous extension $g \in C_+(X, K)$ of f, that is $g|_E(y) = f(y)$ for each $y \in E$, since $\inf A$ and $\sup A$ exist in K due to 2.3(DW) and 2.4(1).

19. Definition. It will be said that a pair (X, K) of a topological space X and a ringoid K has property (CE) if for each closed subset E in X and each continuous function $f : E \to K$, *i.e.*, $f \in C(E, K)$ or $f \in C_+(E, K)$ or $f \in C_-(E, K)$, there exists a continuous extension $g : X \to K$, *i.e.*, $g|_E = f$ so that $g \in C(X, K)$ or $g \in C_+(X, K)$ or $g \in C_-(X, K)$ respectively.

Henceforward, it will be supposed that a pair (X, K) has property (CE).

20. Definitions. If Hausdorff topological spaces X and Y are given and $f : X \to Y$ is a continuous mapping, K_1, K_2 are ordered topological ringoids (or may be particularly semirings) with an order-preserving continuous algebraic homomorphism $u : K_1 \to K_2$ then it induces the mapping $\mathcal{O}(f, u) : \mathcal{O}(X, K_1) \to \mathcal{O}(Y, K_2)$ according to the formula:

(1) $(\mathcal{O}(f, u)(\nu))(g) = u[\nu(g_1(f))]$ for each $g_1 \in C(Y, K_1)$ and $\nu \in \mathcal{O}(X, K_1)$, where $u \circ g_1 = g \in C(Y, K_2)$, $g_1 \in C(Y, K_1)$, $(\mathcal{O}(f, u)(\nu))$ is defined on $(\hat{f}, \hat{u})(C(X, K_1)) = \{t : t \in C(Y, K_2); \forall x \in X\ t(x) = u(h \circ f(x)), h \in C(Y, K_1)\}$.

By $I(f, u)$ will be denoted the restriction of $\mathcal{O}(f, u)$ onto $I(X, K)$. The shorter notations $\mathcal{O}(f)$ and $I(f)$ are used when K is fixed, i.e. $u = id$. When $X = Y$ and $f = id$ we write simply $\mathcal{O}_2(u)$ and $I_2(u)$ respectively omitting $f = id$.

Let $\mathcal{S}$ denote a category such that a family $Ob(\mathcal{S})$ of its objects consists of all topological spaces, a family of morphisms $Mor(X, Y)$ consists of all continuous mappings $f : X \to Y$ for every X, $Y \in Ob(\mathcal{S})$.

Let $\mathcal{K}$ be the category objects of which $Ob(\mathcal{K})$ are all ordered topological ringoids satisfying Conditions 2.3 and 2.4, $Mor(A, B)$ consists of all order-preserving continuous algebraic homomorphisms for each $A, B \in \mathcal{K}$. Then by $\mathcal{K}_w$ we denote its subcategory of well-ordered ringoids and their order-preserving algebraic continuous homomorphisms.

We denote by $\mathcal{OK}$ a category with the families of objects $Ob(\mathcal{OK}) = \{\mathcal{O}(X, K) : X \in Ob(\mathcal{S}), K \in Ob(\mathcal{K}_w)\}$ and morphisms $Mor(\mathcal{O}(X, K_1), \mathcal{O}(Y, K_2))$ for every $X, Y \in Ob(\mathcal{S})$ and $K_1, K_2 \in Ob(\mathcal{K}_w)$. Furthermore, $\mathcal{IK}$ stands for a category with families of objects $Ob(\mathcal{IK}) = \{I(X, K) : X \in Ob(\mathcal{S}), K \in Ob(\mathcal{K}_w)\}$ and morphisms $Mor(I(X, K_1), I(Y, K_2))$ for every $X, Y \in Ob(\mathcal{S})$ and $K_1, K_2 \in Ob(\mathcal{K})$.

By $\mathcal{S}_l$ will be denoted a category objects of which are linearly ordered topological spaces, while $Mor(X, Y)$ consists of all monotone nondecreasing continuous mappings $f : X \to Y$, that is $f(x) \le f(y)$ for each $x \le y \in X$, where $X, Y \in Ob(\mathcal{S}_l)$. Then we put $\mathcal{O}_l(f, u) : \mathcal{O}_l(X, K_1) \to \mathcal{O}_l(Y, K_2)$ for each $X, Y \in Ob(\mathcal{S}_l)$ and $f \in Mor(X, Y)$, $K_1, K_2 \in Ob(\mathcal{K})$, $u \in Mor(K_1, K_2)$ according to the formula:

(2) $(\mathcal{O}_l(f, u)(\nu))(g) = u[\nu(g_1(f))]$ for each $g_1 \in C_+(Y, K_1)$ and $u \circ g_1 = g \in C(Y, K_2)$ and $\nu \in \mathcal{O}_l(X, K_1)$, where $(\mathcal{O}_l(f, u)(\nu))$ is defined on $(\hat{f}, \hat{u})(C_+(X, K_1)) := \{t : t \in C_+(Y, K_2); \forall x \in X\ t(x) = u(h \circ f(x)), h \in C_+(Y, K_1)\}$. Then the category $\mathcal{O}_l\mathcal{K}$ with families of objects $Ob(\mathcal{O}_l\mathcal{K}) = \{\mathcal{O}_l(X, K) : X \in Ob(\mathcal{S}_l), K \in Ob(\mathcal{K})\}$ and morphisms $Mor(\mathcal{O}_l(X, K_1), \mathcal{O}_l(Y, K_2))$ and the category $\mathcal{I}_l\mathcal{K}$ with $Ob(\mathcal{I}_l\mathcal{K}) = \{I_l(X, K) : X \in Ob(\mathcal{S}_l), K \in Ob(\mathcal{K})\}$ and $Mor(I_l(X, K_1), I_l(Y, K_2))$ are defined.

Subcategories of left homogeneous continuous morphisms we denote by $\mathcal{O}_h\mathcal{K}$, $\mathcal{O}_{l,h}\mathcal{K}$, $\mathcal{I}_h\mathcal{K}$, $\mathcal{I}_{l,h}\mathcal{K}$ correspondingly. These morphisms are taken on subcategories $\mathcal{K}_{w,l}$ in $\mathcal{K}$ or $\mathcal{K}_l$ in $\mathcal{K}$ of left distributive topological ringoids.

21. Lemma. *There exist covariant functors $\mathcal{O}$, $\mathcal{O}_h$ and $\mathcal{O}_l$, $\mathcal{O}_{l,h}$ in the categories $\mathcal{S}$ and $\mathcal{S}_l$ respectively.*

Proof. Suppose that $X, Y \in Ob(\mathcal{S})$ and $f \in Mor(X, Y)$, while $g \le h$ in $C(Y, K)$, where $K \in Ob(\mathcal{K}_w)$ (or in $\mathcal{K}_{w,l}$) is marked, then $g \circ f \le h \circ f$ in $C(X, Y)$. Therefore one gets $(\mathcal{O}(f)(\nu))(g) = \nu(g \circ f) \le \nu(h \circ f) = (\mathcal{O}(f)(\nu))(h)$ for each $\nu \in \mathcal{O}(X, K)$. Now if $c \in K$, $g^c \in C(Y, K)$, then $g^c \circ f \in C(X, K)$, but also the equalities are fulfilled $(\mathcal{O}(f)(\nu))(g^c + h) = \nu(g^c \circ f + h \circ f) = c + \nu(h \circ f) = c + (\mathcal{O}(f)(\nu))(h)$ and $(\mathcal{O}(f)(\nu))(h + g^c) = \nu(h \circ f + g^c \circ f) = \nu(h \circ f) + c = (\mathcal{O}(f)(\nu))(h) + c$ for each $h \in C(Y, K)$. Then for $1_X \in Mor(X, X)$, that is $1_X(x) = x$ for each $x \in X$, one deduces $1_X \circ q = q$ for each $q \in Mor(Y, X)$ and $t \circ 1_X = t$ for each $t \in Mor(X, Y)$. On the other hand, $(\mathcal{O}(1_X)(\nu))(g) = \nu(g \circ 1_X) = \nu(g)$ for each $g \in C(X, K)$, i.e. $\mathcal{O}(1_X) = 1_{\mathcal{O}(X)}$. But at the same time, the equalities are valid: $(\mathcal{O}(f \circ s)(\nu))(g) = \nu(g \circ f \circ s) = (\mathcal{O}(s)(\nu)(g \circ f) = ((\mathcal{O}(f) \circ \mathcal{O}(s))(\nu))(g)$, since the composition of continuous mappings is continuous.

Moreover, if $\nu \in \mathcal{O}_h(X, K)$, then $(\mathcal{O}(f)(\nu))(bg) = \nu(bg \circ f) = b\nu(g \circ f) = (b(\mathcal{O}(f)(\nu))(g)$. Furthermore, for the categories $\mathcal{O}_l$ (or $\mathcal{O}_{l,h}$) the proof is analogous with $X, Y \in Ob(\mathcal{S}_l)$, $C_+(X, K)$ and $C_+(Y, K)$, where $K \in Ob(\mathcal{K})$ (or $K \in Ob(\mathcal{K}_l)$) is marked.

22. Proposition. *Suppose that $f \in Mor(X, Y)$ for $X, Y \in Ob(\mathcal{S})$ or in $Ob(\mathcal{S}_l)$. Then*

$\mathcal{O}(f)(I(X, K)) \subseteq I(Y, K)$ and $\mathcal{O}_h(f)(I_h(X, K)) \subseteq I_h(Y, K)$ for $K \in Ob(\mathcal{K}_{w,l})$ or $\mathcal{O}_l(f)(I_h(X, K)) \subseteq I_l(Y, K)$ or $\mathcal{O}_{l,h}(f)(I_{l,h}(X, K)) \subseteq I_{l,h}(Y, K)$ for $K \in Ob(\mathcal{K})$ or $K \in Ob(\mathcal{K}_l)$ correspondingly.

Proof. If $g, h \in C(Y, K)$ are such that $g \vee h$ or $g \wedge h$ exists (see Condition (3) in Lemma 2.12) and $f : X \to Y$ is a continuous mapping, $\nu \in I(X, K)$ (or $I_l(X, K)$), then we infer that

$(\mathcal{O}(f)(\nu))(g \vee h) = \nu(g \circ f \vee h \circ f) = \nu(g \circ f) \vee \nu(h \circ f) = (\mathcal{O}(f)(\nu))(g) \vee (\mathcal{O}(f)(\nu))(h)$ or

$(\mathcal{O}(f)(\nu))(g \wedge h) = \nu(g \circ f \wedge h \circ f) = \nu(g \circ f) \wedge \nu(h \circ f) = (\mathcal{O}(f)(\nu))(g) \wedge (\mathcal{O}(f)(\nu))(h)$.

Furthermore, for each $c \in K$ we deduce that

$(\mathcal{O}(f)(\nu))(g^c \times_2 h) = \nu(g^c \circ f \times_2 h \circ f) = \nu(g^c \circ f) \times_2 \nu(h \circ f) = c \times_2 (\mathcal{O}(f)(\nu))(h)$ and

$(\mathcal{O}(f)(\nu))(h \times_2 g^c) = \nu(h \circ f \times_2 g^c \circ f) = \nu(h \circ f) \times_2 \nu(g^c \circ f) = (\mathcal{O}(f)(\nu))(h) \times_2 c$.

Then for $\nu \in I_h(X,K)$ (or $I_{l,h}(X,K)$) one gets $(\mathcal{O}(f)(\nu))(bg) = \nu(bg \circ f) = b\nu(g \circ f) = (b(\mathcal{O}(f)(\nu)))(g)$.

23. Definitions. A covariant functor $F : \mathcal{S} \to \mathcal{S}$ will be called epimorphic (monomorphic) if it preserves continuous epimorphisms (monomorphisms). If $\phi : A \hookrightarrow X$ is a continuous embedding, then $F(A)$ will be identified with $F(\phi)(F(A))$.

If for each $f \in Mor(X,Y)$ and each closed subset A in Y, the equality $(F(f)^{-1})(F(A)) = F(f^{-1}(A))$ is satisfied, then a covariant functor F is called preimage-preserving. In the case $F(\bigcap_{j\in J} X_j) = \bigcap_{j\in J} F(X_j)$ for each family $\{X_j : j \in J\}$ of closed subsets in $X \in Ob(\mathcal{S})$ (or in $Ob(\mathcal{S}_l)$), the monomorphic functor F is called intersection-preserving.

If a functor F preserves inverse mapping system limits, it is called continuous.

A functor is said to be semi-normal when it is monomorphic, epimorphic, also preserves intersections, preimages and the empty space.

If a functor is monomorphic, epimorphic, also preserves intersections and the empty space, then it is called weakly semi-normal.

24. Proposition. *The functor $\mathcal{O}$ (or $\mathcal{O}_h$, $\mathcal{O}_l$, $\mathcal{O}_{l,h}$) is monomorphic.*

Proof. Let $X, Y \in Ob(\mathcal{S})$ (or in $Ob(\mathcal{S}_l)$ respectively) with a continuous embedding $s : X \hookrightarrow Y$ (order-preserving respectively). Then we suppose that $\nu_1 \neq \nu_2 \in \mathcal{O}(X,K)$ (or in $\mathcal{O}_h(X,K)$, $\mathcal{O}_l(X,K)$, $\mathcal{O}_{l,h}(X,K)$ correspondingly). This means that a mapping $g \in C(X,K)$ (or in $C_+(X,K)$ correspondingly) exists such that $\nu_1(g) \neq \nu_2(g)$. A function $u \in C(Y,K)$ (or in $C_+(Y,K)$ respectively) exists such that $u \circ s = g$, hence $(\mathcal{O}(s)(\nu_k)))(u) = \nu_k(u \circ s) = \nu_k(g)$. Thus $\mathcal{O}(s)(\nu_1) \neq \mathcal{O}(s)(\nu_2)$ (or $\mathcal{O}_h(\nu_1) \neq \mathcal{O}_h(\nu_2)$, $\mathcal{O}_l(\nu_1) \neq \mathcal{O}_l(\nu_2)$, $\mathcal{O}_{l,h}(\nu_1) \neq \mathcal{O}_{l,h}(\nu_2)$ correspondingly).

25. Corollary. *The functors I, I_h, I_l and $I_{l,h}$ are monomorphic.*

Proof. This follows from Proposition 24 and Definitions 20.

26. Proposition. *The functors $\mathcal{O}$, $\mathcal{O}_h$, $\mathcal{O}_l$ and $\mathcal{O}_{l,h}$ are epimorphic, when $X \in H_X$ (see §14 also).*

Proof. Let $f : X \to Y$ be a continuous surjective mapping, $\nu \in \mathcal{O}(Y,K)$ (or in $\mathcal{O}_h(Y,K)$, $\mathcal{O}_l(Y,K)$, $\mathcal{O}_{l,h}(Y,K)$ respectively). The set L of all continuous mappings $g \circ f : X \to K$ with $g \in C(Y,K)$ (or in $C_+(Y,K)$ correspondingly) is the A-subset according to Definitions 6 or the left module over K in $C(X,K)$ (or in $C_+(X,K)$). Then we put $\mu(g \circ f) = \nu(g)$. This continuous morphism has an extension from L to a continuous morphism $\mu \in \mathcal{O}(X,K)$ (or in $\mathcal{O}_h(X,K)$, $\mathcal{O}_l(X,K)$, $\mathcal{O}_{l,h}(X,K)$ correspondingly) due to Lemmas 9, 14 and Corollary 8.

27. Lemma. *Let L be a submodule over K of $C(X,K)$ or $C_+(X,K)$ relative to the operations $\vee$, $\wedge$, $\times_2$ and containing all constant mappings $g^c : X \to K$, where $c \in K$. Let also $\nu : L \to K$ be an idempotent (left homogeneous) continuous morphism. For each $f \in C(X,K) \setminus L$ or $C_+(X,K) \setminus L$ there exists an idempotent (left homogeneous) continuous extension μ_M of ν on a minimal closed submodule M containing L and f.*

Proof. For each $g \in M$ we put

(1) $\mu_M(g) = \nu(g) = \inf\{\nu(h) : g \leq h, h \in L\}$.

This implies that $\nu(g_1) \leq \nu(g_2)$ for each $g_1 \leq g_2 \in M$. Then

$\nu(g^c \times_2 g) = \inf\{\nu(h) : h \in L, g^c \times_2 g \leq h\} =$

$\inf\{\nu(g^c \times_2 q) : q \in L, g^c \times_2 g \leq g^c \times_2 q\} = c \times_2 \inf\{\nu(q) : q \in L, g \leq q\} = c \times_2 \nu(g)$ and

$\nu(g \times_2 g^c) = \inf\{\nu(h) : h \in L, g \times_2 g^c \leq h\} = \inf\{\nu(q \times_2 g^c) : q \in L, q \times_2 g^c \geq g \times_2 g^c\}$

$= \inf\{\nu(q) : q \in L, q \geq g\} \times_2 c = \nu(g) \times_2 c$.

On the other hand for each $g_1, g_2 \in M$ one gets

$\nu(g_1) \vee \nu(g_2) = \inf\{\nu(g) : g \in L, g_1 \leq g\} \vee \inf\{\nu(q) : q \in L, g_2 \leq q\}$

$= \inf\{\nu(g) \vee \nu(q) : g, q \in L, g_1 \leq g, g_2 \leq q\} \geq \inf\{\nu(g \vee q) : g, q \in L, g_1 \vee g_2 \leq g \vee q\} = \nu(g_1 \vee g_2)$.

From the inequalities $g_k \leq g_1 \vee g_2$ for $k = 1$ and $k = 2$ it follows, that $\nu(g_k) \leq \nu(g_1 \vee g_2)$, consequently, $\nu(g_1) \vee \nu(g_2) = \nu(g_1 \vee g_2)$. Then

$\nu(g_1) \wedge \nu(g_2) = \inf\{\nu(g) : g \in L, g_1 \leq g\} \wedge \inf\{\nu(q) : q \in L, g_2 \leq q\}$

$= \inf\{\nu(g) \wedge \nu(q) : g, q \in L, g_1 \leq g, g_2 \leq q\} \leq \inf\{\nu(g \wedge q) : g, q \in L, g_1 \wedge g_2 \leq g \wedge q\} = \nu(g_1 \wedge g_2)$.

But $\nu(g_k) \geq \nu(g_1 \wedge g_2)$, since $g_k \geq g_1 \wedge g_2$ for $k = 1$ and $k = 2$, consequently, $\nu(g_1) \wedge \nu(g_2) = \nu(g_1 \wedge g_2)$. If ν is left homogeneous, then $\inf\{\nu(bh) : bh \geq bg, h \in L\} = \inf\{\nu(bh) : h \geq g, h \in$

$L\} = b \inf\{\nu(h) : h \geq g, h \in L\}$ for each $b \in K$, consequently, ν is left homogeneous on M. If ν is continuous and g_k is a net in M converging to $g \in M$ (see §2.9), then $\nu(g) = \inf\{\nu(h) : g \leq h, h \in L\} = \lim_k \inf\{\nu(h) : g_k \leq h, h \in L\} = \lim_k \nu(g_k)$.

28. Lemma. *If suppositions of Lemma 27 are satisfied, then there exists an idempotent (left homogeneous) continuous morphism λ on $C(X,K)$ or $C_+(X,K)$ respectively such that $\lambda|_L = \nu$.*

Proof. The family of all extensions (M, μ_M) of ν on closed submodules M of $C(X,K)$ or $C_+(X,K)$ respectively is partially ordered by inclusion: $(M, \mu_M) \leq (N, \mu_N)$ if and only if $M \subset N$ and $\nu_N|_M = \nu_M$. In view of the Kuratowski-Zorn lemma [20] there exists the maximal closed submodule P in $C(X,K)$ or $C_+(X,K)$ correspondingly and an idempotent extension ν_P of ν on P. If $P \neq C(X,K)$ or $C_+(X,K)$ correspondingly by Lemma 27 this morphism ν_P could be extended on a module L containing P and some $g \in C(X,K) \setminus P$ or in $C(X,K)_+ \setminus P$ respectively. This contradicts the maximality of (P, ν_P). Thus $P = C(X,K)$ or $C_+(X,K)$ correspondingly.

29. Proposition. *The functors I, I_l and I_h, $I_{l,h}$ are epimorphic.*

Proof. Let a continuous mapping $f : X \to Y$ be epimorphic. We consider the set L of all continuous mappings $g \circ f : X \to K$ such that $g \in C(Y,K)$ or $C_+(Y,K)$. Then L is a submodule of $C(X,K)$ or $C_+(X,K)$ relative to the operations $\vee$, $\wedge$, $\times_2$ and L contains all constant mappings $g^c : X \to K$, where $c \in K$. Then we put $\mu(g \circ f) = \nu(g)$ for $\nu \in I(X,K)$ or in $I_l(X,K)$, $I_h(X,K)$ or $I_{l,h}(X,K)$. In view of Lemma 28 there is a continuous extension of μ from L onto $C(Y,K)$ or $C_+(Y,K)$ such that $\mu \in I(Y,K)$ or in $I_l(Y,K)$, $I_h(Y,K)$ or $I_{l,h}(Y,K)$ correspondingly.

30. Definition. It is said that $\nu \in \mathcal{O}(X,K)$ (or $\nu \in \mathcal{O}_l(X,K)$) is supported on a closed subset E in X, if $\nu(f) = 0$ for each $f \in C(X,K)$ or in $C_+(X,K)$ such that $f|_E \equiv 0$. A support of ν is the intersection of all closed subsets in X on which ν is supported.

31. Proposition. *Let $\nu \in \mathcal{O}(X,K)$ or in $\mathcal{O}_l(X,K)$. Then ν is supported on $E \subset X$ if and only if $\nu(f) = \nu(g)$ for each $f, g \in C(X,K)$ or in $C_+(X,K)$ correspondingly such that $f|_E \equiv g|_E$. Moreover, E is a support of ν if and only if ν is supported on E and for each proper closed subset F in E, i.e. $F \subset E$ with $F \neq E$, there are $f, h \in C(X,K)$ or in $C_+(X,K)$ respectively with $f|_F \equiv h|_F$ such that $\nu(f) \neq \nu(h)$.*

Proof. Consider $\nu \in \mathcal{O}(X,K)$ such that $\nu(f) = \nu(g)$ for each functions $f, g : X \to K$ with $f|_E = g|_E$. A continuous morphism ν induces a continuous morphism $\lambda \in \mathcal{O}(E,K)$ such that $\lambda(h) = \nu(h)$ for each $h \in C(X,K)$ with $h|_{X \setminus E} = 0$. Denote by *id* the identity embedding of a closed subset E into X. Each function $t : E \to K$ has an extension on X with values in K by Condition 19(CE). Then $\mathcal{O}(id)(\lambda) = \nu$, since $\nu(g^0) = 0$ and hence $\nu(s) = 0$ for each $s \in C(X,K)$ such that $s|_E \equiv 0$.

If $\nu \in \mathcal{O}(X,K)$ and ν is supported on E, then by Definition 30 there exists a morphism $\lambda \in \mathcal{O}(E,K)$ such that $\mathcal{O}(id)(\lambda) = \nu$. Therefore the equalities are valid: $\nu(f) = \lambda(f|_E) = \lambda(g|_E) = \nu(g)$ for each functions $f, g \in C(X,K)$ such that $f|_E = g|_E$.

If E is a support of ν, then by the definition this implies that ν is supported on E. Suppose that $F \subset E$, $F \neq E$ and for each $f, g \in C(X,K)$ with $f|_F \equiv g|_F$ the equality $\nu(f) = \nu(g)$ is satisfied, then a support of ν is contained in F, hence E is not a support of ν. This is the contradiction, hence there are $f, g \in C(X,K)$ with $f|_F \equiv g|_F$ such that $\nu(f) \neq \nu(g)$.

If ν is supported on E and for each proper closed subset F in E there are $f, h \in C(X,K)$ with $f|_F \equiv h|_F$ such that $\nu(f) \neq \nu(h)$, then ν is not supported on any such proper closed subset F, consequently, each closed subset G in X on which ν is supported contains E, i.e. $E \subset G$. Thus E is the support of ν.

32. Proposition. *The functors $\mathcal{O}$, I, $\mathcal{O}_l$, I_l, $\mathcal{O}_{l,h}$, $I_{l,h}$ preserve intersections of closed subsets.*

Proof. If E is a closed subset in X, then there is the natural embedding $C(E,K) \hookrightarrow C(X,K)$ (or $C_+(E,K) \hookrightarrow C_+(X,K)$, when $X \in Ob(\mathcal{S}_l)$) due to Condition 19(CE). Therefore, $\mathcal{O}(E \cap F, K) \subset \mathcal{O}(E,K) \cap \mathcal{O}(F,K)$ (or $\mathcal{O}_l(E \cap F, K) \subset \mathcal{O}_l(E,K) \cap \mathcal{O}_l(F,K)$ respectively). For any closed subsets E and F in X and each functions $f, g \in C(X,K)$ (or $C_+(X,K)$) with $f|_{E \cap F} \equiv g|_{E \cap F}$ there exists a function $h \in C(X,K)$ (or $C_+(X,K)$) such that $h|_E = f$ and $h|_F = g$ due to 19(CE). Therefore $\nu(f) = \nu(h)$ and $\nu(g) = \nu(h)$ for each $\nu \in \mathcal{O}(E,K) \cap \mathcal{O}(F,K)$ (or in $\mathcal{O}_l(E,K) \cap \mathcal{O}_l(F,K)$). In view of Proposition 31 the functors $\mathcal{O}$ and $\mathcal{O}_l$ preserve intersections of closed subsets. This implies that the functors I, I_l, $\mathcal{O}_{l,h}$ and $I_{l,h}$ also have this property.

33. Proposition. *Let $\{X_b; p^b_a; V\} =: P$ be an inverse system of topological spaces X_b, where V is a directed set, $p^b_a : X_b \to X_a$ is a continuous mapping for each $a \le b \in V$, $p_b : X = \lim P \to X_b$ is a continuous projection. Then the mappings*

(1) $s = (\mathcal{O}(p_b) : b \in V) : \mathcal{O}(X, K) \to \mathcal{O}(P, K)$ and $s_h = (\mathcal{O}_h(p_b) : b \in V) : \mathcal{O}_h(X, K) \to \mathcal{O}_h(P, K)$
(2) $t = (I(p_b) : b \in V) : I(X, K) \to I(P, K)$ and $t_h = (I_h(p_b) : b \in V) : I_h(X, K) \to I_h(P, K)$
are bijective and surjective continuous algebraic homomorphisms. Moreover, if $X_b \in Ob(\mathcal{S}_l)$ and p^b_a is order-preserving for each $a < b \in V$, then the mappings
(3) $s_l = (\mathcal{O}_l(p_b) : b \in V) : \mathcal{O}_l(X, K) \to \mathcal{O}_l(P, K)$ and $s_{l,h} = (\mathcal{O}_{l,h}(p_b) : b \in V) : \mathcal{O}_{l,h}(X, K) \to \mathcal{O}_{l,h}(P, K)$
(4) $t_l = (I_l(p_b) : b \in V) : I_l(X, K) \to I_l(P, K)$ and $t_{l,h} = (I_{l,h}(p_b) : b \in V) : I_{l,h}(X, K) \to I_{l,h}(P, K)$
also are bijective and surjective continuous algebraic homomorphisms.

Proof. We consider the inverse system $\mathcal{O}(P) = (\mathcal{O}(X_a); \mathcal{O}(p^a_b); V\}$ and its limit space $Y = \lim \mathcal{O}(P)$. Then $\mathcal{O}(p^b_a)\mathcal{O}(p_b) = \mathcal{O}(p_a)$ for each $a \le b \in V$, since $p^b_a \circ p_b = p_a$. Let $q : \mathcal{O}(X, K) \to Y$ denote the limit map of the inverse mapping system $q = \lim\{\mathcal{O}(p_a); \mathcal{O}(p^a_b); V\}$ (see also §2.5 [21]).

A continuous morphism ν is in $\mathcal{O}(X, K)$ if and only if $\mathcal{O}(p_a)(\nu) \in \mathcal{O}(X_a, K)$ for each $a \in V$, since

(5) $f \in C(X, K)$ if and only if $f = \lim\{f_b; p^b_a; V\}$ and
(6) $\mathcal{O}(p_a)(\nu)(f_a) = \nu(f_a \circ p_a) = \nu_a(f_a)$, where $\nu_a \in \mathcal{O}(X_a, K)$, $f_b \in C(X_b, K)$, $f_b = f_a \circ p^b_a$ for each $a \le b \in V$, $p^b_b = id$, $f(x) = \{f_a \circ p_a(x) : \ a \in V\} \in \theta(K)$ for each $x = \{x_a : a \in V\} \in X$, where $\{x_a : a \in V\}$ is a thread of P such that $x_a \in X_a$, $p^b_a(x_b) = x_a$ for each $a \le b \in V$, $\theta : K \to K^X$ is an order-preserving continuous algebraic embedding, $\theta(K)$ is isomorphic with K.

If $\nu, \lambda \in \mathcal{O}(X, K)$ are two different continuous morphisms, then this means that a continuous function $f \in C(X, K)$ exists such that $\nu_1(f) \ne \nu_2(f)$. This is equivalent to the following: there exists $a \in V$ such that $(\mathcal{O}(p_a)(\nu))(f) \ne (\mathcal{O}(p_a)(\lambda))(f)$. Thus the mappings s and analogously t are surjective and bijective.

On the other hand,

(7) $\nu_b(f_b \vee g_b) = \nu_b(f_b) \vee \nu_b(g_b)$ and
(8) $\nu_b(f_b \wedge g_b) = \nu_b(f_b) \wedge \nu_b(g_b)$ for each $b \in V$ and each $\nu_b \in I(X_b, K)$ and every $f_b, g_b \in C(X_b, K)$ such that either $f_b(x) < g_b(x)$ or $f_b(x) = g_b(x)$ or $g_b(x) < f_b(x)$ for each $x \in X_b$, also
(9) $\nu_b(g^c \times_2 f_b) = c \times_2 \nu_b(f_b)$ and
(10) $\nu_b(f_b \times_2 g^c) = \nu_b(f_b) \times_2 c$ for each $c \in K$ and $f_b \in C(X_b, K)$. Taking the inverse limit in Equalities $(5 - 10)$ gives the corresponding equalities for $\nu \in I(X, K)$, where $\nu = \lim\{\nu_a; I(p^b_a); V\}$, hence t is the continuous algebraic homomorphism due to Theorem 2.5.8 [21].

Analogously s preserves Properties $(9, 10)$, that is $\lambda = \lim\{\lambda_a; \mathcal{O}(p^b_a); V\}$ is weakly additive, where $\lambda_b \in \mathcal{O}(X_b, K)$ for each $b \in V$. Suppose that $f \le g \in C(X, K)$, then $f_b \le g_b$ for each $b \in V$ due to (5). From $\lambda_b(f_b) \le \lambda_b(g_b)$ for each $b \in V$, the inverse limit decomposition $\lambda = \lim\{\lambda_b; \mathcal{O}(p^b_a); V\}$ and Formula (6) it follows that λ is order-preserving.

If $X_b \in Ob(\mathcal{S}_l)$ for each $b \in V$, then a topological space X is linearly ordered: $x = \{x_b : b \in V\} \le y = \{y_b : b \in V\}$ if and only if $x_b \le y_b$ for each $b \in V$, where $x, y \in X$ are threads of the inverse system P such that $p^b_a(x_b) = x_a$ for each $a \le b \in V$. Since p^b_a is order-preserving for each $a \le b \in V$ and each f_b is non-decreasing, then f is nondecreasing and hence $f \in C_+(X, K)$ for each $f = \lim\{f_b; p^b_a; V\}$, where $f_b \in C_+(X_b, K)$ and $f_b = f_a \circ p^b_a$ for each $a \le b \in V$ and $x \in X$, $f(x) = \{f_a \circ p_a(x) : a \in V\}$.

Moreover, $\nu \in \mathcal{O}_h(X, K)$ is left homogeneous if and only if $\theta(p_a)(\nu)$ is left homogeneous for each $b \in V$, since $(\mathcal{O}_h(p_a)(\nu))(f_a) = \nu(f_a \circ p_a) = \nu_a(f_a)$. Applying Lemma 2.5.9 [21] one gets properties of mappings in Formulas $(3, 4)$.

34. Lemma. *There exist covariant functors $\mathcal{O}_2$, I_2, and $\mathcal{O}_{l,2}$, $I_{l,2}$ and $\mathcal{O}_{h,2}$, $I_{h,2}$ and $\mathcal{O}_{l,h,2}$, $I_{l,h,2}$ in the categories $\mathcal{K}_w$ and $\mathcal{K}$ and $\mathcal{K}_{w,l}$ and $\mathcal{K}_l$ respectively.*

Proof. If $K_1, K_2, K_3 \in Ob(\mathcal{K}_w)$, $u \in Mor(K_1, K_2)$, $v \in Mor(K_2, K_3)$, $\nu \in I(X, K_1)$, then $(I_2(vu)(\nu))(f) = v \circ u \circ \nu(f_1) = [I_2(v)(I_2(u)(\nu))](f)$ for each $f_1 \in C(X, K_1)$ such that $f(x) = v \circ u \circ f_1(x)$ for each $x \in X$, where $X \in Ob(\mathcal{S})$. That is $I_2(vu) = I_2(v)I_2(u)$. On the other hand, the equality $I_2(id) = 1$ is fulfilled.

If $f(x) \le g(x)$, then $u(f(x)) \le u(g(x))$, where $x \in X$, $f, g \in C(X, K_1)$. Therefore, if a mapping either $f \vee g$ or $f \wedge g$ exists in $C(X, K_1)$, then $u(f \vee g) = u(f) \vee u(g)$ or $u(f \wedge g) = u(f) \wedge u(g)$ in $C(X, K_2)$ respectively. If $f, g \in C(X, K_1)$, then $u(f(x) + g(x)) = u(f(x)) + u(g(x))$ for each $x \in X$, particularly, this is valid for $f = g^c$ or $g = g^c$, where $c \in K_1$. Therefore, $u(g^c \times_2 g) = g^{u(c)} \times_2 u(g)$ and $u(g \times_2 g^c) = u(g) \times_2 g^{u(c)}$. To each $\nu_n \in \mathcal{O}(X, K_n)$ and $u \in Mor(K_n, K_{n+1})$ there corresponds a morphism $u \circ \nu_n$ on $(\hat{id}, \hat{u})(C(X, K_n))$, $(\hat{id}, \hat{u})(C(X, K_n)) \hookrightarrow C(X, K_{n+1})$ (see §20). If $u : K_n \to K_{n+1}$ is not an epimorphism, the image $(\hat{id}, \hat{u})(C(X, K_n))$ is a proper submodule over $u(K_n)$ in $C(X, K_{n+1})$.

If $K_n, K_{n+1} \in Ob(\mathcal{K})$ and $X \in Ob(\mathcal{S}_l)$, $u \in Mor(K_n, K_{n+1})$, then $\hat{u} : C_+(X, K_n) \to C_+(X, K_{n+1})$ is a continuous homomorphism. If $K_n, K_{n+1} \in Ob(\mathcal{K}_l)$ and $X \in Ob(\mathcal{S}_l)$ (or $K_n, K_{n+1} \in Ob(\mathcal{K}_{w,l})$ and $X \in Ob(\mathcal{S})$) and $\nu \in \mathcal{O}_h(X, K_n)$ or in $I_h(X, K_n)$, $u \in Mor(K_n, K_{n+1})$, then $u \circ \nu \in \mathcal{O}_h(X, K_{n+1})$ or in $I_h(X, K_{n+1})$ respectively.

This and the definitions above imply that $\mathcal{O}_2(u) : \mathcal{O}(X, K_1) \to \mathcal{O}(X, K_2)$, $I_2(u) : I(X, K_1) \to I(X, K_2)$ and $\mathcal{O}_{l,2}(u)$, $I_{l,2}(u)$ and $\mathcal{O}_{h,2}(u)$, $I_{h,2}(u)$ and $\mathcal{O}_{l,h,2}(u)$, $I_{l,h,2}(u)$ are the homomorphisms. Thus we deduce that $\mathcal{O}_2 : \mathcal{K}_w \to \mathcal{OK}$ and $\mathcal{O}_{l,2} : \mathcal{K} \to \mathcal{O}_l\mathcal{K}$, $I_2 : \mathcal{K}_w \to \mathcal{IK}$ and $I_{l,2} : \mathcal{K} \to \mathcal{I}_l\mathcal{K}$, $\mathcal{O}_{h,2} : \mathcal{K}_{w,l} \to \mathcal{O}_h\mathcal{K}$, $I_{h,2} : \mathcal{K}_{w,l} \to \mathcal{I}_h\mathcal{K}$, $\mathcal{O}_{l,h,2} : \mathcal{K}_l \to \mathcal{O}_{l,h}\mathcal{K}$ and $\mathcal{I}_{l,h,2} : \mathcal{K}_l \to \mathcal{I}_{l,h}\mathcal{K}$ are the covariant functors on the categories $\mathcal{K}_w$, $\mathcal{K}$, $\mathcal{K}_{w,l}$ and $\mathcal{K}_l$ correspondingly with values in the categories of skew idempotent continuous morphisms, when a set $X \in Ob(\mathcal{S})$ or in $Ob(\mathcal{S}_l)$ correspondingly is marked.

35. Proposition. *The bi-functors I on $\mathcal{S} \times \mathcal{K}_w$, I_l on $\mathcal{S}_l \times \mathcal{K}$, I_h on $\mathcal{S} \times \mathcal{K}_{w,l}$ and $I_{l,h}$ on $\mathcal{S}_l \times \mathcal{K}_l$ preserve pre-images.*

Proof. In view of Proposition 24 and Lemma 34 I, I_l, I_h and $I_{l,h}$ are the covariant bi-functors, *i.e.*, the functors in $\mathcal{S}$ or $\mathcal{S}_l$ and the functors in $\mathcal{K}_w$ or $\mathcal{K}$ or $\mathcal{K}_{w,l}$ or $\mathcal{K}_l$ correspondingly as well. For any functor F the inclusion $F(f^{-1}(B)) \subset (F(f))^{-1}(F(B))$ is satisfied, where, for example, B is closed in $Y \in Ob(\mathcal{S})$.

Suppose the contrary that I does not preserve pre-images. This means that there exist $X, Y \in Ob(\mathcal{S})$ and $K_1, K_2 \in Ob(\mathcal{K}_w)$ or $X, Y \in Ob(\mathcal{S}_l)$ and $K_1, K_2 \in Ob(\mathcal{K})$, $f \in Mor(X, Y)$, $u \in Mor(K_1, K_2)$, $A \subset X$ and $B \subset Y$, where B is closed and hence A is closed when $A = F^{-1}(B)$, $\nu \in I(X, K_1)$ such that $I(f, u)(\nu) \in I(B, K_2)$ but $\nu \notin I(f^{-1}(B), u^{-1}(K_2))$ (or $\nu \in I_l(X, K_1)$, $I_l(f, u)(\nu) \in I_l(B, K_2)$ and $\nu \notin I_l(f^{-1}(B), u^{-1}(K_2))$ respectively). One can choose two functions $g, h \in C(X, K_1)$ such that

(1) $g|_A = h|_A$,
(2) $0 < c_1 = u[\inf_{x \in X} g(x)]$, $0 < c_2 = u[\inf_{x \in X} h(x)]$ and
(3) $u[\nu(g)] \ne u[\nu(h)]$.

There exist functions $s, t \in C(X, K_1)$ such that

(4) $s|_A = g|_A$ and $t|_A = h|_A$, while
(5) $s|_{X \setminus A} = t|_{X \setminus A}$ and
(6) $s(x) \le g(x)$ and $s(x) \le h(x)$ for each $x \in X \setminus A$, where g, h satisfy Conditions $(1-3)$ due to property 19(CE). There are also functions $q, r \in C(X, K_1)$ such that
(7) $q|_{X \setminus A} = g|_{X \setminus A}$ and $r|_{X \setminus A} = h|_{X \setminus A}$ with
(8) $q(x) = r(x)$ and $q(x) \le c$ for each $x \in A$, where
(9) $c \in K_1$, $c < \inf_{x \in X} g(x)$, $c < \inf_{x \in X} h(x)$ such that $u(c) < c_1$ and $u(c) < c_2$.
Evidently, $c_1 \le u[\nu(g)]$ and $c_2 \le u[\nu(h)]$. Then
(10) $\nu(g) = \nu(s \vee q) = \nu(s) \vee \nu(q)$ and
(11) $\nu(h) = \nu(t \vee r) = \nu(t) \vee \nu(r)$ and $u[\nu(q)] \ne u[\nu(r)]$.

On the other hand, there are functions $q_1, r_1 \in C(Y, K_2)$, $q_2, r_2 \in C(Y, K_1)$ such that $q_2 \circ f = q$, $r_2 \circ f = r$, $u \circ q_2 = q_1$, $u \circ r_2 = r_1$ and $q_2|_B = r_2|_B$. Therefore, from Properties $(7-10)$ it follows that

(12) $(I(f,u)(\nu))(q_1) = u[\nu(q)] \leq u(c)$ and $(I(f,u)(\nu))(r_1) = u[\nu(r)] \leq u(c)$. The condition $s = t$ on A and on $X \setminus A$ imply that

(13) $\nu(s) = \nu(t)$. Therefore,

(14) $u(\nu(g)) = u(\nu(s)) \vee u(\nu(q))$ and $u(\nu(h)) = u(\nu(t)) \vee u(\nu(r))$, which follows from $(10, 11)$. But Formulas $(4-6, 12-14)$ contradict the inequality $u[\nu(g)] \neq u[\nu(h)]$, since u is the order-preserving continuous algebraic homomorphism from K_1 into K_2. Thus the bi-functors I and I_l preserve pre-images. The proof in other cases is analogous.

36. Corollary. *If $\nu \in I(X,K)$ or $\nu \in I_l(X,K)$, $f \in Mor(X,Y)$, $u \in Mor(K_1,K_2)$, where $X, Y \in Ob(\mathcal{S})$ and $K_1, K_2 \in Ob(\mathcal{K}_w)$ or $X, Y \in Ob(\mathcal{S}_l)$ and $K_1, K_2 \in Ob(\mathcal{K})$, then $supp(I(f,u)(\nu)) = f(supp(u[\nu]))$ or*

$supp(I_l(f,u)(\nu)) = f(supp(u[\nu]))$ correspondingly.

37. Definitions. Suppose that Q is a category and F, G are two functors in Q. Suppose also that a transformation $p : F \to G$ is defined for each $X \in Q$, that is a continuous mapping $p_X : F(X) \to G(X)$ is given. If $p_Y \circ F(f) = G(f) \circ p_X$ for each mapping $f \in Mor(X,Y)$ and every objects $X, Y \in Ob(Q)$, then the transformation $p = \{p_X : X \in Ob(Q)\}$ is called natural.

If $T : Q \to Q$ is an endofunctor in a category Q and there are natural transformations the identity $\eta : 1_Q \to T$ and the multiplication $\psi : T^2 \to T$ satisfying the relations $\psi \circ T\eta = \psi \circ \eta T = 1_T$ and $\psi \circ \psi T = \psi \circ T\psi$, then one says that the triple $\mathbf{T} := (T, \eta, \psi)$ is a monad.

38. Theorem. *There are monads in the categories $\mathcal{S} \times \mathcal{K}_w$, $\mathcal{S}_l \times \mathcal{K}$, $\mathcal{S} \times \mathcal{K}_{w,l}$ and $\mathcal{S}_l \times \mathcal{K}_l$.*

Proof. Let $\bar{g}(\nu) := \nu(g)$ for $g \in C(X,K)$ and $\nu \in I(X,K)$, where $X \in Ob(\mathcal{S})$ and $K \in Ob(\mathcal{K}_w)$. Therefore, this induces the morphism $\bar{g} : I(X,K) \to K$. Then

$$\overline{g^b \times_2 g}(\nu) = \nu(g^b \times_2 g) = b \times_2 \nu(g) = b \times_2 \bar{g}(\nu) \text{ and}$$

$$\overline{g \times_2 g^b}(\nu) = \nu(g \times_2 g^b) = \nu(g) \times_2 b = \bar{g}(\nu) \times_2 b,$$

where $g^b(x) = b$ for each $x \in X$, that is

(1) $\overline{g \times_2 g^b} = \bar{g} \times_2 \overline{g^b}$ and (1′) $\overline{g^b \times_2 g} = \overline{g^b} \times_2 \bar{g}$
for each $g \in C(X,K)$ and $b \in K$.

Then we get $\overline{g \vee h}(\nu) = \nu(g \vee h) = \nu(g) \vee \nu(h) = \bar{g}(\nu) \vee \bar{h}(\nu) = (\bar{g} \vee \bar{h})(\nu)$. Moreover, we deduce that $\overline{g \wedge h}(\nu) = \nu(g \wedge h) = \nu(g) \wedge \nu(h) = \bar{g}(\nu) \wedge \bar{h}(\nu) = (\bar{g} \wedge \bar{h})(\nu)$. Thus we get the equalities

(2) $\overline{g \vee h} = \bar{g} \vee \bar{h}$ and (2′) $\overline{g \wedge h} = \bar{g} \wedge \bar{h}$.

If additionally ν is left homogeneous and $K \in Ob(\mathcal{K}_{w,l})$, then $\overline{bg} = \nu(bg) = b\nu(g) = b\bar{g}(\nu)$. Therefore, we infer that $\overline{bg} = b\bar{g}$ for every $b \in K$ and $g \in C(X,K)$.

For $\lambda \in I(I(X,K),K)$ we put $\xi_{X,K}(\lambda)(g) = \lambda(\bar{g})$ for each $g \in C(X,K)$. Then $\xi_{X,K}(\lambda)(g^b) = \lambda(\overline{g^b}) = \lambda(q^b) = b$, where $q^b : I(X,K) \to K$ denotes the constant mapping $q^b(y) = b$ for each $y \in I(X,K)$. From Formulas $(1, 1')$ it follows that

$$\xi_{X,K}(\lambda)(g^b \times_2 g) = \lambda(\overline{g^b \times_2 g}) = \lambda(b \times_2 \bar{g}) = b \times_2 \lambda(\bar{g}) = b \times_2 \xi_{X,K}(\lambda)(g) \text{ and}$$

$$\xi_{X,K}(\lambda)(g \times_2 g^b) = \lambda(\overline{g \times_2 g^b}) = \lambda(\bar{g} \times_2 \overline{g^b}) = \lambda(\bar{g}) \times_2 b = \xi_{X,K}(\lambda)(g) \times_2 b.$$

On the other hand, from Formulas $(2, 2')$ we get that

$$\xi_{X,K}(\nu)(g \vee h) = \nu(\overline{g \vee h}) = \nu(\bar{g} \vee \bar{h}) = \nu(\bar{g}) \vee \nu(\bar{h}) = \xi_{X,K}(\nu)(g) \vee \xi_{X,K}(\nu)(h) \text{ and}$$

$$\xi_{X,K}(\nu)(g \wedge h) = \nu(\overline{g \wedge h}) = \nu(\bar{g} \wedge \bar{h}) = \nu(\bar{g}) \wedge \nu(\bar{h}) = \xi_{X,K}(\nu)(g) \wedge \xi_{X,K}(\nu)(h)$$

for each $b \in K$, $g, h \in C(X,K)$. Thus $\xi_{X,K} : I(I(X,K),K) \to I(X,K)$.

If $\lambda \in I_h(I_h(X,K),K)$ for some $K \in Ob(\mathcal{K}_{w,l})$, then $\xi_{X,K}(\lambda)(bg) = \lambda(\overline{bg}) = \lambda(b\bar{g}) = b\lambda(\bar{g})$, hence $\xi_{X,K} : I_h(I_h(X,K),K) \to I_h(X,K)$. Analogously the mapping $\xi_{X,K} : \mathcal{O}(\mathcal{O}(X,K),K) \to \mathcal{O}(X,K)$ is defined for each $X \in Ob(\mathcal{S})$ and $K \in \mathcal{K}_w$, also $\xi_{X,K} : \mathcal{O}_l(\mathcal{O}_l(X,K),K) \to \mathcal{O}_l(X,K)$, $\xi_{X,K} : I_l(I_l(X,K),K) \to I_l(X,K)$ for each $X \in Ob(\mathcal{S}_l)$ and $K \in \mathcal{K}$, $\xi_{X,K} : I_h(I_h(X,K),K) \to I_h(X,K)$ for $X \in Ob(\mathcal{S})$ and $K \in \mathcal{K}_{w,l}$, $\xi_{X,K} : I_{l,h}(I_{l,h}(X,K),K) \to I_{l,h}(X,K)$ for $X \in Ob(\mathcal{S}_l)$ and $K \in \mathcal{K}_l$. One also puts $\eta : Id_Q \to \mathcal{O}$ or $\eta : Id_Q \to I$ for $Q = \mathcal{S} \times \mathcal{K}_w$, also $\eta : Id_Q \to \mathcal{O}_l$ or $\eta : Id_Q \to I_l$ for $Q = \mathcal{S}_l \times \mathcal{K}$ correspondingly.

Next we verify that the transformations η and ξ are natural for each $f \in Mor(X \times K_1, Y \times K_2)$, i.e. $f = (s,u)$, $s \in Mor(X,Y)$, $u \in Mor(K_1,K_2)$:

$$\eta_{(Y,K_2)} \circ \mathcal{O}((s,u)) = \mathcal{O}(id_Y, id_{K_2})) \circ \mathcal{O}((s,u))$$

$$= \mathcal{O}((s,u)) = \mathcal{O}((s,u)) \circ \mathcal{O}(id_X, id_{K_1})) = \mathcal{O}((s,u)) \circ \eta_{(X,K_1)},$$

$$\xi_{(Y,K_2)} \circ \mathcal{O}((s,u))[\mathcal{O}^2(X,K_1)] = \xi_{(Y,K_2)}(\mathcal{O}(\bar{s},\bar{u})[\mathcal{O}(X,K_1)])$$

$$= \mathcal{O}((s,u)) \circ \eta_{(X,K_1)}[\mathcal{O}^2(X,K_1)]),$$

where $\mathcal{O}^{m+1}(X,K) := \mathcal{O}(\mathcal{O}^m(X,K),K)$ for each natural number m (see also §20 and Proposition 35).

For each $\nu \in \mathcal{O}(X,K)$ and $g \in C(X,K)$ one gets

$$\xi_{X,K} \circ \eta_{(\mathcal{O}(X,K),K)}(\nu)(g) = \eta_{(\mathcal{O}(X,K),K)}(\nu)(\bar{g}) = \bar{g}(\nu) = \nu(g) \text{ and}$$

$$\xi_{X,K} \circ \mathcal{O}(\eta_{(X,K)})(\nu)(g) = (\mathcal{O}(\eta_{(X,K)}(\nu))(\bar{g}) = \nu(\bar{g} \circ \eta_{(X,K)}) = \nu(g).$$

Let now $\tau \in \mathcal{O}^3(X,K)$ and $g \in C(X,K)$, then

$$\xi_{(X,K)} \circ \xi_{\mathcal{O}(X,K)}(\tau)(g) = (\xi_{\mathcal{O}(X,K)}(\tau))(\bar{g}) = \tau(\bar{\bar{g}}) \text{ and}$$

$$\xi_{(X,K)} \circ \mathcal{O}(\xi_{(X,K)})(\tau)(g) = (\mathcal{O}(\xi_{(X,K)})(\tau))(\bar{g}) = \tau(\bar{g} \circ \xi_{(X,K)}) = \tau(\bar{\bar{g}}),$$

where $\bar{\bar{g}} \in C(\mathcal{O}^2(X,K),K)$ is prescribed by the formula $(\bar{\bar{g}})(\nu) = \nu(\bar{g})$ for each $\nu \in \mathcal{O}^2(X,K)$. Thus $\mathbf{O} := (\mathcal{O},\eta,\xi)$ is the monad. Since I is the restriction of the functor $\mathcal{O}$, the triple $\mathbf{I} := (I,\eta,\xi)$ is the monad in the category $\mathcal{S} \times \mathcal{K}_w$ as well. Analogously $\mathbf{O}_l := (\mathcal{O}_l,\eta,\xi)$ and $\mathbf{I}_l := (I_l,\eta,\xi)$ form the monads in the category $\mathcal{S}_l \times \mathcal{K}$; $\mathbf{O}_h = (\mathcal{O}_h,\eta,\xi)$ and $\mathbf{I}_h = (I_h,\eta,\xi)$ are the monads in $\mathcal{S} \times \mathcal{K}_{w,l}$; $\mathbf{O}_{l,h} = (\mathcal{O}_{l,h},\eta,\xi)$ and $\mathbf{I}_{l,h} = (I_{l,h},\eta,\xi)$ are the monads in $\mathcal{S}_l \times \mathcal{K}_l$.

39. Proposition. *If a sequence*

(1) ... $\to K_n \to K_{n+1} \to K_{n+2} \to$ ... *in $\mathcal{K}_w$ (or in $\mathcal{K}$) is exact, then sequences*

(2) ... $\to \mathcal{O}_2(X,K_n) \to \mathcal{O}_2(X,K_{n+1}) \to \mathcal{O}_2(X,K_{n+2}) \to$ ... *and*

(3) ... $\to I_2(X,K_n) \to I_2(X,K_{n+1}) \to I_2(X,K_{n+2}) \to$ ... *are exact (analogously for $\mathcal{O}_{l,2}$ and $I_{l,2}$ correspondingly).*

Proof. A sequence

... $\to K_n \to K_{n+1} \to K_{n+2} \to$ ... is exact means that $s_n(K_n) = ker(s_{n+1})$ for each n, where $s_n : K_n \to K_{n+1}$ is an order-preserving continuous algebraic homomorphism, $ker(s_{n+1}) = s_{n+1}^{-1}(0)$. Each continuous homomorphism s_n induces the continuous homomorphism $\mathbf{s}_n : C(X,K_n) \to C(X,K_{n+1})$ point-wise $(\mathbf{s}_n(f))(x) = s_n(f(x))$ for each $x \in X$. Therefore, we get that $\mathbf{s}_n(f \vee g) = \mathbf{s}_n(f) \vee \mathbf{s}_n(g)$ or $\mathbf{s}_n(f \wedge g) = \mathbf{s}_n(f) \wedge \mathbf{s}_n(g)$, when $f \vee g$ or $f \wedge g$ exists, where $f,g \in C(X,K_n)$. Moreover, the equalities $(\mathbf{s}_n(f+g))(x) = \mathbf{s}_n(f(x)+g(x)) = s_n(f(x)) + s_n(g(x)) = [\mathbf{s}_n(f) + \mathbf{s}_n(g)](x)$ and $[\mathbf{s}_n(fg)](x) = s_n(f(x)g(x)) = s_n(f(x))s_n(g(x)) = [(\mathbf{s}_n(f))(\mathbf{s}_n(g))](x)$ are fulfilled, consequently, $\mathbf{s}_n(C(X,K_n)) = \mathbf{s}_{n+1}^{-1}(0)$, since $f_{n+2} \in C(X,K_{n+2})$ is zero if and only if $f_{n+2}(x) = 0$ for each $x \in X$. Thus the sequence

... $\to C(X,K_n) \to C(X,K_{n+1}) \to C(X,K_{n+2}) \to$ ... is exact.

Then a continuous morphism $\lambda_{n+2} \in \mathcal{O}(X, K_{n+2})$ is zero on $\mathbf{s}_{n+1}(C(X, K_{n+1}))$ if and only if $\lambda_{n+2}(f_{n+2}) = 0$ for each $f_{n+2} \in \mathbf{s}_{n+1}(C(X, K_{n+1}))$. Therefore, $\mathbf{s}_{n+1}(\lambda_{n+1}) = 0 = \lambda_{n+2}$ on $\mathbf{s}_{n+1}[\mathbf{s}_n(C(X, K_n))]$ if and only if $\lambda_{n+1}(f_{n+1}) \in s_n(K_n)$ for each $f_{n+1} \in \mathbf{s}_n(C(X, K_n))$. At the same time we have that $\mathbf{s}_{n+1}[\mathbf{s}_n(C(X, K_n))] \subset \mathbf{s}_{n+1}(C(X, K_{n+1}))$, consequently, $\mathcal{O}_2(s_n) = ker\mathcal{O}_2(s_{n+1})$. Thus the sequences $(2,3)$ are exact, analogously for other functors I_2, $\mathcal{O}_{l,2}$ and $I_{2,l}$.

3.3. Lattices Associated with Actions of Groupoids on Topological Spaces

40. Lemma. *Let G be a topological groupoid with a unit acting on a topological space X such that to each element $g \in G$ a continuous mapping $v_g : X \to X$ corresponds having the properties*

(1) *$v_g v_h = v_{gh}$ for each $g, h \in G$ and*

(2) *$v_e = id$, where $e \in G$ is the unit element, $id(x) = x$ for each $x \in X$. If K is a topological ringoid with the associative sub-ringoid L, $L \supset \{0, 1\}$, such that*

(3) *$a(bc) = (ab)c$ for each $a, b \in L$ and $c \in K$, a continuous mapping $\rho : G^2 \to L \setminus \{0\}$ satisfies the cocycle condition*

(4) *$\rho(g, x)\rho(h, v_g x) = \rho(gh, x)$ and*

(5) *$\rho(e, x) = 1 \in K$ for each $g, h \in G$ and $x \in X$, then*

(6) *$T_g f(x) := \rho(g, x)\hat{v}_g f(x)$ is a representation of G by continuous in the $g \in G$ variable mappings T_g of $C(X, K)$ into $C(X, K)$, when f is marked, where $f \in C(X, K)$, $\hat{v}_g f(x) := f(v_g(x))$ for each $g \in G$ and $x \in X$.*

Proof. For each $g, h \in G$ one has $T_g(T_h f(x)) = \rho(g, x)\hat{v}_g[\rho(h, x)\hat{v}_h f(x)] = \rho(gh, x)\hat{v}_{gh} f(x) = T_{gh} f(x)$, hence $T_g T_h = T_{gh}$. Moreover, $T_e f = f$, since $v_e = id$ and $\rho(e, x) = 1$, *i.e.*, $T_e = I$ is the unit operator on $C(X, K)$. Mappings $T_g f(x)$ are continuous in the $g \in G$ variable as compositions and products of continuous mappings.

The continuous mappings T_g are (may be) generally non-linear relative to K. If K is commutative, distributive and associative, then T_g are K-linear on $C(X, K)$.

41. Definition. A continuous morphism ν on $C(X, K)$ or $C_+(X, K)$ we call semi-idempotent, if it satisfies the property:

(1) $\nu(g + f) = \nu(g) + \nu(f)$ for each $f, g \in C(X, K)$ or $C_+(X, K)$ respectively, where $(g + f)(x) = g(x) + f(x)$ for each $x \in X$.

Suppose that G is a topological groupoid with the unit continuously acting on a topological space X and satisfying Conditions 40(1,2). A continuous morphism λ on $C(X, K)$ or $C_+(X, K)$ we call (T, G)-invariant if

(2) $\hat{T}_g \lambda = \lambda$, where $(\hat{T}_g \lambda)(f) := \lambda(T_g f)$ for each $g \in G$ and f in $C(X, K)$ or $C_+(X, K)$ correspondingly.

Let $S_+(G, K)$ denote the family of all semi-idempotent continuous morphisms, when K is commutative and associative relative to the addition for (G, K), let also $S_\vee(G, K)$ (or $S_\wedge(G, K)$) denote the family of all continuous morphisms satisfying Conditions 2(4) (or 2(5) correspondingly) for general K. Denote by $H_+(G, K)$ (or $H_\vee(G, K)$ or $H_\wedge(G, K)$) the family of all G-invariant semi-idempotent (or in $S_\vee(G, K)$ or in $S_\wedge(G, K)$ correspondingly) continuous morphisms for (X, K), when $X = G$ as a topological space. We supply these families with the operations of the addition

(3) $\nu(f) +_i \lambda(f) =: (\nu +_i \lambda)(f)$ in $S_j(G, K)$ for $i = 1, 2, 3$ and $j = +, \vee, \wedge$ respectively and the multiplication being the convolution of continuous morphisms

(4) $(\nu * \lambda)(f) = \nu(\lambda(T_g f))$ in $S_j(G, K)$, where $g \in G$, $j \in \{+, \vee, \wedge\}$.

Then we put $H_h(G, K)$, $S_h(G, K)$, $H_{\vee,h}(G, K)$, $S_{\vee,h}(G, K)$, $H_{\wedge,h}(G, K)$ and $S_{\wedge,h}(G, K)$ for the subsets of all left homogeneous morphisms in $H_+(G, K)$, $S_+(G, K)$, $H_\vee(G, K)$, $S_\vee(G, K)$, $H_\wedge(G, K)$, $S_\wedge(G, K)$ correspondingly.

42. Proposition. *If ν is a (T,G)-invariant semi-idempotent continuous morphism, then its support is contained in $\bigcap_{n=1}^{\infty} T^n(X)$, where*

$$T(A) := \bigcup_{g\in G} supp(\rho(g,x)\hat{v}_g(\chi_A(x)))$$

for a closed subset A in X. Moreover, if K has not divisors of zero a support of ν is G-invariant and contained in $\bigcap_{n=1}^{\infty} P^n(X)$, where

$$P(X) = \bigcup_{g\in G} v_g(X).$$

Proof. If $\nu(f) \neq 0$, then $\nu(T_g f) \neq 0$ for each $g \in G$, when a continuous morphism ν is (T,G)-invariant. On the other hand, if $supp(f) \subset supp(\nu)$, then $supp(\rho(g,x)\hat{v}_g f(x)) \subset supp(\nu)$ for each $g \in G$. At the same time, $\bigcup_{g\in G} supp(T_g f) \subset \bigcup_{g\in G} supp(\hat{v}_g f)$, since $\rho(g,x) \in L \setminus \{0\}$ for each $g \in G$ and $x \in X$. If $f = \chi_{supp(\nu)}$, then $supp(\nu) \subset T(supp(\nu)) \subset T(X)$, hence by induction we deduce that $supp(\nu) \subset T^n(X)$ for each natural number n, where χ_A is the characteristic function of a set A, so that $\chi_A(x) = 1$ for each $x \in A$ while $\chi_A(x) = 0$ for each $x \notin A$.

If K has not divisors of zero, then $supp(\hat{T}_g\nu) = \hat{v}_g supp(\nu) \subset supp(\nu)$ for each element $g \in G$, hence $\bigcup_{g\in G} \hat{v}_g supp(\nu) = supp(\nu)$, since $e \in G$ and $\nu_e = id$. That is $supp(\nu)$ is G-invariant. Since $supp(\nu) \subset X$, then $supp(\nu) \subset P(X)$ and by induction $supp(\nu) \subset P^n(X)$ for each natural number n.

43. Proposition. *If G is a topological groupoid with a unit or a topological monoid, then $S_+(G,K)$, $S_\vee(G,K)$ and $S_\wedge(G,K)$ for general T_g and K (or $S_h(G,K)$, $S_{h,\vee}(G,K)$ and $S_{h,\wedge}(G,K)$ for $T_g \equiv \hat{v}_g$ or when K is commutative and associative relative to the multiplication) supplied with the convolution 41(4) as the multiplication operation are topological groupoids with a unit or monoids correspondingly.*

Proof. Certainly, the definitions above imply the inclusion $S_h(G,K) \subset S_+(G,K)$. If $\nu,\lambda \in I_h(G,K)$, then $(\nu*\lambda)(bf) = \nu(\lambda(T_g(bf))) = \nu(b\lambda(T_g f)) = b((\nu*\lambda)(f))$, when either $T_g \equiv \hat{v}_g$ for each $g \in G$ or K is commutative and associative relative to the multiplication. We mention that the evaluation morphism δ_e at e belongs to $S_h(G,K)$ and has the property $\nu*\delta_e = \delta_e*\nu = \nu$ for each $\nu \in S(G,K)$, where e is a unit element in G, $\delta_x f = f(x)$ for each $f \in C(X,K)$ and $x \in X$. Thus δ_e is the neutral element in $S(G,K)$.

For a topological monoid G one has $\hat{v}_s(\hat{v}_u f(x)) = f(s(ux)) = f((su)x) = \hat{v}_{su} f(x)$ for each $f \in C(G,K)$ and $s,u,x \in G$ so that $f((su)x)$ is a function continuous in the variables s, u and x in G. Since ν and λ are continuous on $C(G,K)$, then $\nu*\lambda$ is continuous on $C(G,K)$.

If G is a topological monoid, then $(\nu*(\lambda*\phi))(f) = \nu^u((\lambda*\phi)(T_u f)) = \nu^u(\lambda^s(\phi(T_s T_u f)) = \nu^u(\lambda^s(\phi(T_{su} f))) = (\nu*\lambda)^{su}(\phi(T_{su} f)) = [(\nu*\lambda)*\phi](f)$ for every $f \in C(G,K)$ and $u,s \in G$ and $\nu,\lambda,\phi \in S_j(G,K)$, where $\nu^u(h)$ means that a continuous morphism ν on a function h acts by the variable $u \in G$, consequently, $\nu*(\lambda*\phi) = (\nu*\lambda)*\phi$. Thus the family $S_j(G,K)$ is associative, when G is associative, where $j \in \{+,\vee,\wedge,h,(h,\vee),(h,\wedge)\}$ for the corresponding T_g and K.

From §§2.3, 2.4, 2.9 and 5 it follows that the mapping $(\nu,\lambda) \mapsto \nu*\lambda$ is continuous.

44. Theorem. *If G is a topological groupoid with a unit or a topological monoid, then $S_+(G,K)$ (for K commutative and associative relative to $+$), $S_\vee(G,K)$ and $S_\wedge(G,K)$ for general T_g (or $S_{\vee,h}(G,K)$ and $S_{\wedge,h}(G,K)$ for either $T_g \equiv \hat{v}_g$ or when K is commutative and associative relative to the multiplication) are topological ringoids or semirings correspondingly.*

Proof. If $f,g \in C(X,K)$ or in $C_+(X,K)$ and $f\vee g$ or $f\wedge g$ exists (see Condition (3) in Lemma 2.12), ν,λ are continuous morphisms satisfying Condition either 2(4) or 2(5) respectively, then

(1) $(\nu +_i \lambda)(f +_i g) = \nu(f +_i g) +_i \lambda(f +_i g) = (\nu(f) +_i \nu(g)) +_i (\lambda(f) +_i \lambda(g)) = (\nu(f) +_i \lambda(f)) +_i (\nu(g) +_i \lambda(g)) = (\nu +_i \lambda)(f) +_i (\nu +_i \lambda)(g)$

for $i = 1,2,3$, where $+_1 = +$, $+_2 = \vee$, $+_3 = \wedge$. That is, the continuous morphism $\nu +_i \lambda$ satisfies Property 41(1) for $i = 1$ or 2(4) for $i = 2$ or 2(5) when $i = 3$ correspondingly. If additionally ν and λ are left homogeneous, then

(2) $(\nu +_i \lambda)(bf) = \nu(bf) +_i \lambda(bf) = b\nu(f) +_i b\lambda(f) = b(\nu +_i \lambda)(f)$ for each $b \in K$.

On the other hand, we deduce that
$((\nu_1 +_i \nu_2) * \lambda)(f) = (\nu_1 +_i \nu_2)(\lambda(T_g f)) = \nu_1(\lambda(T_g f)) +_i \nu_2(\lambda(T_g f))$
$= (\nu_1 * \lambda)(f) +_i (\nu_2 * \lambda)(f)$ and
$(\lambda * (\nu_1 +_i \nu_2))(f) = \lambda((\nu_1 +_i \nu_2)(T_g f)) = \lambda(\nu_1(T_g f)) +_i \lambda(\nu_2(T_g f))$
$= (\lambda * \nu_1)(f) +_i (\lambda * \nu_2)(f)$
for each $\nu_1, \nu_2, \lambda \in S_j(G,K)$ and $f \in C(G,K)$ or in $C_+(G,K)$ correspondingly, for $i = 1,2,3$ and $i = i(j)$ respectively, where $+_1 = +$, $+_2 = \vee$ and $+_3 = \wedge$. Thus, the right and left distributive rules are satisfied:

(3) $(\nu_1 +_i \nu_2) * \lambda = \nu_1 * \lambda +_i \nu_2 * \lambda$ and
(4) $\lambda * (\nu_1 +_i \nu_2) = \lambda * \nu_1 +_i \lambda * \nu_2$
for $i = 1,2,3$ respectively. From the definitions of these operations and Proposition 43 their continuity follows.

Therefore, Formulas $(1-4)$ and Proposition 43 imply that $S_+(G,K)$, $S_\vee(G,K)$, $S_\wedge(G,K)$, $S_{\vee,h}(G,K)$ and $S_{\wedge,h}(G,K)$ are left and right distributive topological ringoids or semirings correspondingly.

45. Theorem. *If G is a topological groupoid with a unit, $X = G$ as a topological space (see §41), then $H_j(G,K)$ is a closed ideal in $S_j(G,K)$, where $j = +$ (for K commutative and associative relative to $+$) or $j = \vee$ or $j = \wedge$ or $j = (\vee,h)$ or $j = (\wedge,h)$ with $\rho(u,x) \equiv 1$; $j = (\vee,h)$ or $j = (\wedge,h)$ for commutative and associative K relative to the multiplication with general T_u.*

Proof. We mention that $\hat{T}_g(b_1\lambda_1 +_i b_2\lambda_2)(f) = b_1\lambda_1(T_g f) +_i b_2\lambda_2(T_g f)$, where the operation denoted by the addition $+_i$ is either $+$ or $\vee$ or $\wedge$ for $i = 1$ or $i = 2$ or $i = 3$ correspondingly (and also below in this section), consequently, $b_1\lambda_1 +_i b_2\lambda_2 \in H_j(G,K)$ for each $\lambda_1, \lambda_2 \in H_j(G,K)$ and $b_1, b_2 \in K$, $i = i(j)$.

In Formula 41(4) after the action of a morphism λ on a continuous function $T_g f(x)$ in the variable x one gets that $\lambda(T_g f) =: h(g)$ is a continuous function in the variable g and ν is acting on this function, i.e. $(\nu * \lambda)(f) = \nu(h(x))$, where $x, g \in G$. This implies that
$\nu * (\lambda(f +_i t)) = \nu * (\lambda(f) +_i \lambda(t)) = \nu(\lambda(T_g f) +_i \lambda(T_g t))$
$= \nu(\lambda(T_g f)) +_i \nu(\lambda(T_g t)) = (\nu * \lambda)(f) +_i (\nu * \lambda)(t)$ for $i = 1,2,3$,
consequently, the convolution operation maps from $S_j(G,K)^2$ into $S_j(G,K)$.

The property being G-invariant provides closed subsets in $S_j(G,K)$, since if a net of continuous mappings g_k converges to a continuous mapping g an each g_k is G-invariant, then $g = \lim_k g_k$ is G-invariant as well.

If $\lambda \in H_j(G,K)$ and $\nu \in S_j(G,K)$, then
$(\hat{T}_s(\nu * \lambda))(f) = \hat{T}_s(\nu^u(\lambda^x(T_u f(x)))) = \nu^u(\lambda^x(T_s(T_u f(x)))$
$= \nu^u(\lambda^x(T_u f(x)))) = (\nu * \lambda)(f)$ and
$(\hat{T}_s(\lambda * \nu))(f) = \hat{T}_s(\lambda^u(\nu^x(T_u f(x)))) = \lambda^u(\nu^x(T_s(T_u f(x))))$
$= \lambda^u(T_s(\nu^x(T_u f(x)))) = (\lambda * \nu)(f)$,
since $\lambda^u(T_s g(u)) = \lambda^u(g(u)) = \lambda(g)$, particularly with $g(x) = T_u f(x)$ or $g(u) = \nu^x(T_u f(x))$ correspondingly, whilst $T_s \equiv \hat{v}_s$ in the cases $j = +$ or $j = \vee$ or $j = \wedge$ with $\rho \equiv 1$, or for general $T_u f(x) = \rho(u,x)\hat{v}_s f(x)$ in the cases of homogeneous continuous morphisms $j = (\vee,h)$ or $j = (\wedge,h)$ (see §43 also), hence $\nu * \lambda$, $\lambda * \nu \in H_j(G,K)$. Therefore, the latter formula and Theorem 44 imply that
$(\nu +_i H_j(G,K)) * H_j(G,K) \subset (\nu * H_j(G,K)) +_i (H_j(G,K) * H_j(G,K))$
$\subset H_j(G,K) +_i H_j(G,K) \subset H_j(G,K)$ and
$H_j(G,K) * (\nu +_i H_j(G,K)) \subset (H_j(G,K) * \nu) +_i (H_j(G,K) * H_j(G,K))$
$\subset H_j(G,K) +_i H_j(G,K) \subset H_j(G,K)$
for each $\nu \in S_j(G,K)$ and $+_i$ corresponding to j, that is $H_j(G,K)$ is the right and left closed ideal in $S_j(G,K)$.

4. Conclusions

Skew continuous morphisms of ordered ringoids, semirings, algebroids and non-associative algebras can be used for studies of their structures and representations.

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