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Article

Characteristic Variety of the Gauss–Manin Differential Equations of a Generic Parallelly Translated Arrangement

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Abstract: We consider a weighted family of n generic parallelly translated hyperplanes in $\mathbb{C}^k$ and describe the characteristic variety of the Gauss–Manin differential equations for associated hypergeometric integrals. The characteristic variety is given as the zero set of Laurent polynomials, whose coefficients are determined by weights and the Plücker coordinates of the associated point in the Grassmannian $\mathrm{Gr}(k, n)$. The Laurent polynomials are in involution.

Keywords: Master function; Lagrangian variety; Characteristic variety; Bethe ansatz

1. Introduction

There are three places where a flat connection depending on a parameter appears:

- KZ equations, $\kappa\frac{\partial I}{\partial z_i}(z) = K_i(z)I(z)$, $z = (z_1, \dots, z_n)$, $i = 1, \dots, n$. Here κ is a parameter, $I(z)$ a V-valued function, where V is a vector space from representation theory, $K_i(z) : V \to V$ are linear operators, depending on z. The connection is flat for all κ, see for example [1,2].
- Differential equations for hypergeometric integrals associated with a family of weighted arrangements with parallelly translated hyperplanes, $\kappa\frac{\partial I}{\partial z_i}(z) = K_i(z)I(z)$, $z = (z_1, \dots, z_n)$, $i = 1, \dots, n$. The connection is flat for all κ, see for example [3,4].
- Quantum differential equations, $\kappa\frac{\partial I}{\partial z_i}(z) = p_i *_z I(z)$, $z = (z_1, \dots, z_n)$, $i = 1, \dots, n$. Here $p_1, \dots, p_n$ are generators of some commutative algebra H with quantum multiplication $*_z$ depending on z. The connection is flat for all κ. These equations are part of the Frobenius structure on the quantum cohomology of a variety, see [5,6].

If $\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z)I(z)$, $i = 1, \ldots, n$, is a system of V-valued differential equations of one of these types, then its characteristic variety is

$$\mathrm{Spec} = \{(z, p) \in T^*\mathbb{C}^n \mid \exists v \in V \text{ with } K_j(z)v = p_j v,\ j = 1, \ldots, n\}$$

It is known that the characteristic varieties of the first two types of differential equation are interesting. For example, the characteristic variety of the quantum differential equation of the flag variety is the zero set of the Hamiltonians of the classical Toda lattice, according to [7,8], and the characteristic variety of the $\mathfrak{gl}_N$ KZ equations with values in the tensor power of the vector representation is the zero set of the Hamiltonians of the classical Calogero–Moser system, according to [9].

In this paper we describe the characteristic variety of the Gauss–Manin differential equations for hypergeometric integrals associated with a weighted family of n generic parallelly translated hyperplanes in $\mathbb{C}^k$. The characteristic variety is given as the zero set of Laurent polynomials, whose coefficients are determined by weights and the Plücker coordinates of the associated point in the Grassmannian $\mathrm{Gr}(k, n)$. The Laurent polynomials are in involution.

It is known that the KZ differential equations can be identified with Gauss–Manin differential equations of certain weighted families of parallelly translated hyperplanes, see [10], and that some quantum differential equations can be identified with Gauss–Manin differential equations of certain weighted families of parallelly translated hyperplanes, see [11]. Therefore, the results in this paper on the characteristic variety of the Gauss–Manin differential equations associated with a family of generic parallelly translated hyperplanes can be considered as a first step to studying characteristic varieties of more general KZ and quantum differential equations that admit integral hypergeometric representations.

The Laurent polynomials, defining our characteristic variety, are regular functions of the Plücker coordinates of the associated point in $\mathrm{Gr}(k, n)$. Therefore they can be used to study the characteristic varieties of more general Gauss–Manin differential equations for multidimensional hypergeometric integrals.

Our description of the characteristic variety is based on the fact, proved in [12], that the characteristic variety of the Gauss–Manin differential equations is generated by the master function of the corresponding hypergeometric integrals, that is, the characteristic variety coincides with the Lagrangian variety of the master function. That fact is a generalization of Theorem 5.5 in [13], proved with the help of the Bethe ansatz, that the local algebra of a critical point of the master function associated with a $\mathfrak{gl}_N$ KZ equation can be identified with a suitable local Bethe algebra of the corresponding $\mathfrak{gl}_N$ module.

In Section 2, we consider the algebra of functions on the critical set of the master function and describe it by generators and relations.

In Section 3, we show that these relations give us equations defining the Lagrangian variety of the master function. We show that the corresponding functions are in involution. We define coordinate systems $(z_I, p_{\bar{I}})$ on the Lagrange variety and for each of them a function $\Phi(z_I, p_{\bar{I}})$ also generating the Lagrangian variety. We describe the Hessian of the master function lifted to the Lagrangian variety and relate it to the Jacobian of the projection of the Lagrangian variety to the base of the family.

In Section 4, we remind the identification from [12] of the Lagrangian variety of the master function and the characteristic variety of the Gauss–Manin differential equations.

2. Algebra of Functions on the Critical Set

2.1. An Arrangement in $\mathbb{C}^n \times \mathbb{C}^k$

Let $n > k$ be positive integers. Denote $J = \{1, \ldots, n\}$. Consider $\mathbb{C}^k$ with coordinates $t_1, \ldots, t_k$, $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. Fix n linear functions on $\mathbb{C}^k$, $g_j = \sum_{m=1}^k b_j^m t_m$, $j \in J$, $b_j^m \in \mathbb{C}$. For $i_1, \ldots, i_k \subset J$, denote $d_{i_1,\ldots,i_k} = \det_{\ell,m=1}^k (b_{i_\ell}^m)$. We assume that all the numbers $d_{i_1,\ldots,i_k}$ are nonzero if $i_1, \ldots, i_k$ are distinct. In other words, we assume that the collection of functions $g_j, j \in J$, is generic. We define n linear functions on $\mathbb{C}^n \times \mathbb{C}^k$, $f_j = z_j + g_j$, $j \in J$. We define the arrangement of hyperplanes $\tilde{\mathcal{C}} = \{\tilde{H}_j \mid j \in J\}$ in $\mathbb{C}^n \times \mathbb{C}^k$, where $\tilde{H}_j$ is the zero set of f_j. Denote by $U(\tilde{\mathcal{C}}) = \mathbb{C}^n \times \mathbb{C}^k - \cup_{j\in J}\tilde{H}_j$ the complement.

For every $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, the arrangement $\tilde{\mathcal{C}}$ induces an arrangement $\mathcal{C}(z)$ in the fiber over z of the projection $\pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. We identify every fiber with $\mathbb{C}^k$. Then $\mathcal{C}(z)$ consists of hyperplanes $H_j(z), j \in J$, defined in $\mathbb{C}^k$ by the equations $f_j = 0$. Denote by $U(\mathcal{C}(z)) = \mathbb{C}^k - \cup_{j\in J} H_j(z)$ the complement.

The arrangement $\mathcal{C}(z)$ is with normal crossings if and only if $z \in \mathbb{C}^n - \Delta$,

$$\Delta = \cup_{\{i_1<\cdots<i_{k+1}\}\subset J} H_{i_1,\ldots,i_{k+1}} \tag{1}$$

where $H_{i_1,\ldots,i_{k+1}}$ is the hyperplane in $\mathbb{C}^n$ defined by the equation $f_{i_1,\ldots,i_{k+1}}(z) = 0$,

$$f_{i_1,\ldots,i_{k+1}}(z) = \sum_{m=1}^{k+1} (-1)^{m-1} d_{i_1,\ldots,\widehat{i_m},\ldots,i_{k+1}} z_{i_m} \tag{2}$$

We have the following identify

$$\sum_{m=1}^{k+1} (-1)^{m-1} d_{i_1,\ldots,\widehat{i_m},\ldots,i_{k+1}} (z_{i_m} - f_{i_m}(z,t)) = 0 \tag{3}$$

Lemma 2.1. *Consider the $\mathbb{C}$-span S of the linear functions $f_{i_1,\ldots,i_{k+1}}$, where $\{i_1, \ldots, i_{k+1}\}$ runs through all $k+1$-element subsets of J. Then $\dim S = n - k$.*

Proof. The dimension of S equals the codimension in $\mathbb{C}^n$ of $X_1 = \{z \in \mathbb{C}^n \mid f_I(z) = 0 \text{ for all } I\}$. The subspace X_1 is the image of the subspace $X_2 = \{(z,t) \in \mathbb{C}^n \times \mathbb{C}^k \mid f_j(z,t) = 0 \text{ for all } j \in J\}$ under the projection $\pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. Clearly the subspace X_2 is k-dimensional and the projection $\pi|_{X_2} : X_2 \to X_1$ is an isomorphism. Hence $\dim X_1 = k$ and $\dim S = n - k$. □

2.2. Plücker Coordinates

The matrix (b_j^m) is an $n\times k$-matrix of rank k. The matrix defines a point in the Grassmannian $\mathrm{Gr}(k, n)$ of k-planes in $\mathbb{C}^n$. The numbers $d_{i_1,\ldots,i_k}$ are Plücker coordinates of this point. Most of objects in this paper are determined in terms of these Plücker coordinates. We will use the following Plücker relation.

Lemma 2.2. *For arbitrary sequences $j_1, \ldots, j_{k+1}$ and $i_1, \ldots, i_{k-1}$ in J, we have*

$$\sum_{m=1}^{k+1} (-1)^{m-1} d_{j_1,\ldots,\widehat{j_m},\ldots,j_{k+1}} d_{j_m,i_1\ldots,i_{k-1}} = 0 \tag{4}$$

See this statement, for example, in [14].

2.3. Algebra $A_\Phi(z)$

Assume that nonzero weights $(a_j)_{j\in J} \subset \mathbb{C}^\times$ are given. Denote $|a| = \sum_{j\in J} a_j$. Assume that $|a| \neq 0$.

Each arrangement $\mathcal{C}(z)$ is weighted, meaning that to every hyperplane $H_j(z), j \in J$, we assign weight a_j. The master function of the weighted arrangement $\mathcal{C}(z)$ in $\mathbb{C}^k$ is the function

$$\Phi(z,t) = \sum_{j\in J} a_j \log f_j(z,t) \tag{5}$$

The critical point equations are

$$\partial\Phi/\partial t_i = \sum_{j\in J} b_j^i a_j/f_j = 0, \qquad i = 1,\dots,k \tag{6}$$

We have

$$\partial\Phi/\partial z_j = a_j/f_j, \qquad i \in J \tag{7}$$

Denote by $\mathcal{I}(z) \subset \mathcal{O}(U(\mathcal{C}(z)))$ the ideal generated by the functions $\partial\Phi/\partial t_j$, $j \in J$. The algebra of functions on the critical set is

$$A_\Phi(z) = \mathcal{O}(U(\mathcal{C}(z)))/\mathcal{I}(z) \tag{8}$$

For a function $g \in \mathcal{O}(U(\mathcal{C}(z)))$, denote by $[g]$ its projection to $A_\Phi(z)$. Denote

$$p_j = [a_j/f_j], \qquad j \in J$$

We introduce the following polynomials in $z_1,\dots,z_n,p_1,\dots,p_n$. For every subset $I = \{i_1,\dots,i_{k-1}\}$ of distinct elements in J, we set

$$F_I(p_1,\dots,p_n) = \sum_{j\in J} d_{j,i_1,\dots,i_{k-1}} p_j \tag{9}$$

For every subset $I = \{i_1,\dots,i_{k+1}\}$ of distinct elements in J, we set

$$F_I(z_1,\dots,z_n,p_1,\dots,p_n) = \\ p_{i_1}\dots p_{i_{k+1}} f_{i_1,i_2,\dots,i_{k+1}}(z) + \sum_{m=1}^{k+1} (-1)^m a_{i_m} d_{i_1,\dots,\widehat{i_m},\dots,i_{k+1}} p_{i_1}\dots\widehat{p_{i_m}}\dots p_{i_{k+1}} \tag{10}$$

The following lemma collects the properties of the elements $p_1,\dots,p_n$.

Lemma 2.3. *Let $z \in \mathbb{C}^n - \Delta$.*

(i) The elements $p_j, j \in J$, generate the algebra $A_\Phi(z)$.

(ii) For every subset $I = \{i_1,\dots,i_{k-1}\}$ of distinct elements in J, we have

$$F_I(p_1,\dots,p_n) = 0 \tag{11}$$

Relation Equation (11) will be called the I-relation of first kind.

(iii) For every subset $I = \{i_1, \dots, i_{k+1}\}$ of distinct elements in J, we have

$$F_I(z_1, \dots, z_n, p_1, \dots, p_n) = 0 \tag{12}$$

Relation Equation (12) will be called the I-relation of second kind.

(iv) In $A_\Phi(z)$, we have

$$1 = \frac{1}{|a|}\sum_{j\in J} z_j p_j \tag{13}$$

(v) We have $\dim A_\Phi(z) = \binom{n-1}{k}$, and for any $j_1 \in J$, the set of monomials $p_{i_1}\dots p_{i_k}$, with $i_1 < \dots < i_k$ and $j_1 \notin \{i_1, \dots, i_k\}$, is a $\mathbb{C}$-basis of $A_\Phi(z)$.

Part (i) is Lemma 2.5 in [12]. Parts (ii), (iii), (iv) are Lemmas 6.7, 6.8, 2.5 in [15], respectively. The first statement of part (v) is ([12], Lemma 4.2) that follows from ([15], Lemma 6.5). The second statement of part (v) is Theorem 6.11 in [15].

Note that the polynomials F_I in Equations (11) and (12) are homogeneous if we put

$$\deg p_j = 1, \qquad \deg z_j = -1 \qquad \text{for all } j \tag{14}$$

2.4. Relations of Second Kind

For $j \in J$, denote

$$G_j(z_j, p_j) = z_j - a_j/p_j \tag{15}$$

Then the projection to $A_\Phi(z)$ of the left hand side of Equation (3) can be written as

$$\begin{aligned} G_I(z,p) &= \sum_{m=1}^{k+1}(-1)^{m-1} d_{i_1,\dots,\widehat{i_m},\dots,i_{k+1}} G_{i_m}(z_{i_m}, p_{i_m}) \\ &= \sum_{m=1}^{k+1}(-1)^{m-1} d_{i_1,\dots,\widehat{i_m},\dots,i_{k+1}} \left(z_{i_m} - \frac{a_{i_m}}{p_{i_m}}\right) \end{aligned} \tag{16}$$

where $I = \{i_1, \dots, i_{k+1}\}$. Hence in $A_\Phi(z)$ we have

$$G_I(z,p) = 0 \tag{17}$$

Notice that $F_I(z,p) = p_{i_1}\dots p_{i_{k+1}} G_I(z,p)$ and the functions p_j are nonzero at every point of the critical set of the master function.

2.5. New Presentation for $A_\Phi(z)$

Fix $z \in \mathbb{C}^n - \Delta$. Consider $(\mathbb{C}^\times)^n$ with coordinates $p_1, \ldots, p_n$. Consider the polynomials $F_I(p)$ in Equation (11) and polynomials $F_I(z,p)$ in Equation (12) as elements of $\mathcal{O}((\mathbb{C}^\times)^n)$. Let $\tilde{\mathcal{I}}(z) \subset \mathcal{O}((\mathbb{C}^\times)^n)$ be the ideal generated by all F_I with $|I| = k-1,\ k+1$.

Notice that all polynomials $F_I(p)$, $|I| = k-1$, in Equation (11) and all functions $G_I(z,p)$, $|I| = k+1$, in Equation (16) also generate $\tilde{\mathcal{I}}(z)$.

Let $\tilde{A}(z) = \mathcal{O}((\mathbb{C}^\times)^n)/\tilde{\mathcal{I}}(z)$ be the quotient algebra.

Theorem 2.4. *The natural homomorphism $\tilde{A}(z) \to A_\Phi(z)$, $p_j \mapsto [a_j/f_j]$, is an isomorphism.*

Example. If $k = 1$ and $f_j = t_1 + z_j$, then the ideal $\mathcal{I}(z)$ is generated by the function $\sum_{j\in J} a_j/(t_1+z_j)$, while the ideal $\tilde{\mathcal{I}}(z)$ is generated by the functions

$$p_1 + \cdots + p_n, \qquad (z_i - z_j)p_ip_j - a_ip_j + a_jp_i, \quad 1 \leqslant i < j \leqslant n$$

or by the functions

$$p_1 + \cdots + p_n, \qquad (z_i - a_i/p_i) - (z_j - a_j/p_j), \quad 1 \leqslant i < j \leqslant n$$

2.6. Proof of Theorem 2.4

Lemma 2.5. *Let $I = \{i_1, \ldots, i_k\}$ be a subset of distinct elements. Then in $\tilde{A}(z)$, we have*

$$\sum_{j\in J} z_jp_j = \frac{1}{d_{i_1,\ldots,i_k}} \sum_{j\in J-I} f_{j,i_1,\ldots,i_k}(z)\, p_j \tag{18}$$

Proof. The statement easily follows from Equation (11), that is, from relations of first kind. For example, if $k = 2$ and $I = \{1,2\}$, then the two relations of first kind $p_1 = \frac{1}{d_{2,1}}\sum_{j>2} d_{j,2}p_j$ and $p_2 = \frac{1}{d_{1,2}}\sum_{j>2} d_{j,1}p_j$ transform $\sum_{j\in J} z_jp_j$ to $\frac{1}{d_{1,2}}\sum_{j>2} f_{1,2,j}(z)p_j$. □

Lemma 2.6. *In $\tilde{A}(z)$, we have $1 = \frac{1}{|a|}\sum_{j\in J} z_jp_j$.*

Proof. We have

$$\begin{aligned} p_1 \ldots p_k \sum_{j\in J} z_jp_j &= p_1 \ldots p_k \frac{1}{d_{1,\ldots,k}} \sum_{j>k} f_{j,1,\ldots,k}(z)p_j \\ &= \sum_{j>k} \Big[a_jp_1 \ldots p_k + \sum_{m=1}^{k} (-1)^m a_m \frac{d_{j,1,\ldots,\widehat{m},\ldots,k}}{d_{1,\ldots,k}} p_jp_1 \ldots \widehat{p_m} \ldots p_k\Big] = |a|\, p_1 \ldots p_k \end{aligned}$$

where the first equality follows from Lemma 2.5, the second equality follows from the relations of second kind, and the third equality follows from the relations of first kind. Denote by $C(z) \subset (\mathbb{C}^\times)^n$ the zero set of the ideal $\tilde{\mathcal{I}}(z)$. Then the function $p_1 \ldots p_k$ is nonvanishing on $C(z)$. The previous calculation shows that the multiplication of the invertible function $p_1 \ldots p_k$ by $\frac{1}{|a|}\sum_{j\in J} z_jp_j$ does not change the invertible function. This gives the lemma. □

Lemma 2.7. *Let $s \leqslant k$ be a natural number and $M = \prod_{j\in J} p_j^{s_j}$, $\sum_{j\in J} s_j = s$, a monomial of degree s. Let $J_{k-s+1} = \{j_1, \dots, j_{k-s+1}\}$ be any subset in J with distinct elements. Then by using the relations of first kind only, the monomial M can be represented as a $\mathbb{C}$-linear combination of monomials $p_{i_1} \dots p_{i_s}$ with $1 \leqslant i_1 < \dots < i_s \leqslant n$ and $\{i_1, \dots, i_s\} \cap J_{k-s+1} = \emptyset$.* □

C.f. the proof of Lemma 6.9 in [15].

Lemma 2.8. *Let $s \leqslant k$ be a natural number and $M = \prod_{j\in J} p_j^{s_j}$ a monomial of degree s. Fix an element $j_1 \in J$. Then by using the relations of first kind and the relation $1 = \frac{1}{|a|}\sum_{j\in J} z_j p_j$ only, the monomial M can be represented as a linear combination of monomials $p_{i_1} \dots p_{i_k}$ with $1 \leqslant i_1 < \dots < i_k \leqslant n$ and $j_1 \notin \{i_1, \dots, i_s\}$, where the coefficients of the linear combination are homogeneous polynomials in z of degree $s - k$.* □

Recall the $\deg z_j = -1$ for all $j \in J$.

Lemma 2.9. *Let $s > k$ be a natural number and $M = \prod_{j\in J} p_j^{s_j}$ a monomial of degree s. Then by using the relations of first and second kinds, the monomial M can be represented as a linear combination of monomials $p_{i_1} \dots p_{i_k}$ of degree k, where the coefficients of the linear combination are rational functions in z, regular on $\mathbb{C}^n - \Delta$ and homogeneous of degree $s - k$.* □

Let us finish the proof of Theorem 2.4. Let $P(p_1, \dots, p_n)$ be a polynomial. Fix $j_1 \in J$. By using the relations of first and second kinds only, the polynomial can be represented as a linear combination $\tilde{P}$ of monomials $p_{i_1} \dots p_{i_k}$ with $1 \leqslant i_1 < \dots < i_k \leqslant n$ and $j_1 \notin \{i_1, \dots, i_s\}$, see Lemmas 2.7–2.9. Assume that $P(p_1, \dots, p_n)$ projects to zero in $A_\Phi(z)$, then all coefficients of that linear combination $\tilde{P}$ must be zero, see part (v) of Lemma 2.3. This means that P lies in the ideal $\tilde{\mathcal{I}}(z)$. Theorem 2.4 is proved.

3. Lagrangian Variety of the Master Function

3.1. Critical Set Recall the projection $\pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. For any $z \in \mathbb{C}^n - \Delta$, the arrangement $\mathcal{C}(z)$ in $\pi^{-1}(z)$ has normal crossings. Recall the complement $U(\tilde{\mathcal{C}}) \subset \mathbb{C}^n \times \mathbb{C}^k$ to the arrangement $\tilde{\mathcal{C}}$ in $\mathbb{C}^n \times \mathbb{C}^k$. Denote

$$U^0 = U(\tilde{\mathcal{C}}) \cap \pi^{-1}(\mathbb{C}^n - \Delta) \subset \mathbb{C}^n \times \mathbb{C}^k \tag{19}$$

Consider the master function $\Phi(z, t)$, defined in Equation (5), as a function on U^0. Denote by C_Φ the critical set of Φ with respect to variables t,

$$C_\Phi = \{(z, t) \in U^0 \mid \partial\Phi/\partial t_i(z, t) = 0,\ i = 1, \dots, k\} \tag{20}$$

Lemma 3.1. *The set C_Φ is a smooth n-dimensional subvariety of U^0.*

Proof. For any subset $I = \{1 \leqslant i_1 < \dots < i_k \leqslant n\} \subset J$, the $k \times k$-determinant

$$\det{}_{l,m=1}^{k} \left(\frac{\partial^2\Phi}{\partial t_l \partial z_{j_m}}\right) = -d_{i_1,\dots,i_k} \prod_{m=1}^{k} \frac{a_{j_m}}{f_{j_m}^2(z, t)}$$

is nonzero on U^0. □

Denote by $\mathcal{I} \subset \mathcal{O}(U^0)$ the ideal generated by the functions $\partial\Phi/\partial t_j$, $j \in J$. The algebra of functions on C_Φ is the quotient algebra

$$A_\Phi = \mathcal{O}(U^0)/\mathcal{I} \tag{21}$$

Consider $(\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n$ with coordinates $z_1, \ldots, z_n, p_1, \ldots, p_n$. Consider the polynomials $F_I(p)$ in Equation (11) and polynomials $F_I(z,p)$ in Equation (12) as elements of $\mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n)$. Let $\tilde{\mathcal{I}} \subset \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n)$ be the ideal generated by all F_I with $|I| = k-1,\ k+1$. Notice that all polynomials $F_I(p)$, $|I| = k-1$, in Equation (11) and all functions $G_I(z,p)$, $|I| = k+1$, in Equation (16) also generate $\tilde{\mathcal{I}}(z)$. Let

$$\tilde{A} = \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n)/\tilde{\mathcal{I}} \tag{22}$$

be the quotient algebra.

Theorem 3.2. *The natural homomorphism $\tilde{A} \to A_\Phi$, $p_j \mapsto [a_j/f_j]$, is an isomorphism.*

The proof is the same as the proof of Theorem 2.4.

3.2. Lagrangian Variety Consider the cotangent bundle $T^*(\mathbb{C}^n - \Delta)$ with dual coordinates $z_1, \ldots, z_n$, $p_1, \ldots, p_n$ with respect to the standard symplectic form $\omega = \sum_{j=1}^n dp_j \wedge dz_j$. Consider the open subset $(\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n \subset T^*(\mathbb{C}^n - \Delta)$ of all points with nonzero coordinates $p_1, \ldots, p_n$. Consider the map

$$\varphi : C_\Phi \ \to \ (\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n,\ (z,t) \mapsto \Big(z_1, \ldots, z_n, p_1 = \frac{\partial\Phi}{\partial z_1}(z,t), \ldots, p_n = \frac{\partial\Phi}{\partial z_n}(z,t) \Big)$$

Denote by Λ the image $\varphi(C_\Phi)$ of the critical set. The set Λ is invariant with respect to the action of $\mathbb{C}^\times$, which multiplies all coordinates p_j and divides all coordinates z_j by the same number. Denote by $\hat{\mathcal{I}} \subset \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n)$ the ideal of functions that equal zero on Λ.

Theorem 3.3. *The ideal $\tilde{\mathcal{I}} \subset \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n)$ coincides with the ideal $\hat{\mathcal{I}}$. The subset $\Lambda \subset (\mathbb{C}^n - \Delta) \times (\mathbb{C}^\times)^n$ is a smooth Lagrangian subvariety.*

Proof. It is clear that $\tilde{\mathcal{I}} \subset \hat{\mathcal{I}}$. The proof of the inclusion $\hat{\mathcal{I}} \subset \tilde{\mathcal{I}}$ is basically the same as the proof of Theorem 2.4. This gives the first statement of the theorem.

It is clear that $\dim \Lambda = n$. To prove that Λ is smooth, it is enough to show that at any point of Λ, the span of the differentials of the functions $F_I(p), |I| = k-1$, and $G_I(z,p), |I| = k+1$ is at least n-dimensional. By Lemma 2.1, the span of the z-parts of the differentials of the functions $G_I(z,p)$, $I = |I| = k+1$, is $n-k$-dimensional. It is easy to see that the span of the differentials of the functions $F_I(p)$, $I = |I| = k+1$, is at least k-dimensional (c.f. the example in the proof of Lemma 2.5). Hence Λ is smooth.

By the definition of φ, the set Λ is isotropic. Hence Λ is Lagrangian. □

Let $I = \{i_1, \ldots, i_k\} \subset J$ be a k-element subset and $\bar{I}$ its complement. Then the functions $z_I = \{z_i \,|\, i \in I\}$, $p_{\bar{I}} = \{p_j \,|\, j \in \bar{I}\}$, form a system of coordinates on Λ. Indeed, we have

$$\begin{aligned} p_{i_m} &= -\frac{1}{d_{i_m,i_1,\ldots,\widehat{i_m},\ldots,i_k}} \sum_{j\in\bar{I}} d_{j,i_1,\ldots,\widehat{i_m},\ldots,i_k} p_j, \qquad m = 1, \ldots, k \\ z_j &= \frac{a_j}{p_j} + \frac{1}{d_{i_1,\ldots,i_k}} \sum_{m=1}^{k} (-1)^{k-m} d_{j,i_1,\ldots,\widehat{i_m},\ldots,i_k} \Big(z_{i_m} - \frac{a_{i_m}}{p_{i_m}} \Big), \qquad j \in \bar{I} \end{aligned} \tag{23}$$

where in the second line the functions p_{i_m} must be expressed in terms of the functions $p_j, j \in \bar{I}$, by using the first line.

We order the functions of the coordinate system $z_I, p_{\bar{I}}$ according to the increase of the low index. For example, if $k = 3, n = 6, I = \{1, 3, 6\}$, then the order is $z_1, p_2, z_3, p_4, p_5, z_6$.

Lemma 3.4. *Let $I = \{i_1, \dots, i_k\}$ and $I' = \{i'_1, \dots, i'_k\}$ be two k-element subsets of J. Consider the corresponding ordered coordinate systems z_I, $p_{\bar{I}}$ and $z_{I'}$, $p_{\bar{I}'}$. Express the coordinates of the second system in terms of the coordinates of the first system and denote by $\mathrm{Jac}_{I,\bar{I}'}(z_I, p_{\bar{I}})$ the Jacobian of this change. Then*

$$\mathrm{Jac}_{I,\bar{I}'}(z_I, p_{\bar{I}}) = (d_{i'_1,\dots,i'_k}/d_{i_1,\dots,i_k})^2$$

Proof. It is enough to check this formula for the case $I = \{1, 3, \dots, k+1\}$ and $I' = \{2, 3, \dots, k+1\}$. Then

$$p_1 = -\frac{d_{2,3,\dots,k+1}}{d_{1,3,\dots,k+1}} p_2 + \dots, \qquad z_2 = \frac{a_2}{p_2} + \frac{d_{2,3,\dots,k+1}}{d_{1,3,\dots,k+1}} z_1 + \dots$$

where the first dots denote the terms that do not depend on z_1, p_2 and the second dots denote the terms that do not depend on z_1. According to these formulas the 2×2 Jacobian of the dependence of p_1, z_2 on z_1, p_2 equals $(d_{2,3,\dots,k+1}/d_{1,3,\dots,k+1})^2$ and hence $\mathrm{Jac}_{I,\bar{I}'}(z_I, p_{\bar{I}}) = (d_{2,3,\dots,k+1}/d_{1,3,\dots,k+1})^2$. □

3.3. Generating Functions

Consider the function

$$\Psi = \sum_{j \in J} a_j \ln p_j - \sum_{i \in I} z_i p_i \tag{24}$$

of $n + k$ variables $z_j, j \in I,\ \ p_j, j \in J$. Express in Ψ the variables $p_i, i \in I$, according to Equation (23). Denote by $\Psi(z_I, p_{\bar{I}})$ the resulting function of variables $z_I, p_{\bar{I}}$.

Theorem 3.5. *The function $\Psi(z_I, p_{\bar{I}})$ is a generating function of the Lagrangian variety Λ. Namely, Λ lies in the image of the map*

$$(z_I, p_{\bar{I}}) \ \mapsto \ \Big(z_I, z_{\bar{I}} = \frac{\partial \Psi_I}{\partial p_{\bar{I}}}(z_I, p_{\bar{I}}), p_I = -\frac{\partial \Psi_I}{\partial z_I}(z_I, p_{\bar{I}}), p_{\bar{I}}\Big) \tag{25}$$

Proof. The proof that these formulas give Equations (23) is by straightforward verification. □

3.4. Integrals in Involution

Consider the standard Poisson bracket on $T^*(\mathbb{C}^n)$,

$$\{M, N\} = \sum_{j=1}^{n} \Big(\frac{\partial M}{\partial z_j}\frac{\partial N}{\partial p_j} - \frac{\partial M}{\partial p_j}\frac{\partial N}{\partial z_j}\Big)$$

for $M, N \in \mathcal{O}(T^*(\mathbb{C}^n))$. The functions are in involution if $\{M, N\} = 0$.

Theorem 3.6. *All functions $F_I(p)$, $|I| = k-1$, and $G_I(z,p)$, $|I| = k+1$, are in involution.*

Proof. Clearly, $\{F_I, F_{I'}\} = 0$, since $F_I, F_{I'}$ depend on z only. If $I = \{j_1,\dots,j_{k+1}\}$ and $I' = \{i_1,\dots,i_{k-1}\}$, then

$$\{G_I, F_{I'}\} = \sum_{m=1}^{k+1} (-1)^{m-1} d_{j_1,\dots,\widehat{j_m},\dots,j_{k+1}} d_{j_m,i_1\dots,i_{k-1}} = 0$$

due to the Plücker relation (4).

Recall the function $G_j(z_j, p_j)$ in Equation (15). It is clear that $\{G_j, G_{j'}\} = 0$ for all $j, j' \in J$. Now $\{G_I, G_{I'}\} = 0$ for all I, I' with $|I| = |I'| = k+1$, since $G_I, G_{I'}$ are linear combination of G_j with constant coefficients. □

All the functions F_I, G_I define commuting Hamiltonian flows, preserving Λ and giving symmetries of Λ. For $I = \{i_1,\dots,i_{k-1}\}$, the flow φ_I^t of the function $F_I(p)$ has the form

$$(z_1,\dots,z_n,p) \mapsto (z_1 + d_{1,i_1,\dots,i_{k-1}}t,\dots,z_n + d_{n,i_1,\dots,i_{k-1}}t, p)$$

For $I = \{j_1,\dots,j_{k+1}\}$, the flow φ_I^t of the function $G_I(z,p)$ does not change the pair of coordinate (z_j, p_j) of a point, if $j \notin I$, and transforms the pair (z_{j_m}, p_{j_m}) to the pair

$$(z_{j_m} - \frac{a_{j_m}}{p_{j_m}} + \frac{a_{j_m}}{p_{j_m} + (-1)^m d_{j_1,\dots,\widehat{j_m},\dots,i_{k+1}} t},\ p_{j_m} + (-1)^m d_{j_1,\dots,\widehat{j_m},\dots,i_{k+1}} t)$$

for $m = 1,\dots,k+1$.

Remark. An interesting property of the Hamiltonians F_I, G_I is that they are regular with respect the Plücker coordinates $d_{i_1,\dots,i_k}$. Hence, they can be used to study the Lagrange varieties of the arrangements in $\mathbb{C}^n \times \mathbb{C}^k$ associated with not necessarily generic matrices (b_j^i).

3.5. Hessian as a Function on the Lagrange Variety

Let $z \in \mathbb{C}^n - \Delta$ and let t^0 be a critical point of the master function $\Phi(z,\,\cdot\,)$. An important characteristic of the critical point is the Hessian

$$\text{Hess } \Phi(z,t^0) = \det_{i,j=1}^{k} \left(\frac{\partial^2 \Phi}{\partial t_i \partial t_j}(z,t^0) \right)$$

see, for example, [2,16–18].

For a subset $I = \{i_1,\dots,i_k\} \subset J$, we denote by d_I^2 the number $(d_{i_1,\dots,i_k})^2$.

Lemma 3.7. *We have*

$$\text{Hess } \Phi \ = \ (-1)^k \sum_{I \subset J, |I|=k} d_I^2 \prod_{i \in I}^{k} \frac{p_i^2}{a_i} \tag{26}$$

Proof. In [18], the formula Hess $\Phi = (-1)^k \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} d_{i_1,\dots,i_k}^2 \prod_{m=1}^{k} a_{i_m}/f_{i_m}^2$ is given, which is the right hand side of Equation (26). The formula itself is obvious. □

3.6. Hessian and Jacobian Let $M = \{m_1, \dots, m_k\} \subset J$ be a k-element subset and $z_M, p_{\bar{M}}$ the corresponding ordered coordinate system on Λ. The functions $z_1, \dots, z_n$ form an ordered coordinate system on $\mathbb{C}^n - \Delta$. Consider the projection $\Lambda \mapsto \mathbb{C}^n - \Delta$, $(z, p) \mapsto z$, and the Jacobian $\mathrm{Jac}_M(z_M, p_{\bar{M}})$ of the projection with respect to the chosen coordinate systems.

Theorem 3.8. *As a function on* Λ, *the function* $d_M^2 \,\mathrm{Jac}_M$ *does not depend on* M *and*

$$d_M^2 \,\mathrm{Jac}_M = (-1)^{n-k} \sum_{L \subset J, \, |L| = n-k} d_{\bar{L}}^2 \prod_{j \in L} \frac{a_j}{p_j^2} \tag{27}$$

Proof. The function $d_M^2 \,\mathrm{Jac}_M$ does not depend on M by Lemma 3.4.

Consider the function $\tilde{\Psi} = \sum_{j \in J} a_j \ln p_j$ of n variables p_j. Express in $\tilde{\Psi}$ the variables p_M in terms of variables $p_{\bar{M}}$ by formulas Equation (23). Denote by $\tilde{\Psi}_M(p_{\bar{M}})$ the resulting function. By Theorem 3.5, $\mathrm{Jac}_M = \det\left(\frac{\partial^2 \tilde{\Psi}_M}{\partial p_{\bar{M}} \partial p_{\bar{M}}}\right)$. This implies that $d_M^2 \,\mathrm{Jac}_M$ is a polynomial in $a_j, j \in J$, of the form

$$d_M^2 \,\mathrm{Jac}_M = \sum_{L \subset J, \, |L| = n-k} c_L \prod_{j \in L} \frac{a_j}{p_j^2}$$

where c_L are numbers independent of M. Our goal is to show that $c_L = (-1)^{n-k} d_{\bar{L}}^2$ but this is clear for $L = M$. This proves the theorem. □

Corollary 3.9. *We have*

$$d_M^2 \,\mathrm{Jac}_M = (-1)^n \mathrm{Hess}\, \Phi \prod_{j \in J} \frac{a_j}{p_j^2} \tag{28}$$

4. Characteristic Variety of the Gauss–Manin Differential Equations

4.1. Space Sing V

Consider the complex vector space V generated by vectors $v_{i_1, \dots, i_k}$ with $i_1, \dots, i_k \in J$ subject to the relations $v_{i_{\sigma(1)}, \dots, i_{\sigma(k)}} = (-1)^{\sigma} v_{i_1, \dots, i_k}$ for any $i_1, \dots, i_k \in J$ and $\sigma \in S_k$. The vectors $v_{i_1, \dots, i_k}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n$ form a basis of V. If $v = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} c_{i_1, \dots, i_k} v_{i_1, \dots, i_k}$ is a vector of V, we introduce the numbers $c_{i_1, \dots, i_k}$ for all $i_1, \dots, i_k \in J$ by the rule: $c_{i_{\sigma(1)}, \dots, i_{\sigma(k)}} = (-1)^{\sigma} c_{i_1, \dots, i_k}$. We introduce the subspace $\mathrm{Sing}\, V \subset V$ of singular vectors by the formula

$$\mathrm{Sing}\, V = \left\{ \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} c_{i_1, \dots, i_k} v_{i_1, \dots, i_k} \mid \sum_{j \in J} a_j \, c_{j, j_1, \dots, j_{k-1}} = 0 \text{ for all } \{j_1, \dots, j_{k-1}\} \subset J \right\}$$

The symmetric bilinear contravariant form on V is defined by the formulas: $S(v_{i_1, \dots, i_k}, v_{j_1, \dots, j_k}) = 0$, if $\{i_1, \dots, i_k\} \neq \{i_1, \dots, i_k\}$, and $S(v_{i_1, \dots, i_k}, v_{i_1, \dots, i_k}) = \prod_{m=1}^k a_{i_m}$, if $i_1, \dots, i_k$ are distinct. Denote by $s^{\perp} : V \to \mathrm{Sing}\, V$ the orthogonal projection with respect to the contravariant form.

4.2. Differential Equations

Consider the master function $\Phi(z,t)$ as a function on $U^0 \subset \mathbb{C}^n \times \mathbb{C}^k$. Let κ be a nonzero complex number. The function $e^{\Phi(z,t)/\kappa}$ defines a rank one local system $\mathcal{L}_\kappa$ on U^0 whose horizontal sections over open subsets of $\tilde{U}$ are univalued branches of $e^{\Phi(z,t)/\kappa}$ multiplied by complex numbers. The vector bundle

$$\cup_{z\in\mathbb{C}^n-\Delta} H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))}) \ \to \ \mathbb{C}^n - \Delta$$

has the canonical flat Gauss–Manin connection. For a horizontal section $\gamma(z) \in H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))})$, consider the V-valued function

$$I_\gamma(z) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \Big(\int_{\gamma(z)} e^{\Phi(z,t)/\kappa} d\ln f_{i_1} \wedge \cdots \wedge d\ln f_{i_k} \Big) v_{i_1,\dots,i_k}$$

For any horizontal section $\gamma(z)$, the function $I_\gamma(z)$ takes values in $\mathrm{Sing}\, V$ and satisfies the Gauss–Manin differential equations

$$\kappa \frac{\partial I_\gamma}{\partial z_j} = K_j(z) I_\gamma, \qquad j \in J \tag{29}$$

where $K_j(z) \in \mathrm{End}(\mathrm{Sing}\, V)$ are suitable linear operators independent of κ and γ. Formulas for $K_j(z)$ can be seen, for example, in ([12], Formula (5.3)).

For $z \in \mathbb{C}^n - \Delta$, the subalgebra $\mathcal{B}(z) \subset \mathrm{End}(\mathrm{Sing}\, V)$ generated by the identity operator and the operators $K_j(z), j \in J$, is called the Bethe algebra at z of the Gauss–Manin differential equations. The Bethe algebra is a maximal commutative subalgebra of $\mathrm{End}(\mathrm{Sing}\, V)$, see ([12], Section 8).

We define the characteristic variety of the κ-dependent D-module associated with the Gauss–Manin differential Equation (29) as

$$\mathrm{Spec} = \{(z,p) \in T^*(\mathbb{C}^n - \Delta) \mid \exists v \in \mathrm{Sing}\ V \text{ with } K_j(z)v = p_j v,\ j \in J\}$$

4.3. Identification

Let $z \in \mathbb{C}^n - \Delta$. By Lemma 2.3, given $j_1 \in J$, the monomials $p_{i_1} \dots p_{i_k}$, with $i_1 < \cdots < i_k$ and $j_1 \notin \{i_1, \dots, i_k\}$, form a $\mathbb{C}$-basis of $A_\Phi(z)$. Consider the linear map $\mu : A_\Phi(z) \to \mathrm{Sing}\ V$ that sends $d_{i_1,\dots,i_k} p_{i_1} \dots p_{i_k}$ to $s^\perp(v_{i_1,\dots,i_k})$ for all $i_1 < \cdots < i_k$ with $j_1 \notin \{i_1, \dots, i_k\}$.

Theorem 4.1. *([15], Corollary 6.16) The linear map μ does not depend on j_1 and is an isomorphism of complex vector spaces. For any $j \in J$, the isomorphism μ identifies the operator of multiplication by p_j on $A_\Phi(z)$ and the operator $K_j(z)$ on* $\mathrm{Sing}\ V$.

Corollary 4.2. *The characteristic variety* Spec *of the Gauss–Manin differential equations coincides with the Lagrangian variety of the master function.*

Thus the statements in Section 3 give us information on the characteristic variety of the Gauss–Manin differential equations. In particular, equations in $A_\Phi(z)$ are satisfied in $\mathcal{B}(z)$, for example,

$$f_{i_1,i_2,\dots,i_{k+1}}(z) K_{i_1}(z) \dots K_{i_{k+1}}(z) = \sum_{m=1}^{k+1} (-1)^{m-1} a_{i_m} d_{i_1,\dots,\widehat{i_m},\dots,i_{k+1}} K_{i_1}(z) \dots \widehat{K_{i_m}(z)} \dots K_{i_{k+1}}(z)$$

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Conflicts of Interest

The author declares no conflict of interest.

References

1. Etingof, P.; Frenkel, I.; Kirillov, A. *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*; Mathematical Surveys and Monographs 58; AMS: Providence, RI, USA, 1998.
2. Varchenko, A. *Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups*; Advanced Series in Mathematical Physics; World Scientific: Singapore, 1995; Volume 21.
3. Orlik, P.; Terao, H. Arrangements and hypergeometric integrals. *Math. Soc. Jpn.* **2001**, *9*, 112.
4. Varchenko, A. Arrangements and Frobenius like structures. *Ann. Fac. Sci. Toulouse Math.* **2012**, *1*, 1–55.
5. Dubrovin, B. Springer lecture notes in mathematics. *Geometry of 2D Topological Field Theories*; Integrable Systems and Quantum Groups, Francaviglia, M., Greco, S., Eds.; Springer: Berlin, Germany, 1996; pp. 120–348.
6. Manin, Y.I. *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*; American Mathematical Society Colloquium Publications: Providence, RI, USA, 1999; Volume 47.
7. Givental, A. Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, topics in singularity theory. *AMS Transl. Ser. 2* **1997**, *180*, 103–115.
8. Givental, A.; Kim, B. Quantum cohomology of flag manifolds and Toda lattices. *Commun. Math. Phys.* **1995**, *168*, 455–675.
9. Mukhin, E.; Tarasov, V.; Varchenko, A. KZ Characteristic Variety as the Zero Set of Classical Calogero–Moser Hamiltonians. SIGMA Symmetry Integrability Geom. *Methods Appl.* **2012**, *8*, 11.
10. Schechtman, V.; Varchenko, A. Arrangements of hyperplanes and lie algebra homology. *Invent. Math.* **1991**, *106*, 139–194.
11. Tarasov, V.; Varchenko, A. Hypergeometric solutions of the quantum differential equation of the cotangent bundle of a partial flag variety. *Cent. Eur. J. Math.* **2014**, *12*, 694–710.
12. Varchenko, A. Bethe ansatz for arrangements of hyperplanes and the Gaudin model. *Mosc. Math. J.* **2006**, *6*, 195–210, 223–224.
13. Mukhin, E.; Tarasov, V.; Varchenko, A. *Bethe Algebra of the $\mathfrak{gl}_{N+1}$ Gaudin Model and Algebra of Functions on the Critical Set of the Master Function*; New Trends in Quantum Integrable Systems; World Science Publisher: Hackensack, NJ, USA, 2011; pp. 307–324.
14. Kleiman, S.L.; Laksov, D. Schubert calculus. *Am. Math. Mon.* **1972**, *79*, 1061–1082. Available online: http://www.jstor.org/stable/2317421 (accessed on 19 June 2014).

15. Varchenko, A. Quantum integrable model of an arrangement of hyperplanes, SIGMA symmetry integrability geom. *Methods Appl.* **2011**, *7*, 55.
16. Arnol'd, V.I.; Gusein-Zade, S.M.;Varchenko, A.N. *Singularities of Differential Maps*; Nauka: Moscow, Russia, 1982; Birkhauser: Boston, MA, USA, 1985; Volume I.
17. Mukhin, E.; Varchenko, A. Norm of a bethe vector and the hessian of the master function. *Compos. Math.* **2005**, *141*, 1012–1028.
18. Varchenko, A. Critical points of the product of powers of linear functions and families of bases of singular vectors. *Compos. Math.* **1995**, *97*, 385–401.