mathematics

Article

Derivations of Incidence Algebras

Piotr Krylov [1,†] and Askar Tuganbaev [2,3,*,†]

[1] Department of Algebra, National Research Tomsk State University, Tomsk 634050, Russia; krylov@math.tsu.ru
[2] Department of Higher Mathematics, National Research University 'MPEI', Moscow 111250, Russia
[3] Department of Higher Algebra, Lomonosov Moscow State University, Moscow 119991, Russia
* Correspondence: tuganbaev@gmail.com
† These authors contributed equally to this work.

Abstract: We study the derivations of the incidence algebra $I(X, R)$, where X is a preordered set and R is an algebra over some commutative ring T. A satisfactory description of the T-module of derivations and the T-module of outer derivations of this algebra is given.

Keywords: incidence algebra; derivation

MSC: 16W20; 16S50

1. Introduction

Much attention has been paid to the derivations of incidence rings, including ordinary and Lie derivations and Jordan derivations (see [1–9]). We point to a very informative introduction in [9] with a history of derivation studies and an extensive list of references. Note that other important linear maps of incidence rings are also systematically studied: automorphisms, anti-automorphisms, and involutions (see [10–14]).

This paper is devoted to the derivations of the incidence algebra $I(X, R)$ for an arbitrary preordered set X and an arbitrary algebra R over some commutative ring T. The initial stage of research into the derivations of incidence rings can be found in the book [15]. The final result of this stage is Theorem 7.1.4 [15]. It describes the derivations of the R-algebra $I(X, R)$, where X is a partially ordered set and R is a commutative ring (see Corollary 6 of the given paper).

Later, various questions about the derivations of incidence rings were considered in interesting informative papers [2,6]. The first paper is devoted to additive derivations of the F-algebra $I(X, F)$, where X is a finite partially ordered set and F is a field. In this case, the set X is considered in the standard way as a directed graph. In the paper [6], the author studied the derivations of finitary incidence rings. Such rings are more general objects compared to incidence rings. Theorem 5 in [6] reveals the structure of the Lie algebra of the outer derivations of a finitary incidence ring.

In our previous papers, we used and developed an approach that has proven itself to well describe the automorphisms of formal matrix rings with zero trace ideals [16,17]. Namely, an incidence algebra is written as a splitting extension of some ideal M by means of some subring L. In other words, there is a direct decomposition of $I(X, R) = L \oplus M$. Here, L is the product of square matrix rings over R and M is the product of rectangular matrix groups over R. We note that the ideal M is contained in the Jacobson radical of the ring $I(X, R)$, while M is a nilpotent ideal in the case of a formal matrix ring. The derivations of the algebra $I(X, R)$ are represented by certain 2×2 matrices with respect to the direct sum $L \oplus M$. This helps greatly when studying the derivations of the algebra $I(X, R)$.

The final result of this work is Theorem 4, which states that any derivation of the algebra I(X, R) can be written as a sum of an inner derivation, an additive derivation, and a ring derivation. The nature of the action of each of these three derivations on the

check for updates

Citation: Krylov, P.; Tuganbaev, A. Derivations of Incidence Algebras. *Mathematics* **2024**, *12*, 999. https://doi.org/10.3390/math12070999

Academic Editors: Seok-Zun Song and Qing-Wen Wang

Received: 1 February 2024
Revised: 7 March 2024
Accepted: 23 March 2024
Published: 27 March 2024

elements of the algebra $I(X, R)$ is completely clear. Therefore, we can claim that Theorem 4 satisfactorily describes the derivations of the algebra $I(X, R)$.

Incidence algebras were defined by Rota in his well-known paper [18] as a tool for solving some problems of combinatorics and, above all, for studying generalizations of the Möbius inversion formula in number theory in a unified way. Over time, incidence algebras themselves, regardless of their applications in combinatorics and other areas of mathematics, have turned out to be a meaningful algebraic object. Many works are dedicated to them, including the book [15].

In this paper, we consider only associative rings with a non-zero unit.

If S is a ring (or an algebra), then $M(n, S)$ is the usual ring of all $n \times n$ matrices with values in S.

Using Der S, we denote the group (or the module) of the derivations of the algebra S; In(Der$S$) is the subgroup of inner derivations; Out S is the group of outer derivations of the algebra S, i.e., Out S is the quotient group Der S/In(Der$S$).

2. On Preordered Sets

We briefly outline some initial information about preordered sets (one can get acquainted with them in more detail in [15]).

Let X be an arbitrary set, and let $\leq$ be a reflexive and transitive relation on X. In this case, the system $\langle X, \leq\rangle$ is called a preordered set, and $\leq$ is a preorder on X. If the relation $\leq$ is also antisymmetric, then $\langle X, \leq\rangle$ is a partially ordered set.

We further assume that $\langle X, \leq\rangle$ is a preordered set. For any two elements $x, y \in X$, we denote by $[x, y]$ the set of $\{z \in X \mid x \leq z \leq y\}$. It is called an interval in X. An interval of the form $[x, x]$ is denoted by $[x]$. There are the following two useful properties of intervals:

(a) For any $y, z \in [x]$, we have the relation $[y, z] = [x]$.
(b) If $x < y$, then $s < t$ for arbitrary elements $s \in [x]$ and $t \in [y]$.

We define a binary relation $\sim$ on X by setting $x \sim y \Leftrightarrow x \leq y$ and $y \leq x$. It is clear that $\sim$ is an equivalence relation on X. The corresponding equivalence classes have the form $[x]$ for all possible $x \in X$. It follows from (b) that the preorder relation $\leq$ is consistent with the equivalence relation $\sim$. Consequently, the induced relation $\leq$ appears on the quotient set $\overline{X} = X/\sim$, and $\langle\overline{X}, \leq\rangle$ is a partially ordered set.

A directed graph can be associated with the preordered set X, as well as with the partially ordered set $\overline{X}$ (we do not take into account the loops that arise in this case). It is more convenient to proceed from the set $\overline{X}$. When necessary, we consider $\overline{X}$ as a simple graph correlated with the directed graph $\overline{X}$. At the same time, we use standard concepts of graph theory: a connected component, a semipath, and its length.

We agree that all intervals in X are finite. In this case, X is called a locally finite preordered set.

For an interval in a locally finite partially ordered set, the length of the interval is the largest of the chain lengths in this interval.

In what follows, we denote by x the elements of a partially ordered set $\overline{X}$, i.e., equivalence classes of the form $[x]$. In other words, to denote the class $[x]$, we use some representative of it. This should not lead to confusion. In a particular situation, it is always clear what elements of which set ($\overline{X}$ or X) we are talking about.

3. Some Ideals and Subbimodules in Incidence Algebras

Starting from this section, the symbol R denotes an algebra over some commutative ring T. However, the ring T itself is almost never used.

An incidence algebra is a certain ring of functions. Let $\langle X, \leq\rangle$ be an arbitrary locally finite preordered set. We set $I(X, R) = \{f : X \times X \to R \mid f(x, y) = 0, \text{ if } x \nleq y\}$. The functions are added pointwise. The product of the functions $f, g \in I(X, R)$ is given by the relation:

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z) \cdot g(z, y) \qquad (*)$$

for any $x, y \in X$. Since X is a locally finite set, it is possible to write $z \in X$ in $(*)$ instead of $x \leq z \leq y$. For any $t \in T$ and $x, y \in X$, we also assume $(tf)(x,y) = tf(x,y)$. As a result, we obtain a T-algebra $I(X,R)$, called the incidence algebra or an incidence ring, of the preordered set X over the ring R. In what follows, the specific algebra $I(X,R)$ is denoted by the symbol K.

We introduce some special functions from $I(X,R)$. For a given $x \in X$, we set $e_{[x]}(t,t) = 1$ for all $t \in [x]$ and $e_{[x]}(z,y) = 0$ for the remaining pairs (z,y). The system $\{e_{[x]} \mid x \in X\}$ consists of pairwise orthogonal central elements in L idempotents (the ring L is defined in the next paragraph). According to the agreement, we write e_x instead of $e_{[x]}$ at the end of Section 2.

We define a subring L and an ideal M in K. We set $L = \{f \in K \mid f(x,y) = 0, \text{ if } x \not\sim y\}$ and $M = \{f \in K \mid f(x,y) = 0, \text{ if } x \sim y\}$. We have a T-module direct sum $K = L \oplus M$, i.e., the ring K is a splitting extension of the ideal M with the use of the subring L. The ideal M is naturally considered as an L-L-bimodule. In addition, M is a nonunital algebra.

Let us have an arbitrary interval $[x]$. We denote by $R_{[x]}$ the set of functions $f \in K$ for which $f(z,y) = 0$ if $z \not\sim x$ or $y \not\sim x$. As in the case of idempotents e_x, we write R_x instead of $R_{[x]}$. The following relations:

$$R_x = e_x K e_x = e_x L e_x$$

are true. We conclude that R_x is a ring with identity element e_x. If we go to the restrictions of functions from K to $[x] \times [x]$, then, in fact, R_x is the algebra of all functions $[x] \times [x] \to R$ with pointwise addition and the product of the convolution type as in $(*)$. We choose any numbering of the interval $[x]$: $[x] = \{x_1, \ldots, x_n\}$. After that, if the functions $f \in R_x$ match the matrix $(f(x_i, x_j))$, then we come to the isomorphism of the algebras $R_x \cong M(n,R)$. Now, we take two different intervals $[x]$, $[y]$ and set

$$M_{xy} = \{f \in K \mid f(s,t) = 0, \text{ if } s \not\sim x \text{ or } t \not\sim y\}.$$

Then, $M_{xy} = e_x K e_y$, and therefore, M_{xy} is an R_x-R_y-bimodule. The relation $M_{xy} = e_x M e_y$ is also true.

We clarify that $M_{xy} = 0$ if $x \not\leq y$. For $x < y$, there is a canonical isomorphism:

$$M_{xy} \cong M(n \times m, R), \quad n = |[x]|, \; m = |[y]|,$$

with respect to the above isomorphisms $R_x \cong M(n,R)$ and $R_y \cong M(m,R)$. After identifying all algebras R_x with $M(n,R)$ and bimodules M_{xy} with $M(n \times m, R)$, it becomes clear that the actions of the rings R_x and R_y on M_{xy} will be ordinary matrix multiplications. It is also clear that M_{xy} is an L-L-bimodule. The action of L on M_{xy} is reduced to the action of R_x on the left and R_y on the right.

We note once again that we mean $[x]$ and $[x][y]$ in the subscripts of R_x and M_{xy}, respectively (see Section 2).

The product $\prod_{x,y \in X} M_{xy}$ has the L-L-bimodule structure. Exactly, if $f \in L$ and $(g_{xy}) \in \prod_{x,y \in X} M_{xy}$, then

$$f(g_{xy}) = (f_x g_{xy}) \text{ and } (g_{xy})f = (g_{xy} f_y), \tag{1}$$

where $f_x = e_x f e_x$ and $f_y = e_y f e_y$.

Proposition 1. *There are canonical algebra isomorphisms $L \cong \prod_{x \in X} R_x$, as well as L-L-bimodule isomorphisms and algebra isomorphisms $M \cong \prod_{x,y \in X} M_{xy}$.*

Proof. We define the mapping $\omega \colon L \to \prod_{x \in X} R_x$, assuming $\omega(f) = (f_x)$ for each $f \in L$, where $f_x = e_x f e_x$. Then, ω is an algebra isomorphism.

The mapping $\varepsilon \colon M \to \prod_{x,y \in X} M_{xy}$, $\varepsilon(g) = (g_{xy})$, where $g_{xy} = e_x g e_y$, is an isomorphism of L-L-bimodules and algebras. $\square$

In what follows, we do not distinguish the corresponding objects with respect to the isomorphisms ω and ε.

Remark 1. *The algebra K can be viewed as an algebra of functions and as an abstract ring represented as a splitting extension of $L \oplus M$. These two approaches can be called "functional" and "abstract"; they are, of course, equivalent. We use both of these approaches.*

4. Representation of Derivations by Matrices

We use all the material of Sections 1–3. In particular, the incidence algebras $I(X, R)$ are usually denoted by the letter K. We do not impose any preconditions on the locally finite preordered set X, the ring R, which is a T-algebra, and the algebra K itself.

A description of the derivation group of the algebra K is obtained (Section 7). We note that, in Sections 4–6, we consider that K is either an incidence algebra or a formal matrix algebra with zero trace ideals. Proofs are carried out for incidence algebras, but they allow the transfer to formal matrix algebras (with appropriate corrections). The technique of working with formal matrix algebras with zero trace ideals is well reflected in [16,17].

Let A be an algebra over some commutative ring T. A mapping $d\colon A \to A$ is called a derivation of the algebra A if d is a linear mapping, i.e., an endomorphism of the T-module A, and $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. All derivations of the algebra A form a T-module. We denote it by Der A.

For an element $c \in A$, we define a mapping d_c from A to A by assuming that $d_c(a) = ac - ca$, $a \in A$. Then, d_c is a derivation called inner. One says that d_c is defined by an element c. Inner derivations of the algebra A form a submodule of the T-module Der A. We denote it by In (Der A). To designate the factor module Der A/In (Der A), we use the symbol Out A. The elements of Out A are called outer derivations of the algebra A, and Out A is the module of the outer derivations of the algebra A.

There is a notion of a derivation in more general form. Let M be an A-A-bimodule. A derivation of the algebra A with values in the bimodule M is the homomorphism of T-modules $d\colon A \to M$, which satisfies the equality $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. Such a derivation d is said to be inner if there is an element $c \in M$ such that $d(a) = ac - ca$, $a \in A$.

It is known that the T-module Der A has the structure of a Lie algebra. In this algebra, the multiplication $\circ$ is defined by the relation $d_1 \circ d_2 = d_1 d_2 - d_2 d_1$. Here, we focus on the additive structure of this algebra.

We assume that the algebra A is a splitting extension of its ideal M with the use of some of the algebra L, i.e., $A = L \oplus M$, where $\oplus$ is the sign of the group direct sum. The ideal M is a natural L-L-bimodule. Additionally, M is a non-unital algebra.

We know that every incidence algebra is such an extension. A formal matrix algebra with zero trace ideals also can be represented in the form of an indicated splitting extension (see [16,17]).

We take an arbitrary derivation d of the algebra $A = L \oplus M$. Every additive endomorphism, d, can be represented by the matrix $\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$ with respect to the direct decomposition $A = L \oplus M$. Here,

$$\alpha\colon L \to L,\ \beta\colon M \to M,\ \gamma\colon M \to L,\ \delta\colon L \to M$$

are T-module homomorphisms and

$$d(a+b) = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha(a) + \gamma(b) \\ \delta(a) + \beta(b) \end{pmatrix}$$

for all $a \in L$ and $b \in M$.

We do not distinguish a derivation d and the matrix corresponding to it. We write a "triangular derivation d" if $\gamma = 0$, and a "diagonal derivation d" if $\gamma = 0 = \delta$.

In the case of triangular derivations, it is possible to obtain very meaningful information about the group Der A. We will soon see that, for incidence algebras and formal matrix algebras with zero trace ideals, this is always the case.

So, let $x+y,\ s+t \in A = L \oplus M$. We write down the relation:

$$d((x+y)(s+t)) = d(x+y)\cdot(s+t) + (x+y)\cdot d(s+t). \qquad (**)$$

Sequentially assigning values to arguments:

$$y = 0 = t,\ x = 0 = s,\ x = 0 = t,\ s = 0 = y$$

and calculating the left and right parts in $(**)$, we obtain the equalities:

$$\alpha(xs) = \alpha(x)s + x\alpha(s), \quad \delta(xs) = \delta(x)s + x\delta(s),$$

$$\gamma(ys) = \gamma(y)s, \quad \gamma(xt) = x\gamma(t),$$
$$\beta(yt) = \beta(y)t + y\beta(t) + \gamma(y)t + y\gamma(t),$$
$$\beta(xt) = \alpha(x)t + x\beta(t) + \delta(x)t,$$
$$\beta(ys) = \beta(y)s + y\alpha(s) + y\delta(s).$$

The first two relations mean that α is a derivation of the algebra L and δ is a derivation of the algebra L with values in the bimodule M. It follows from the third relation that γ is an L-L-bimodule homomorphism.

The converse is also true. Let

$$\alpha\colon L \to L,\ \beta\colon M \to M,\ \gamma\colon M \to L,\ \delta\colon L \to M$$

be T-module homomorphisms such that the above relations hold. Then, the transformation of the algebra A defined by the matrix $\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$, i.e.,

$$\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha(x) + \gamma(y) \\ \delta(x) + \beta(y) \end{pmatrix},\ x \in L,\ y \in M,$$

is its derivation.

Now, we will see that any derivation of an incidence algebra or a formal matrix algebra with zero trace ideals is triangular. As established in Section 3, an incidence algebra is a splitting extension of $L \oplus M$. A formal matrix algebra with zero trace ideals has a similar property; see [16,17]. Next, the letter K denotes one of these two algebras. Let $d = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$ be some derivation of the algebra K.

Lemma 1. *The relation $\gamma = 0$ holds.*

Proof. We assume that K is an incidence algebra. We assume that $\gamma \neq 0$, where, as we know, γ is an L-L-bimodule homomorphism $M \to L$. There is an idempotent e_x such that $e_x(\gamma M) = \gamma(e_x M) \neq 0$. Next, we have

$$\gamma(e_x M) = e_x\gamma(e_x M) \subseteq e_x L = R_x \ \text{ and } \ \gamma(e_x M) \subseteq R_x.$$

The last inclusion leads to contradictory relations

$$0 = \gamma(e_x M e_x) = \gamma(e_x M)e_x \neq 0.$$

Therefore, $\gamma = 0$. $\square$

Remark 2. *In the above proof, we implicitly use the fact that the algebra K has at least two distinct idempotents e_x (they appeared in Section 3). Otherwise, the situation is degenerating, i.e., K is simply some matrix ring $M(n, R)$.*

We summarize this section.

Corollary 1. *Every derivation d of an incidence algebra or a formal matrix algebra with zero trace ideals is triangular, $d = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$. Here, α is a derivation of the algebra L, β is a derivation of the algebra M, and δ is a derivation of the algebra L with values in the bimodule M.*

5. Some Properties of Derivations

As was shown in the previous section, the letter K denotes either an incidence algebra or a formal matrix algebra with zero trace ideals. However, the notation used in this case refers to incidence algebras. By Corollary 1, every derivation of both algebras is triangular. We usually mean the equality $K = L \oplus M$.

We focus on inner derivations of the algebra K. We denote by $\mathrm{In}_0(\mathrm{Der}\, K)$ (respectively, $\mathrm{In}_1(\mathrm{Der}\, K)$) the submodule of inner derivations defined by elements of L (respectively, M).

Lemma 2. *There is a direct sum:*

$$In\ (Der\ K) = In_0(Der\ K) \oplus In_1(Der\ K).$$

Proof. It is not quite obvious that the defined submodules have a zero intersection. Let $b \in L$, $c \in M$, and $d_b = d_c$ (these symbols are defined in Section 4). For any $x \in X$, the relations:

$$d_b(e_x) = e_x b - b e_x = d_c(e_x) = e_x c - c e_x$$

hold. Therefore,

$$e_x c - c e_x = 0, \quad e_x c = c e_x = e_x c e_x = 0.$$

Therefore, $c = 0$ and $d_b = d_c = 0$. $\square$

Now, we define one homomorphism and several derivation modules. Namely, we denote by f the module homomorphism $\mathrm{Der}\, K \to \mathrm{Der}\, L$ such that $f(d) = \alpha$ for every derivation $d = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$. $\mathrm{Ker}\, f$ consists of derivations of the form $\begin{pmatrix} 0 & 0 \\ \delta & \beta \end{pmatrix}$. The image of this homomorphism is denoted by Ω. Next, let Λ be the submodule of diagonal derivations, i.e., derivations of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. The derivations, which are representable by matrices $\begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}$, are called additive. The letter Ψ denotes the submodule of all additive derivations, and Ψ_0 is the submodule of the inner derivations defined by the central elements of the ring L. The last of the submodules is the submodule Φ, equal to $\left\{ \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix} \,\middle|\, \alpha \in \mathrm{In}\ (\mathrm{Der}\, L) \right\}$.

The following inclusions hold:

$$\mathrm{Ker}\, f \subseteq \Phi, \quad \Psi \subseteq \Phi, \quad \mathrm{In}_0(\mathrm{Der}\, K) \subseteq \Lambda,$$

$$\mathrm{In}_1(\mathrm{Der}\, K) \subseteq \mathrm{Ker}\, f, \quad \mathrm{In}\ (\mathrm{Der}\, K) \subseteq \Phi.$$

As will be seen from the following, information about the submodules introduced is extremely important for understanding the structure of the entire module $\mathrm{Der}\, K$.

We highlight the following research directions on the problem of finding the structure of the derivation module $\mathrm{Der}\, K$:

1. Calculation of the module Ω.
2. Calculation of the submodules Ψ and Ψ_0.

3. Calculation of the submodule Φ.

If $\begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$ is a derivation of the algebra K, then α is a derivation of the algebra L. Since L is the product of algebras R_x, $x \in X$ (Proposition 1), we need information about derivations of ring products. They are arranged quite simply.

Proposition 2. *Let S_i, $i \in I$, be some algebras, $S = \prod_{i\in I} S_i$, and let d be a derivation of the algebra S. Then, $d(S_i) \subseteq S_i$ for any $i \in I$. In addition, d acts on the product $\prod_{i\in I} S_i$ coordinatewise, and there is a module isomorphism Der $S \cong \prod_{i\in I}$ Der S_i.*

Proof. We denote by e_i the identity element of the ring S_i. Then,

$$d(e_i) = d(e_i)e_i + e_i d(e_i) \in S_i.$$

For any element $a \in S_i$, we have

$$d(a) = d(e_i a) = d(e_i)a + e_i d(a) \in S_i.$$

Therefore, $d(S_i) \subseteq S_i$, and we can set $d_i = d|_{S_i}$, where $d_i \in \text{Der } S_i$.

The coordinatewise action of the derivation d means that $d(a) = (d_i(a_i))$ for any element $a = (a_i) \in S$.

We take an arbitrary element $a = (a_i) \in S$. We fix a subscript $k \in I$ and write down $a = a_k + b$, where $b \in (1 - e_k)S$. With the use of the idempotent $1 - e_k$, it is easy to obtain that the derivation d leaves the ring $\prod_{i\neq k} S_i$ in place. We write down

$$d(a) = d(a_k) + d(b),\ d(a) = (c_i) = c_k + g. \text{ where } g \in (1 - e_k)S.$$

Then, $d(a_k) = c_k$, which confirms the fact of the coordinate action of d.

Now, suppose that, for each i, we have a derivation d_i of the algebra S_i. Assuming $d(a) = (d_i(a_i))$ for the element $a = (a_i) \in S$, we obtain a derivation d of the algebra S.

From all the above, we obtain that we have the canonical isomorphism Der $S \cong \prod_{i\in I}$ Der S_i. □

For an algebra L equal to $\prod_{x\in X} R_x$, we can write down the following useful fact.

Corollary 2. *If $d = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$ is a derivation of the algebra K, then for the derivation α of the algebra L, all assertions of Proposition 2 are true. Therefore, relations $\alpha(e_x) = 0$ and $d(e_x) = \delta(e_x)$ are true for every $x \in X$.*

In the rest of the section, we pay attention to diagonal derivations. For them, Corollary 1 allows for amplification.

Corollary 3. *Let $d = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ be a derivation. Then, α is a derivation of the algebra L, β is a derivation of the algebra M, and the relations:*

$$\beta(xt) = \alpha(x)t + x\beta(t), \quad \beta(ys) = \beta(y)s + y\alpha(s)$$

hold for all $x, s \in L$ and $y, t \in M$. The converse is also true. If some endomorphisms α and β of T-modules L and M, respectively, satisfy the above properties, then $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ is a derivation of the algebra K.

Proposition 3. *A derivation* $d = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$ *is a diagonal derivation if and only if* $d(e_x) = 0$ *for every* $x \in X$.

Proof. The derivation d is diagonal if and only if $e_x d(a) e_y = 0$ for all $a \in L$ and all non-equivalent $x, y \in X$.

We assume that $d(e_x) = 0$ for every $x \in X$. Then, for any $a, b \in L$, we have

$$d(e_x a) = d(e_x)a + e_x d(a) = e_x d(a) \text{ and similarly } d(be_y) = d(b)e_y.$$

We obtain

$$e_x d(a) e_y = d(e_x a) e_y = d(e_x a e_y) = d(0) = 0.$$

Now, let d be a diagonal derivation. Then,

$$d(e_x) = \alpha(e_x) + \delta(e_x) = \alpha(e_x) = 0.$$

□

Corollary 4. *If* $d = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, *then the relation:*

$$\beta(e_x c e_y) = e_x \beta(c) e_y, \quad \textit{where } c \in M$$

holds.

Proof. Taking into account Corollary 3 and Proposition 3, we write down the relations:

$$\beta(e_x c e_y) = \alpha(e_x) c e_y + e_x \beta(c) e_y + e_x c \alpha(e_y) = e_x \beta(c) e_y.$$

□

Proposition 4. *We take an arbitrary diagonal derivation* $d = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ *of the algebra K. For the derivation* β *of the algebra M, we have an inclusion* $\beta M_{xy} \subseteq M_{xy}$ *for all* $x, y \in X$. *In addition,* β *acts coordinatewise on the product* $\prod_{x,y \in X} M_{xy}$.

Proof. The inclusion $\beta M_{xy} \subseteq M_{xy}$ follows from the relation $M_{xy} = e_x M e_y$ and Corollary 4. We can map a vector (β_{xy}) to the derivation β, where $\beta_{xy} = \beta|_{M_{xy}}$. We show that, for $b = (b_{xy}) \in M = \prod_{x,y \in X} M_{xy}$, we have $\beta(b) = (\beta_{xy}(b_{xy}))$, which means that β acts coordinatewise. Let $\beta(b) = (c_{xy})$ for all x, y. We verify that $\beta_{xy}(b_{xy}) = c_{xy}$ for all x, y. We fix elements x, y and write down $M = M_{xy} \oplus N$, where N is the product of all bimodules M_{st} besides M_{xy}. Let $b = b_{xy} + f$, where $f \in N$. Then,

$$\beta(b) = \beta(b_{xy}) + \beta(f), \text{ where } \beta(b_{xy}) \in M_{xy}.$$

In addition, we write down

$$\beta(b) = c_{xy} + g, \text{ where } c_{xy} \in M_{xy}, \ g \in N.$$

By Corollary 4, we have $e_x \beta(f) e_y = \beta(e_x f e_y) = 0$. Therefore, $\beta(f)$ has the zero projection in M_{xy}. We obtain that $\beta_{xy}(b_{xy}) = c_{xy}$, which is required.

The rest of the inclusion follows from what has already been said. □

6. Main Decompositions and Isomorphisms for Module Der K

As before, K denotes an incidence algebra $I(X, R)$ or a formal matrix algebra with zero trace ideals represented in the form $K = L \oplus M$, as in Section 3. We preserve the previously accepted notation. It is important that the derivations of the algebra K are triangular (Corollary 1).

We recall one more very useful equality from Lemma 2:

$$\mathrm{In}\,(\mathrm{Der}\,K) = \mathrm{In}_0(\mathrm{Der}\,K) \oplus \mathrm{In}_1(\mathrm{Der}\,K).$$

We also recall that the modules listed below appeared at the beginning of Section 5.

It is great that the derivations of the algebra K can be diagonalized in a certain sense.

Theorem 1. *There are the following module relations:*

1. *Der* $K = \mathit{In}_1(\mathit{Der}\ K) \oplus \Lambda$.
2. *Ker* $f = \mathit{In}_1(\mathit{Der}\ K) \oplus \Psi$.
3. $\Phi = \mathit{In}\ (\mathit{Der}\ K) + \Psi = \mathit{In}_1(\mathit{Der}\ K) \oplus (\mathit{In}_0(\mathit{Der}\ K) + \Psi)$.

Proof. 1. For a derivation $d = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$ of the algebra K, we define a function $g \in I(X, R)$, by setting

$$g(x, y) = \begin{cases} d(e_y)(x, y), & \text{if } x \leq y, \\ 0, & \text{if } x \not\leq y. \end{cases}$$

We also can write $g = (g_{xy}) = (d(e_y)_{xy})$. In addition, $g \in M$, since $d(e_y) \in M$ (Corollary 2). So, $g_{xy} = d(e_y)_{xy}$.

For every $x \in X$, the relation $ge_x = d(e_x)e_x$ holds. We write down the element g in the form $(d(e_y)e_y)_{y \in X}$. We also use a similar form for other elements. If $x \neq y$, then

$$e_x e_y = 0,\ 0 = d(e_x e_y) = d(e_x)e_y + e_x d(e_y),\ e_x d(e_y) = -d(e_x)e_y.$$

We set $d' = d + d_g$, where d_g is the inner derivation of the algebra K defined by the element g (the designation d_g was given at the beginning of Section 4). To verify that d' is a diagonal derivation, we make the following transformations:

$$e_x g = e_x (d(e_y)e_y)_{y \neq x} = (e_x d(e_y)e_y)_{y \neq x} =$$

$$= -((d(e_x)e_y)e_y)_{y \neq x} = -(d(e_x)e_y)_{y \neq x}.$$

Next, we have the relation:

$$d_g(e_x) = e_x g - g e_x = -(d(e_x)e_y)_{y \neq x} - d(e_x)e_x =$$

$$= -(d(e_x)e_y)_{y \in X} = -d(e_x).$$

We obtain that $d'(e_x) = 0$ and d' is a diagonal derivation by Proposition 3. Thus, $d = -d_g + d'$, where $d_g \in \mathrm{In}_1(\mathrm{Der}\,K)$, $d' \in \Lambda$. We take an arbitrary derivation d from the intersection $\mathrm{In}_1(\mathrm{Der}\,K) \cap \Lambda$. Let $d = d_g$, where $g \in M$. It follows from Proposition 3 that $d(e_x) = 0$ for all x. Therefore, we obtain:

$$e_x g - g e_x = 0 \text{ and } e_x g = e_x g e_x = 0 \text{ for any } x.$$

Therefore, $g = 0$, $d = 0$, and we have proven **1**.

2. The assertion follows from **1** and the relations:

$$\mathrm{In}_1(\mathrm{Der}\,K) \subseteq \mathrm{Ker}\ f \text{ and } \mathrm{Ker}\ f \cap \Lambda = \Psi.$$

3. We take an arbitrary derivation $d = \begin{pmatrix} \alpha & 0 \\ \delta & \beta \end{pmatrix}$ from Φ. Let an inner derivation α of the algebra L be defined by an element $c \in L$. Then, we have $d - d_c \in \mathrm{Ker}\, f$, where d_c is the inner derivation of the algebra K defined by the element c. Thus,

$$d \in \mathrm{In}_0(\mathrm{Der}\, K) + \mathrm{Ker}\, f, \quad \Phi = \mathrm{In}_0(\mathrm{Der}\, K) + \mathrm{Ker}\, f =$$

$$= \mathrm{In}_0(\mathrm{Der}\, K) + (\mathrm{In}_1(\mathrm{Der}\, K) \oplus \Psi) = \mathrm{In}\, (\mathrm{Der}\, K) + \Psi.$$

It remains to verify that $\mathrm{In}_1(\mathrm{Der}\, K) \cap (\mathrm{In}_0(\mathrm{Der}\, K) + \Psi) = 0$. Let $d_b = d_a + \gamma$, where

$$d_b \in \mathrm{In}_1(\mathrm{Der}\, K),\ d_a \in \mathrm{In}_0(\mathrm{Der}\, K),\ \gamma \in \Psi.$$

Since $d_a, \gamma \in \Lambda$, we have $d_b \in \Lambda$. Therefore, if $d_b = \begin{pmatrix} 0 & 0 \\ \delta & \beta \end{pmatrix}$, then $\delta = 0$. Thus, $fb - bf = 0$ for all elements $f \in L$. In particular, $e_x b - b e_x = 0$. Therefore, $e_x b = e_x b e_x = 0$ for any x. Consequently, $b = 0$ and $d_b = 0$. The equality $\Phi = \mathrm{In}_1(\mathrm{Der}\, K) \oplus (\mathrm{In}_0(\mathrm{Der}\, K) + \Psi)$ is proven. □

We gather several useful equalities and isomorphisms.

Proposition 5. *The following relations and isomorphisms hold:*

1. $\Psi \cap In\ (Der\ K) = \Psi \cap In_0(Der\ K) = \Psi_0$.
2. $\Lambda/(In_0(Der\ K) + \Psi) \cong \Omega/In\ (Der\ L)$.
3. $\Phi/Ker\ f \cong In_0(Der\ K)/\Psi_0 \cong In\ (Der\ L)$.
4. $\Phi/In\ (Der\ K) \cong \Psi/\Psi_0$.

Proof. 1. We only verify the inclusion $\Psi \cap \mathrm{In}\, (\mathrm{Der}\, K) \subseteq \Psi_0$. We take an arbitrary derivation d, equal to $d_1 + d_0$, where $d \in \Psi$, $d_1 \in \mathrm{In}_1(\mathrm{Der}\, K)$, $d_0 \in \mathrm{In}_0(\mathrm{Der}\, K)$. Let d_1 be defined by an element $c \in M$, and let d_0 be defined by an element $a \in L$. Then, d is defined by an element $a + c$.

For any element $b \in L$, we have $d(b) = (bc - cb) + (ba - ab)$. Since $d \in \Psi$, we have $ba - ab = 0$. Consequently, $a \in C(L)$, i.e., a is a central element of L. We also have $bc - cb = 0$. In particular, $e_x c - c e_x = 0$ for all x. Similar to the end of the proof of Theorem 1(3), we obtain $c = 0$. So, $d_1 = 0$ and $d = d_0$. Since $a \in C(L)$, we have $d \in \Psi_0$.

2. First, we remark that the submodule $\mathrm{In}\, (\mathrm{Der}\, L)$ is contained in Ω. We denote by π the canonical epimorphism $\Omega \to \Omega/\mathrm{In}\, (\mathrm{Der}\, L)$. The kernel of the homomorphism $\pi f|_\Lambda \colon \Lambda \to \Omega/\mathrm{In}\, (\mathrm{Der}\, L)$ is equal to $\Phi \cap \Lambda = \mathrm{In}_0(\mathrm{Der}\, K) + \Psi$ (see the proof of Theorem 1(3)).

3. By considering Theorem 1 and **1**, we write down the relations:

$$\Psi/\mathrm{Ker}\, f \cong (\mathrm{In}_0(\mathrm{Der}\, K) + \Psi)/\Psi \cong \mathrm{In}_0(\mathrm{Der}\, K)/(\mathrm{In}_0(\mathrm{Der}\, K) \cap \Psi) =$$

$$= \mathrm{In}_0(\mathrm{Der}\, K)/\Psi_0 \cong \mathrm{In}\, (\mathrm{Der}\, L).$$

4. Again, taking into account Theorem 1 and **1**, we have the relations:

$$\Phi/\mathrm{In}\, (\mathrm{Der}\, K) = (\mathrm{In}\, (\mathrm{Der}\, K) + \Psi)/\mathrm{In}\, (\mathrm{Der}\, K) \cong \Psi/(\mathrm{In}\, (\mathrm{Der}\, K) \cap \Psi) \cong \Psi/\Psi_0.$$

□

The following statement is directly derived from Theorem 1 and Proposition 5.

Corollary 5. *The following isomorphisms are true:*

1. $Der\ K/Ker\ f \cong \Omega \cong \Lambda/\Psi$.
2. $Der\ K/\Phi \cong \Omega/In(Der\ L)$.

In conclusion of the section, we note that the module Ω is studied in the next section.

7. Structure of Modules Der K and Out K

In Section 7, the letter K denotes only some incidence algebra (and not the incidence algebra or a formal matrix algebra, as in Sections 4–6).

It is essentially used here in the sense that all R_x are ordinary matrix rings and all M_{xy} are ordinary matrix groups. Therefore, it is impossible to directly transfer the results of Section 7 to formal matrix algebras.

We formulate a number of questions concerning the structure of the modules Ω, Der K, and Out K:

1. Which derivations from Der L belong to Ω?
2. What is the structure of the modules Ω and $\Omega/\mathrm{In}(\mathrm{Der}\ L)$?
3. What is the structure of the modules Der K and Out K?

Satisfactory answers to all three questions will be given.

Proposition 6. *Let H be some T-algebra, and let α and γ be derivations of H. An endomorphism β of the T-module H such that*

$$\beta(ab) = \alpha(a)b + a\beta(b) = \beta(a)b + a\gamma(b) \ \textit{for all} \quad a, b \in H$$

exists if and only if $\gamma - \alpha$ is an inner derivation.

Proof. We assume that the indicated endomorphism β exists. By giving the elements a and b in turn a value of 1, we obtain the relations:

$$\beta(c) = \beta(1)c + \gamma(c) = \alpha(c) + c\beta(1)$$

for every $c \in H$. Therefore, $(\gamma - \alpha)(c) = c\beta(1) - \beta(1)c$. In other words, $\gamma - \alpha$ is the inner derivation defined by the element $\beta(1)$.

Now, let $\gamma - \alpha$ be the inner derivation defined by the element $d \in H$. Then,

$$(\gamma - \alpha)(c) = cd - dc \ \text{ and } \ dc + \gamma(c) = \alpha(c) + cd$$

for any $c \in H$. We define an endomorphism β of the T-module H by setting

$$\beta(c) = \alpha(c) + cd = dc + \gamma(c), \ c \in H.$$

For arbitrary elements $a, b \in H$, we have

$$\beta(ab) = \alpha(ab) + abd = dab + \gamma(ab) =$$

$$= \alpha(a)b + a\alpha(b) + abd = dab + \gamma(a)b + a\gamma(b).$$

We also have

$$\alpha(a)b + a\beta(b) = \alpha(a)b + a\alpha(b) + abd,$$

$$\beta(a)b + a\gamma(b) = dab + \gamma(a)b + a\gamma(b).$$

These equalities imply the required result. □

For positive integers k and ℓ, we set

$$P = M(k, R), \ Q = M(\ell, R), \ H = M(c, R), \ \text{where } \ c = \mathrm{LCM}\ (k, \ell),$$

and finally, $V = M(k \times \ell, R)$. Let $\ell' = c/k$ and $k' = c/\ell$.

The ring H can be represented as a ring of block matrices in two ways: as a ring of block matrices over P of order ℓ' and as a ring of block matrices over Q of order k'. It is also a P-Q-bimodule of block matrices over V of size $\ell' \times k'$.

Let α and γ be derivations of the algebras P and Q, respectively. They induce derivations $\overline{\alpha}$ and $\overline{\gamma}$ of the algebra H, respectively. Namely, $\overline{\alpha}(A) = (\alpha(a_{ij}))$ for any matrix $A = (a_{ij}) \in H$, $a_{ij} \in P$, and similarly for $\overline{\gamma}$. The derivations $\overline{\alpha}$ and $\overline{\gamma}$ are called ring block derivations (they are also called induced derivations). We keep the notation α and γ for them. This agreement is already in effect in the following proposition, which extends Proposition 6 for matrix rings.

Proposition 7. *If $\alpha \in Der\ P$ and $\gamma \in Der\ Q$, then the existence of an endomorphism β of the T-module V such that the relations:*

$$\beta(pa) = \alpha(p)a + p\beta(a) \text{ and } \beta(aq) = \beta(a)q + a\gamma(q)$$

hold for all $p \in P$, $q \in Q$, and $a \in V$ is equivalent to the property that $\gamma - \alpha$ is an inner derivation of the algebra H.

Proof. Let β be a T-endomorphism of the module for which the equalities written in the proposition are fulfilled. It is clear that β induces a (block) endomorphism $\overline{\beta}$ of a T-module H, where $\overline{\beta}(A) = (\beta(A_{ij}))$ for each matrix $A = (A_{ij})$ from H. It is assumed that the matrix A is represented in block form, i.e., A_{ij} are $\ell' \times k'$ blocks. For $\overline{\beta}$, the equalities from Proposition 6 hold. So, $\gamma - \alpha$ is the inner derivation of the algebra H.

Now, we assume that $\gamma - \alpha$ is an inner derivation of the algebra H. We denote by β the endomorphism of the module H, which exists by Proposition 6.

Let $e_1, \dots, e_{\ell'}$ and $f_1, \dots, f_{k'}$ be diagonal matrix units corresponding to two block partitions of matrices from H. We have $\alpha(e_i) = 0$ for $i = 1, \dots, \ell'$ and $\gamma(f_j) = 0$ for $j = 1, \dots, k'$, and β satisfies the equalities from Proposition 6. Therefore, the endomorphism β induces an endomorphism on the T-module e_iHf_j for any unequal i and j. Multiplications in H are made in block form (this applies to all three block decompositions). Therefore, the restriction of β to e_iHf_j, i.e., in fact, on V, satisfies the equalities from the proposition. □

The derivation α of the algebra L is contained in Ω exactly when there is a derivation of β of the algebra M satisfying the equalities from Corollary 3. Before Proposition 1, the formulas of the bimodule multiplication in M and the multiplication in M are given. Given these formulas and the fact of the coordinate action of β (Proposition 4), it is not difficult to make sure that β satisfies the equality:

$$\beta(cd) = \beta(c)d + c\beta(d), \tag{2}$$

where $c \in M_{xz}$, $d \in M_{zy}$, and $x < z < y$, and the relations

$$\beta(ac) = \alpha(a)c + a\beta(c), \quad \beta(db) = \beta(d)b + d\alpha(b), \tag{3}$$

where $a \in R_x$, $c, d \in M_{xy}$, and $b \in R_y$.

Let n_x be the order of the matrices from the rings R_x. We set $c_{xy} = \mathrm{LCM}(n_x, n_y)$ for all $x, y \in X$ such that $x < y$. We denote by H_{xy} the matrix ring $M(c_{xy}, R)$. It can be represented as a block matrix ring over the rings R_x and R_y and as an R_x-R_y-bimodule of block matrices over M_{xy}. For the derivations of rings R_x and R_y, we assume ring (block) derivations of the rings H_{xy}.

Proposition 8. *A derivation $\alpha = (\alpha_x)$ of the algebra $L = \prod_{x \in X} R_x$ is contained in the module Ω if and only if $\alpha_y - \alpha_x$ is an inner derivation of the algebra H_{xy} for all $x, y \in X$ such that $x < y$.*

Proof. Necessity: If $\alpha \in \Omega$, then there is derivation $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ such that β satisfies the relations (3) by Corollary 3. By Proposition 7, $\alpha_y - \alpha_x \in \mathrm{In}\ (\mathrm{Der}\ H_{xy})$.

Sufficiency: By Proposition 7, there exists an endomorphism β_{xy} of the T-module M_{xy} for any x and y such that the relations (3) hold.

Let β be an endomorphism of the T-module M, which maps an element (d_{xy}) to $(\beta_{xy}(d_{xy}))$ for every $(d_{xy}) \in M = \prod_{x,y \in X} M_{xy}$. This β satisfies two relations from Corollary 3. In order for the transformation $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ of the algebra $K = L \oplus M$ to be its derivation, it remains to verify that β is a derivation of the algebra M. Thus, it comes down to checking the equality (2).

We fix three elements x, z, y such that $x < z < y$. We introduce for consideration one more matrix ring. We set $H = M(d, R)$, where $d = \mathrm{LCM}(n_x, n_z, n_y)$. The ring H is a block matrix ring over each of the rings H_{xz}, H_{zy}, H_{xy}, R_x, R_z, and R_y. It can also be considered as a bimodule of block matrices over each of the bimodules M_{xz}, M_{zy}, and M_{xy}. The derivations α_x, α_z, and α_y are considered ring (block) derivations of the algebra H. The T-endomorphisms β_{xz}, β_{zy}, and β_{xy} are considered as (block) endomorphisms of the bimodule H. The last ones satisfy the relations with respect to derivations α_x and α_z, α_z and α_y, and α_x and α_y (of the type recorded in Proposition 6).

The differences $\alpha_z - \alpha_x$, $\alpha_y - \alpha_z$, and $\alpha_y - \alpha_x$ are inner derivations of the algebra H. Naturally, they induce (in the sense disclosed in the proof of Proposition 6) the same mappings β_{xz}, β_{zy}, and β_{xy} as appeared above.

Let inner derivations $\alpha_z - \alpha_x$, $\alpha_y - \alpha_z$, and $\alpha_y - \alpha_x$ be defined by elements d_{xz}, d_{zy}, and d_{xy} of the algebra H, respectively. Then, $d_{xy} = d_{xz} + d_{zy}$. We write out from the proof of Proposition 6 how endomorphisms β_{xz}, β_{zy}, and β_{xy} act:

$$\beta_{xz}(a) = \alpha_x(a) + ad_{xz} = d_{xz}a + \alpha_z(a),$$

$$\beta_{zy}(b) = \alpha_z(b) + bd_{zy} = d_{zy}b + \alpha_y(b),$$

$$\beta_{xy}(c) = \alpha_x(c) + cd_{xy} = d_{xy}c + \alpha_y(c),$$

where $a, b, c \in H$.

We verify the following relation:

$$\beta_{xy}(ab) = \beta_{xz}(a)b + a\beta_{zy}(b) \qquad (***)$$

in H for arbitrary $a, b \in H$. To do this, we transform the right part of the equality $(***)$ to the left part:

$$\begin{aligned} \beta_{xz}(a)b + a\beta_{zy}(b) &= \alpha_x(a)b + ad_{xz}b + a\alpha_z(b) + abd_{zy} = \\ &= \alpha_x(a)b + a(d_{xz}b + \alpha_z(b)) + abd_{zy} = \\ &= \alpha_x(a)b + a(\alpha_x(b) + bd_{xz}) + abd_{zy} = \\ &= \alpha_x(a)b + a\alpha_x(b) + abd_{xz} + abd_{zy} = \\ &= \alpha_x(a)b + a\alpha_x(b) + abd_{xy} = \\ &= \alpha_x(ab) + abd_{xy} = \beta_{xy}(ab). \end{aligned}$$

The equality $(***)$ is proven. The multiplications occurring in it are performed over blocks. This implies that the relation (2) holds. $\square$

We pass to the question of the structure of the modules Ω and $\mathrm{Der}\, K$. The concept of a ring derivation of the algebra $M(n, R)$ is actually already given above. We clarify that, if $\alpha \in \mathrm{Der}\, R$, then the derivation of the algebra $M(n, R)$, which maps the matrix (a_{ij}) to the matrix $(\alpha(a_{ij}))$, is called a ring (or induced) derivation. The following result is known.

Theorem 2 ([4]). *Every derivation of the matrix algebra $M(n, R)$ is the sum of a ring derivation and an inner derivation.*

Let ε be some derivation of the algebra R. Then, ε provides a ring derivation ε_x of the algebra R_x for every $x \in X$. We denote by ε_L the derivation (ε_x) of the algebra L. For any

$x, y \in X$ with $x < y$, the derivation ε similarly induces a mapping ε_{xy} on the bimodule M_{xy}. Let $\varepsilon_M = (\varepsilon_{xy})$ be an endomorphism of the T-module M, which acts coordinatewise. We set $\bar{\varepsilon} = \begin{pmatrix} \varepsilon_L & 0 \\ 0 & \varepsilon_M \end{pmatrix}$. It follows from Corollary 3 or Proposition 8 that $\bar{\varepsilon}$ is a derivation of the algebra K.

The derivations ε_L, ε_M, and $\bar{\varepsilon}$ are also called ring derivations. We denote by D the submodule of all ring derivations of the algebra L. We denote the submodule of all ring derivations of the algebra K by the same letter D. It is clear that $D \subseteq \Lambda$ and $D \cap \Psi = 0$ in Der K, and the module D is canonically isomorphic to Der R. Next, let D_0 be the submodule in Der L consisting of ring derivations ε_L of the algebra L such that $\varepsilon \in$ In (Der R). Let D_0 denote a similar submodule in Der K. These submodules are isomorphic to In (Der R).

One simple result will be useful to us.

Lemma 3. *If d is a ring derivation and an inner derivation of the algebra $M(n, R)$, then it is defined by a scalar matrix. In other words, d is a ring derivation defined by an inner derivation of the algebra R.*

We can use Lemma 3 to directly verify the following lemma.

Lemma 4. ***1.*** *There is an isomorphism $D/D_0 \cong Out\ R$ in Der L.*
2. *$D_0 = D \cap In\ (Der\ L)$ is true in Der L.*
3. *In Der K, we have $D_0 = D \cap In_0(Der\ K) = D \cap (In_0(Der\ K) + \Psi)$:*

$$(\Psi \oplus D) \cap In_0(Der\ K) = \Psi_0 \oplus D_0.$$

We recall that, in Section 2, we agreed to consider a preordered set X as a directed graph.

We formulate the main result of this paper. It contains complete information about the structure of the derivation of modules Der K and Out K.

Theorem 3. *Let X be a connected set. For the algebra K, where $K = I(X, R)$, we have the following relations and isomorphisms:*
1(a). *$\Omega = D + In\ (Der\ L)$, $\Omega / In\ (Der\ L) \cong Out\ R$.*
1(b). *$\Lambda = In_0(Der\ K) + (\Psi \oplus D)$, $Der\ K = In\ (Der\ K) + (\Psi \oplus D) =$*

$$= In_1(Der\ K) \oplus (In_0(Der\ K) + (\Psi \oplus D)).$$

2. *$Out\ K \cong \Psi/\Psi_0 \oplus Out\ R$.*

Proof. 1(a). We take an arbitrary derivation $\alpha = (\alpha_x)$ in Ω. It follows from Theorem 2 that, for every $x \in X$, we have $\alpha_x = \rho_x + \mu_x$, where ρ_x is a ring derivation and μ_x is an inner derivation of the algebra R_x. We form derivations $\rho = (\rho_x)$ and $\mu = (\mu_x)$ of the ring L and obtain the relation $\alpha = \rho + \mu$ in Der L. Since $\alpha, \mu \in \Omega$, we have $\rho \in \Omega$.

We will prove the following property: up to an inner derivation, it is possible to obtain that $\rho_x = \rho_y$ for all $x, y \in X$.

We fix some element $t \in X$. We take an arbitrary element $x \in X$ and choose a semipath from t to x in X. By applying Proposition 8 several times or by applying the induction with respect to the length of the selected semipath, we can obtain that $\rho_x = \rho_t + \nu_x$ for some inner ring derivation ν_x. We limit ourselves to a short comment. Equalities of the form $\rho_s = \rho_z + \nu_s$ arise in the ring H_{sz} (see Proposition 8). We set $c = \mathrm{LCM}(n_t, n_{z_1}, \ldots, n_{z_m}, n_x)$, where $z_1, \ldots, z_m$ are the vertices of the selected semipath from t to x. Then, all ring derivations that appear can be considered as derivations of any ring R_x.

Now, we obtain $\alpha_x = \rho_x + \mu_x = \rho_t + (\nu_x + \mu_x)$ for every $x \in X$. By setting $\gamma = (\nu_x + \mu_x)$ and $\rho = (\rho_x)$, we obtain $\alpha = \rho + \gamma$, where $\rho \in D$, $\gamma \in$ In (Der L). Therefore, the relation $\Omega = D +$ In (Der L) is proven. By using Lemma 4, we obtain $\Omega/$In (Der $L) \cong D/D_0 \cong$ Out R.

1(b). Let $d = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in \Lambda$. Then, $\alpha \in \Omega$, and we have $\alpha = \rho + \gamma$, where $\rho \in D$ and $\gamma \in \text{In}\,(\text{Der } L)$ (see **1(a)**). Derivations ρ and γ induce a ring derivation and an inner derivation, respectively (it is contained in $\text{In}_0(\text{Der } K)$). We keep the designations ρ and γ for them. We denote by ψ the difference $d - \rho - \gamma$. Then, $d = \gamma + \psi + \rho$, where $\gamma \in \text{In}_0(\text{Der } K)$, $\psi \in \Psi$, and $\rho \in D$. This leads to the first equality and also leads to the second equality if we take into account Theorem 1.

2. With the use of Theorem 1 and Lemmas 2 and 4, we obtain the relations:

$$\text{Out } K = \frac{\text{In}_1(\text{Der } K) \oplus \Lambda}{\text{In}_1(\text{Der } K) \oplus \text{In}_0(\text{Der } K)} \cong$$

$$\frac{\text{In}_0(\text{Der } K) + (\Psi \oplus D)}{\text{In}_0(\text{Der } K)} \cong \frac{\Psi \oplus D}{(\Psi \oplus D) \cap \text{In}_0(\text{Der } K)} =$$

$$= \frac{\Psi \oplus D}{\Psi_0 \oplus D_0} \cong \Psi/\Psi_0 \oplus D/D_0 \cong \Psi/\Psi_0 \oplus \text{Out } R.$$

□

We extend Theorem 3 to the incidence algebras $I(X, R)$, where X is an arbitrary preordered set. As always, let $K = I(X, R)$, and let X_i, $i \in I$, be all connected components of the set X. We have $K = \prod_{i \in I} K_i$, where $K_i = I(X_i, R)$. Next, by considering Proposition 2, we obtain the relations:

$$\text{Der } K = \prod_{i \in I} \text{Der } K_i, \quad \text{In }(\text{Der } K) = \prod_{i \in I} \text{In }(\text{Der } K_i),$$

$$\text{Out } K = \prod_{i \in I} \text{Out } K_i$$

and similar relations for $\text{In}_0(\text{Der } K)$ and $\text{In}_1(\text{Der } K)$.

The symbols L_i, M_i, Λ_i, Ω_i, Ψ_i, and D_i have a clear meaning in relation to the algebra K_i. Again, the equalities $L = \prod_{i \in I} L_i$ and $M = \prod_{i \in I} M_i$ and similar equalities are valid for the remaining modules Λ, Ω, Ψ, and D.

With the use of the above relations, it is not difficult to prove the following theorem.

Theorem 4. *For an arbitrary incidence algebra K, there are the following relations:*

1(a). $\Omega = D + In\,(Der\ L)$, $\Omega/In\,(Der\ L) \cong \prod_{|I|} Out\ R$.

1(b). $\Lambda = In_0(Der\ K) + (\Psi \oplus D)$,
$Der\ K = In\,(Der\ K) + (\Psi \oplus D) = In_1(Der\ K) \oplus (In_0(Der\ K) + (\Psi \oplus D))$.

2. $Out\ K \cong \Psi/\Psi_0 \oplus \prod_{|I|} Out\ R$.

As noted in the Introduction, in the paper [6], the structure of the Lie algebra of the outer derivations of a finitary incidence algebra was found (this algebra is the factor algebra of the algebra of derivations by the ideal of inner derivations). It has been proven (Theorem 5)that this algebra is equal to the semidirect product of two Lie algebras. Moreover, one of the factors is some first cohomology group associated with the set X. In Item 2 of Theorem 4 of this work, the structure of the module of outer derivations of the incidence algebra is given. Item 2 is actually a consequence of Item 1(b), which contains complete information about the structure of the module of derivations of an arbitrary incidence algebra. In addition, Item 2 does not use the first cohomology group.

Corollary 6 ([15])**.** *Let X be a partially ordered set, and let R be a commutative ring. We consider only derivations of the R-algebra K. Then, we have the following relations and isomorphism:*

$$\Omega = 0,\ D = 0,\ \Psi_0 = In_0(Der\ K),\ Der\ K = In_1(Der\ K) \oplus \Psi,\ Out\ K \cong \Psi/\Psi_0.$$

Example 1. *It is convenient to illustrate the main concepts and results of the paper if we turn to the incidence ring $I(X, R)$ for a finite set X. So, let X be a finite preordered set. Note that the elements of the set X admit a numbering $x_1, \ldots, x_n$ such that $x_i \leq x_j$ implies $i \leq j$ ([15], Lemma 1.2.5). We assume that this is the chosen numbering of elements from X. Let $B = (b_{ij})$ be a Boolean matrix of order n corresponding to the set X. It is implied that*

$$b_{ij} = \begin{cases} 1, & x_i \leq x_j; \\ 0, & x_i \not\leq x_j. \end{cases}$$

Let us put

$$M(n, B, R) = \{(c_{ij}) \in M(n, R) \mid b_{ij} = 0 \Rightarrow c_{ij} = 0\}.$$

Then, $M(n, B, R)$ is a subring of $M(n, R)$. If f is an arbitrary function from $I(X, R)$, then we associate with it the matrix $(f(x_i, x_j))$, in which the element $f(x_i, x_j)$ is at position (i, j). This correspondence leads to the ring isomorphism $I(X, R) \cong M(n, B, R)$. For a finite set X, the incidence ring is usually called the structural matrix ring*; see [14]. Sometimes, this ring is immediately defined as the ring $M(n, B, R)$. If we go to the ring $M(n, B, R)$, then the material of our paper can be interpreted in a more familiar matrix language.*

8. Conclusions

In the paper, we found the structure of the module of the derivations of the incidence algebra $I(X, R)$ for an arbitrary preordered set X and any T-algebra R (Theorems 3 and 4). As a corollary, the structure of the module of outer derivations of this algebra is indicated.

The ideas, results, and proof techniques of this paper can serve as an example for studying other derivations of incidence rings. For example, we can consider Lie and Jordan derivations, as well as derivations of higher orders. In addition, all this can be used to study the derivations of various generalizations of incidence rings.

Thus, the authors plan to continue the study of additive derivations begun in [2]. The purpose of this study is to calculate the dimensions of the spaces of additive derivations and inner additive derivations.

There is a class of objects that are more general than incidence algebras. We are talking about reduced-incidence algebras. A specific reduced-incidence algebra is a certain subalgebra of the incidence algebra. It is defined using some equivalence relation on the set of all intervals of the partially ordered set X. This subalgebra consists of functions that are constants on equivalence classes. Reduced-incidence algebras are much more complex than incidence algebras. The theory of reduced-incidence algebras is presented in the book [15]. Finding the structure of the module of the derivation of reduced-incidence rings seems to be a very promising problem.

Another area for the research of this paper may turn out to be reduced-incidence coalgebras (including ordinary-incidence coalgebras), bialgebras, and Hopf-incidence algebras (these objects can be found in the book [15]). The derivation of any coalgebra induces the derivation of the algebra dual to it. For a reduced-incidence coalgebra, the dual algebra is canonically isomorphic to some reduced-incidence algebra. This opens the way for applying the results of this paper to the study of derivations of various incidence coalgebras.

The authors thank the reviewers for their careful reading of the paper and useful advice on how to improve it.

Author Contributions: Conceptualization, P.K. and A.T.; investigation, P.K. and A.T.; writing—original draft preparation, P.K. and A.T.; writing—review and editing, P.K. and A.T.; supervision, P.K. and A.T.; validation, P.K. and A.T.; writing—original draft, P.K. and A.T. All authors have read and agreed to the published version of the manuscript.

Funding: The work is supported by the Russian Scientific Foundation.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Fornaroli, É.Z.; Khrypchenko, M. Skew derivations of incidence algebras. *arXiv* **2023**, arXiv:2307.08439v1.
2. Fornaroli, É.Z.; Pezzott, R.E.M. Additive derivations of incidence algebras. *Commun. Algebra* **2021**, *49*, 1816–1828. [CrossRef]
3. Ghahramani, H.; Ghosseiri, M.N.; Rezali, T. Anti-derivations on triangular matrices. *Khayyam J. Math.* **2023**, *9*, 30–37.
4. Jöndrup, S. Automorphisms and derivations of upper triangular matrix rings. *Linear Algebra Its Appl.* **1995**, *221*, 205–218. [CrossRef]
5. Kaygorodov, I.; Khrypchenko, M.; Wei, F. Higher Derivations of finitary incidence algebras. *Algebr. Represent. Theory* **2019**, *22*, 1331–1341. [CrossRef]
6. Khrypchenko, M. Derivations of finitary incidence rings. *Commun. Algebra* **2012**, *40*, 2503–2522. [CrossRef]
7. Khrypchenko, M. Isomorphisms and derivations of partial flag incidence algebras. *Int. J. Algebra Comput.* **2022**, *32*, 193–209. [CrossRef]
8. Xiao, Z. Jordan derivations of incidence algebras. *Rocky Mt. J. Math.* **2015**, *45*, 1357–1368. [CrossRef]
9. Yang, Y.; Wei, F. Nonlinear Lie-Type derivations of finitary incidence algebras and related topics. *arXiv* **2021**, arXiv:2105.10685.
10. Brusamarello, R.; Fornaroli, É.Z.; Santulo, E.A., Jr. Anti-automorphisms and involutions on (finitary) incidence algebras. *Linear Multilinear Algebra* **2012**, *60*, 181–188. [CrossRef]
11. Brusamarello, R.; Fornaroli, É.Z.; Santulo, E.A., Jr. Classification of involutions on finitary incidence algebras. *Int. J. Algebra Comput.* **2014**, *24*, 1085–1098. [CrossRef]
12. Brusamarello, R.; Fornaroli, É.Z.; Santulo, E.A., Jr. Multiplicative automorphisms of incidence algebras. *Commun. Algebra* **2015**, *43*, 726–736. [CrossRef]
13. Brusamarello, R.; Lewis, D.W. Automorphisms and involutions on incidence algebras. *Linear Multilinear Algebra* **2011**, *59*, 1247–1267. [CrossRef]
14. Coelho, S.P. Automorphism groups of certain structural matrix rings. *Commun. Algebra* **1994**, *22*, 5567–5586. [CrossRef]
15. Spiegel, E.; O'Donnell, C.J. *Incidence Algebras*; Marcel Dekker: New York, NY, USA, 1997.
16. Krylov, P.A.; Tuganbaev, A.A. Automorphism groups of formal matrix rings. *J. Math. Sci.* **2021**, *258*, 222–249. [CrossRef]
17. Krylov, P.; Tuganbaev, A. Formal matrix rings: Automorphisms. *Int. J. Algebra Comput.* **2023**, *33*, 687–698. [CrossRef]
18. Rota, G.-C. On the foundations of combinatorial theory I. Theory of Möbius Functions. *Z. Wahrscheinlichkeitstheorie Verwandte Geb.* **1964**, *2*, 340–368. [CrossRef]