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Group Classification of the Unsteady Axisymmetric Boundary Layer Equation

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Abstract: Unsteady equations of flat and axisymmetric boundary layers are considered. For the unsteady axisymmetric boundary layer equation, the problem of group classification is solved. It is shown that the kernel of symmetry operators can be extended by no more than four-dimensional Lie algebra. The kernel of symmetry operators of the unsteady flat boundary layer equation is found and it is shown that it can be extended by no more than a five-dimensional Lie algebra. The non-existence of the unsteady analogue of the Stepanov–Mangler transformation is proved.

Keywords: symmetry operator; Lie algebra; group classification; axisymmetric boundary layer; Stepanov–Mangler transformation

MSC: 35B06; 35G20; 35Q35; 76D10; 76M60

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Contents

1. Introduction

1.1. Preliminary Remarks

The equations of flat and axisymmetric laminar boundary layers in incompressible fluid are considered in this paper. The boundary layer equations were first presented by Ludwig Prandtl at the Third Mathematical Congress in Heidelberg in 1904 as a simplification of the system of Navier–Stokes equations. The classical equations of the unsteady flat boundary layer in an incompressible viscous fluid have the form [1,2]

$$\begin{aligned} &u_t + uu_x + vu_y = -\frac{1}{\rho}\, p_x + \nu u_{yy}\,, \\ &u_x + v_y = 0\,. \end{aligned} \tag{1}$$

Here, $u = u(x, y, t)$, $v = v(x, y, t)$ are the components of the velocity vector; $\rho = \text{const}$ is the density; $p = p(x, t)$ is the pressure determined through the external flow; ν is the kinematic viscosity coefficient. By introducing the stream function $\Psi(x,y,t)$, defined by the equalities $u = \Psi_y$, $v = -\Psi_x$, the system of Equation (1) simplifies to a single equation (which coincides with Equation (5) for $r_0(x) = 1$).

The system of Equation (1) is generalized to the case of a curved streamlined surface. For an axisymmetric boundary layer on a body of rotation, the system of equations has the form [1–3]

$$\begin{aligned} &u_t + uu_z + vu_r = -\frac{1}{\rho}\,p_z + \nu u_{rr}\,, \\ &(r_0u)_z + (r_0v)_r = 0\,, \end{aligned} \tag{2}$$

where $r_0 = r_0(z)$ is the surface equation of a streamlined body of rotation. Introducing the stream function $\Psi = \Psi(z,r,t)$ defined by the equalities

$$u = \frac{1}{r_0}(r_0\Psi)_r\,, \qquad v = -\frac{1}{r_0}(r_0\Psi)_z\,,$$

the system of Equation (2) becomes a single equation (Equation (3)).

In [4,5], the transformation (4) shows that the equivalence of the steady equations of flat and axisymmetric boundary layers was obtained. In the present paper, the existence of such a transformation for steady equations is investigated.

One of the most important applications of boundary layer theory is the calculation of the friction drag of bodies in a flow, e.g., the drag of a flat plate at zero incidence, the friction drag of a ship, an airfoil, the body of an airplane, or a turbine blade. Due to its practical importance, the usage of different methods of studying nonlinear equations to boundary layer equations is widely covered in the literature. In particular, the group properties have been investigated by Ovsiannikov [6]. Self-similar solutions of boundary layer equations were considered in [7,8]. Some partially invariant solutions of the boundary layer equations were considered in [9]. The study of boundary layer equations using the direct Clarkson–Kruskal method [10] was described in [11]. In [12–15], reductions of the boundary layer equations were obtained. The application of the method of non-classical symmetries and the finding of some other solutions for the equations of flat and axisymmetric boundary layers were described in [16–20]. Exact solutions classes of boundary layer equations using the method of functional and generalized separation of variables are given in [21–27].

1.2. The Main Results

The main results of this article are the following:

- Group classification of the unsteady axisymmetric boundary layer equation is carried out; it is shown that the kernel of symmetry operators can be extended by no more than a four-dimensional Lie algebra;
- It is obtained that the kernel of symmetry operators of the flat unsteady boundary layer equation can be extended by no more than a five-dimensional Lie algebra;
- It is shown that there is no unsteady analogue of the Stepanov–Mangler transformation.

2. Basic Equations

Consider an equation describing the unsteady axisymmetric motion of a viscous incompressible fluid in a laminar boundary layer on the surface of a body of rotation [1,2]

$$u_{yt} + u_y u_{xy} - \left(u_x + \frac{r_0'(x)}{r_0(x)}u\right)u_{yy} - u_{yyy} - f_0(x,t) = 0\,. \tag{3}$$

Equation (3) is written in dimensionless variables. Here, $u(x,y,t)$ is the stream function; $f_0(x,t) = -\partial p/\partial x$ is the given function; $p(x,t)$ is the pressure; the function $r_0(x)$ defines the shape of the streamlined surface.

To simplify further calculations, we use the Stepanov–Mangler transformation [3–5], which transforms the equation of a steady axisymmetric boundary layer into the equation of a steady flat boundary layer. This transformation is given by

$$\bar{x} = \int_0^x r_0^2(s)ds, \qquad \bar{y} = r_0(x)y, \qquad \bar{u} = r_0(x)u\,. \tag{4}$$

Then, substituting (4) into Equation (3), we obtain

$$\frac{1}{r_0^2(x)}\bar{u}_{\bar{y}t} + \bar{u}_{\bar{y}}\bar{u}_{\bar{x}\bar{y}} - \bar{u}_{\bar{x}}\bar{u}_{\bar{y}\bar{y}} - \bar{u}_{\bar{y}\bar{y}\bar{y}} - \frac{f_0(x,t)}{r_0^2(x)} = 0\,.$$

Let us introduce the notation $\bar{r}(\bar{x}) = 1/r_0^2(x(\bar{x}))$, $\bar{f}(\bar{x},t) = f_0(x(\bar{x}),t)/r_o^2(x(\bar{x}))$. Omitting the bar, we obtain the following equation:

$$r(x)u_{yt} + u_y u_{xy} - u_x u_{yy} - u_{yyy} - f(x,t) = 0\,. \tag{5}$$

In the following, we will consider Equation (5) as the main equation.

Remark 1. *Transformation (4) is a point transformation and, therefore, Equations (3) and (5) are equivalent. It is more convenient to find symmetries for Equation (5).*

3. Group Classification

3.1. The System of Determining Equations

We are looking for the symmetry operator of Equation (5) in the following form:

$$X = \xi^x(x,y,t,u)\frac{\partial}{\partial x} + \xi^y(x,y,t,u)\frac{\partial}{\partial y} + \xi^t(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial t}\,.$$

Using the invariance criterion [6] (see also [28,29]) and substituting the expression for the highest derivative u_{yyy}

$$u_{yyy} = r(x)u_{yt} + u_y u_{xy} - u_x u_{yy} - f(x,t),$$

into it, we obtain the following relation:

$$\begin{aligned}
&- 3u_{xyy}u_y\xi^x_u - 3u_{xyy}\xi^x_y - 3u_{yyt}\xi^t_y - 3u_{yyt}u_y\xi^t_u + u_{xx}u_y\xi^x_y + u_{xx}u_y^2\xi^x_u\\
&+ u_{xy}u_t u_y\xi^t_u + u_{xy}\left(\xi^t_y + r(x)\xi^x_u\right) - u_{xy}u_x\xi^x_y - u_{xy}u_y^2\left(\xi^y_u + 3\xi^x_{uu}\right)\\
&+ u_{xy}u_y\left(\xi^x_x - \xi^y_y - 6\xi^x_{yu} - \eta^u_u\right) - 3u_{xy}u_{yy}\xi^x_u + u_{xy}\left(r(x)\xi^x_t - 3\xi^x_{yy}\right)\\
&- u_{yy}u_t u_x\xi^t_u - 3u_{yy}u_t u_y\xi^t_{uu} + u_{yy}u_t\left(r(x)\xi^y_u - 3\xi^t_{yu} - \xi^t_x\right) - u_{yy}u_x^2\xi^x_u\\
&+ u_{yy}u_x u_y\left(\xi^y_u - 3\xi^x_{uu}\right) + u_{yy}u_x\left(\xi^y_y + \eta^u_u - \xi^x_x - 3\xi^x_{yu}\right) - 6u_{yy}u_y^2\xi^y_{uu}\\
&+ 3u_{yy}u_y\left(\eta^u_{uu} - 3\xi^y_{yu}\right) - 3u_{yy}u_{yt}\xi^t_u + u_{yy}\left(3\eta^u_{yu} + r(x)\xi^y_t + \eta^u_x - 3\xi^y_{yy}\right)\\
&+ u_{xt}u_y^2\xi^t_u + u_{xt}u_y\left(\xi^t_y + r(x)\xi^x_u\right) + u_{xt}r(x)\xi^x_y + u_{yt}u_t r(x)\xi^t_u - 2u_{yt}u_x\xi^t_y\\
&- u_{yt}u_x u_y\xi^t_u + u_{yt}u_t r(x)\xi^t_u - 3u_{yt}u_y^2\xi^t_{uu} + u_{yt}u_y\left(\xi^t_x - 2r(x)\xi^y_u - 6\xi^t_{yu}\right)\\
&- u_{yt}\left(2r(x)\xi^y_y + 3\xi^t_{yy} + \xi^x r'(x) - r(x)\xi^t_t\right) - u_{yy}u_t u_x\xi^t_u - 3u_{yy}u_t u_y\xi^t_{uu}\\
&+ u_{yy}u_t\left(r(x)\xi^y_u - \xi^t_x - 3\xi^t_{yu}\right) - u_{yy}u_x^2\xi^x_u + u_{yy}u_x\left(\eta^u_u + \xi^y_y - \xi^x_x - 3\xi^x_{yu}\right)\\
&+ u_{yy}u_x u_y\left(\xi^y_u - 3\xi^x_{uu}\right) - 6u_{yy}u_y^2\xi^y_{uu} + 3u_{yy}u_y\left(\eta^u_{uu} - 3\xi^y_{yu}\right)\\
&+ u_{yy}\left(3\eta^u_{yu} + r(x)\xi^y_t + \eta^u_x - 3\xi^y_{yy}\right) + u_{tt}u_y r(x)\xi^t_u + u_{tt}r(x)\xi^t_y
\end{aligned} \tag{6}$$

$$
\begin{aligned}
&-u_y^4\xi^y_{uuu}+u_y^3\left(\eta^u_{uuu}+\xi^y_{xu}-3\xi^t_{uuu}\right)-u_y^3u_x\xi^x_{uuu}-u_y^3u_t\xi^t_{uuu}\\
&+u_y^2u_t\left(\xi^t_{xu}+r(x)\xi^y_{uu}-3\xi^t_{yuu}\right)+u_y^2u_x\left(\xi^x_{xu}-\xi^y_{yu}-3\xi^x_{yuu}\right)\\
&+u_y^2\left(r(x)\xi^y_{tu}+3\eta^u_{yuu}+\xi^y_{xy}-\eta^u_{xu}-3\xi^y_{yyu}\right)+u_yu_t^2r(x)\xi^t_{uu}\\
&+u_yu_tu_x\left(r(x)\xi^x_{uu}-\xi^t_{yu}\right)+u_yu_t\left(r(x)\xi^t_{tu}-r(x)\eta^u_{uu}+r(x)\xi^y_{yu}+\xi^t_{xy}\right.\\
&\left.-3\xi^t_{yyu}\right)-u_yu_x^2\xi^x_{yu}+u_yu_x\left(\xi^x_{xy}+\eta^u_{yu}+r(x)\xi^x_{tu}-3\xi^x_{yyu}-\xi^y_{yy}\right)\\
&+u_y\left(3\eta^u_{yyu}+r(x)\xi^y_{ty}+4f(x,t)\xi^y_u-\eta^u_{xy}-r(x)\eta^u_{tu}-\xi^y_{yyy}\right)\\
&+u_t^2r(x)\xi^t_{yu}+u_tu_x\left(r(x)\xi^x_{yu}-\xi^t_{yy}\right)+u_t\left(f(x,t)\xi^t_u+r(x)\xi^t_{yt}-\xi^t_{yyy}-\eta^u_{yu}\right)\\
&-u_x^2\xi^x_{yy}+u_x(r(x)\xi^x_{yt}+\eta^u_{yy}+f(x,t)\xi^x_u-\xi^x_{yyy})+\eta^u_{yyy}-r(x)\eta^u_{yt}\\
&\qquad\qquad-f(x,t)\eta^u_u+\xi^tf_t(x,t)+\xi^xf_x(x,t)+3f(x,t)\xi^y_y=0\,.
\end{aligned}
$$

Relation (6) is fulfilled for all values of the derivatives of the function $u(x,y,t)$. Splitting it into various partial derivatives of $u(x,y,t)$, we obtain a system of determining equations

$$
\begin{aligned}
&\xi^x_u=\xi^t_u=\xi^y_u=\xi^x_y=\xi^t_y=\xi^t_x=\eta_{uu}=\eta_{yu}=\eta_{yy}=\xi^y_{yy}=\eta_{xu}=0\,,\\
&\xi^y_{xy}=r(x)\xi^y_t+\eta_x=r(x)\xi^x_t-\eta_y=-r'(x)\xi^x+r(x)\xi^t_t-2r(x)\xi^y_y\\
&\quad=\xi^x-\eta_u-\xi^y_y=-\eta_{xy}+r(x)\xi^y_t-r(x)\eta_{ut}\\
&\quad=-r(x)\eta_{yt}+\xi^tf_t(x,t)+\xi^xf_x(x,t)+(3\xi^y_y-\eta_u)f(x,t)=0\,.
\end{aligned}
\tag{7}
$$

3.2. Solving the System of Determining Equations

It follows from the system of determining Equation (7) that the components of the allowed symmetry operators can be searched in the following form:

$$
\begin{aligned}
&\xi^x=a_1(t)x+a_2(t)\,,\qquad \xi^y=b_1(t)y+b_2(x,t)\,,\\
&\xi^t=c_1(t)\,,\qquad \eta^u=d_1(x,t)y+d_2(t)u+d_3(x,t)\,.
\end{aligned}
\tag{8}
$$

Then, the system of determining Equation (7) can be written as follows:

$$
\begin{aligned}
&d_2(t)-a_1(t)+b_1(t)=0\,,\qquad r(x)(a_1'(t)x+a_2'(t))-d_1(x,t)=0\,,\\
&r(x)b_1'(t)+d_{1x}(x,t)=0\,,\qquad r(x)d_2'(t)+d_{1x}(x,t)-r(x)b_1'(t)=0\,,\\
&r(x)c_1'(t)-(a_1(t)x+a_2(t))r'(x)-2r(x)b_1(t)=0\,,\\
&c_1(t)f_t(x,t)+(a_1(t)x+a_2(t))f_x(x,t)\\
&\qquad\qquad+(3b_1(t)-d_2(t))f(x,t)-r(x)d_{1t}(x,t)=0\,.
\end{aligned}
\tag{9}
$$

From the third and fifth equations of system (9), one can obtain

$$d_2'(t)=2b_1'(t)$$

or

$$d_2(t)=2b_1(t)+a\,,\qquad a=\text{const}\,. \tag{10}$$

Given relation (10), the first equation of the system of Equation (9) will take the form

$$a_1(t)=3b_1(t)+a.$$

Further, excluding $d_1(x,t)$ from the third and sixth equations of the system by virtue of the second, system (9) can be rewritten as follows:

$$\begin{aligned}
&d_1(x,t) = r(x)(3b_1'(t)x + a_2'(t)),\\
&4r(x)b_1'(t) + r'(x)(3b_1'(t)x + a_2'(t)) = 0,\\
&r(x)(c_1'(t) - 2b_1(t)) - r'(x)((3b_1(t) + a)x + a_2(t)) = 0,\\
&c_1(t)f_t(x,t) + ((3b_1(t) + a)x + a_2(t))f_x(x,t)\\
&\qquad + (b_1(t) - a)f(x,t) - r^2(x)(3b_1''(t)x + a_2''(t)) = 0.
\end{aligned} \tag{11}$$

3.3. Group Classification Results

The analysis of the system of determining Equation (11) shows that it has a solution only in the following cases.

1. $r(x)$, $f(x,t)$ are arbitrary functions. In this case, the kernel of the symmetry operators of Equation (5) is infinite-dimensional and consists of symmetry operator

$$X = b_2(x,t)\frac{\partial}{\partial y} + d_3(x,t)\frac{\partial}{\partial u}, \tag{12}$$

where the functions $b_2(x,t)$, $d_3(x,t)$ satisfy the relation

$$r(x)b_{2t}(x,t) + d_{3x}(x,t) = 0.$$

2. $r(x) = \alpha(x+\beta)^{-4/3}$.

2.1. $f(x,t) = \alpha^2(x+\beta)^{-5/3}q(t)$. In this case, the kernel of symmetry operators is expanded by operators

$$\begin{aligned}
X_i = &-\frac{3c_i'(t)}{2}(x+\beta)\frac{\partial}{\partial x} - \frac{c_i'(t)}{2}y\frac{\partial}{\partial y} + c_i(t)\frac{\partial}{\partial t}\\
&-\left(\frac{3\alpha}{2}(x+\beta)^{-\frac{1}{3}}c_i''(t)y + c_i'(t)u\right)\frac{\partial}{\partial u}, \qquad i = 1,2,3,\\
X_4 = &\ (x+\beta)\frac{\partial}{\partial x} + \frac{2y}{3}\frac{\partial}{\partial y} + \frac{u}{3}\frac{\partial}{\partial u}.
\end{aligned}$$

Here, $c_i(t)$ are linearly independent solutions of the ordinary differential equation

$$3c'''(t) + 4c'(t)q(t) + 2c(t)q'(t) = 0.$$

2.2. $f(x,t) = g(x)(t+\delta)^{-2}$, $g(x) \neq \varepsilon(x+\beta)^{\kappa}$. In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = y\frac{\partial}{\partial y} + 2(t+\delta)\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

3. $r(x) = \alpha(x+\beta)^{\gamma}$, $\gamma \neq -4/3; 0$.

3.1. $f(x,t) = 0$. In this case, the kernel of symmetry operators is expanded by the operators

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \quad X_2 = y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},\\
&X_3 = 2(x+\beta)\frac{\partial}{\partial x} - \gamma y\frac{\partial}{\partial y} + (\gamma+2)u\frac{\partial}{\partial u}.
\end{aligned} \tag{13}$$

3.2. $\boldsymbol{f(x,t) = const,\ const \neq 0}$**.** In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = 2(x+\beta)\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + (2\gamma+1)t\frac{\partial}{\partial t} + \frac{3u}{2}\frac{\partial}{\partial u}.$$

3.3. $\boldsymbol{f(x,t) \neq \delta(x+\beta)^{\kappa}}$**.** In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = \frac{\partial}{\partial t}.$$

3.4. $\boldsymbol{f(x,t) = \delta(x+\beta)^{\kappa}}$**.** In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = 2(x+\beta)\frac{\partial}{\partial x} - \frac{(\kappa-1)y}{2}\frac{\partial}{\partial y} + (2\gamma+1-\kappa)t\frac{\partial}{\partial t} + \frac{(\kappa+3)u}{2}\frac{\partial}{\partial u}.$$

3.5. $\boldsymbol{f(x,t) = \delta(t+\varepsilon)^{-2}}$**.** In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = (x+\beta)\frac{\partial}{\partial x} + \frac{y}{4}\frac{\partial}{\partial y} + \frac{3u}{4}\frac{\partial}{\partial u},$$
$$X_2 = \frac{y}{2} + (t+\varepsilon)\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}.$$

3.6. $\boldsymbol{f(x,t) = \delta(t+\varepsilon)^{\kappa},\ \kappa \neq -2}$**.** In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = (x+\beta)\frac{\partial}{\partial x} - \frac{(\kappa\gamma-1)}{2(\kappa+2)}\frac{\partial}{\partial y} + \frac{(2\gamma+1)(t+\varepsilon)}{\kappa+2}\frac{\partial}{\partial t} + \frac{(3+\kappa(\gamma+2))}{2(\kappa+2)}\frac{\partial}{\partial u}.$$

3.7. $\boldsymbol{f(x,t) = \delta\exp(\varepsilon t),\ \varepsilon \neq 0}$**.** In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = (x+\beta)\frac{\partial}{\partial x} - \frac{\gamma y}{2}\frac{\partial}{\partial y} + \frac{2\gamma+1}{\varepsilon}\frac{\partial}{\partial t} + \frac{\gamma+2}{2}\frac{\partial}{\partial u}.$$

3.8. $\boldsymbol{f(x,t) = g(t)(x+\beta)^{2\gamma+1}}$**.** In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = (x+\beta)\frac{\partial}{\partial x} - \frac{\gamma y}{2}\frac{\partial}{\partial y} + \frac{(\gamma+2)u}{2}\frac{\partial}{\partial u}.$$

3.9. $\boldsymbol{f(x,t) = g(x)(t+\varepsilon)^{-2}}$**.** In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = \frac{y}{2}\frac{\partial}{\partial y} + (t+\varepsilon)\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}.$$

3.10. $\boldsymbol{f(x,t) = (x+\beta)^{2\kappa+2\gamma+1}g((x+\beta)^{\kappa}(t+\varepsilon))}$**.** In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = -2(x+\beta)\frac{\partial}{\partial x} + y(\gamma+\kappa)\frac{\partial}{\partial y} + 2\kappa(t+\varepsilon)\frac{\partial}{\partial t} - (\gamma+\kappa+2)u\frac{\partial}{\partial u}.$$

4. $r(x) = \alpha \exp(\beta x),\ \alpha\beta \neq 0.$

4.1. $f(x,t) = 0.$ In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$X_3 = 2\frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} + \beta u\frac{\partial}{\partial u}.$$

4.2. $f(x,t) = const,\ const \neq 0.$ In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} + \beta t\frac{\partial}{\partial t}.$$

4.3. $f(x,t) = \delta \exp(\gamma x).$ In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = -4\frac{\partial}{\partial x} + \gamma y\frac{\partial}{\partial y} + 2(\gamma - 2\beta)t\frac{\partial}{\partial t} - \gamma u\frac{\partial}{\partial u}.$$

4.4. $f(x,t) = \gamma(t+\varepsilon)^{\delta}.$ In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = \frac{\delta+2}{\beta}\frac{\partial}{\partial x} - \frac{\delta y}{2}\frac{\partial}{\partial y} + 2(t+\varepsilon)\frac{\partial}{\partial t} + \frac{\delta u}{2}\frac{\partial}{\partial u}.$$

4.5. $f(x,t) = \gamma \exp(\varepsilon t).$ In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = \frac{\varepsilon}{\beta}\frac{\partial}{\partial x} - \frac{\varepsilon y}{2}\frac{\partial}{\partial y} + 2\frac{\partial}{\partial t} + \frac{\varepsilon u}{2}\frac{\partial}{\partial u}.$$

4.6. $f(x,t) = g(t)\exp(2\beta x).$ In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = 2\frac{\partial}{\partial x} - \beta y\frac{\partial}{\partial y} + \beta u\frac{\partial}{\partial u}.$$

4.7. $f(x,t) = g(x)(t+\varepsilon)^{-2}.$ In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = y\frac{\partial}{\partial y} + 2(t+\varepsilon)\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

5. $r(x)$ **is arbitrary,** $r(x) \neq \alpha(x+\beta)^{\gamma},\ r(x) \neq \alpha\exp(\beta x).$

5.1. $f(x,t) = 0.$ In this case, the kernel of symmetry operators is expanded by the operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

5.2. $f(x,t) = const,\ const \neq 0$ **or** $f(x,t) = g(x).$ In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = \frac{\partial}{\partial t}.$$

5.3. $f(x,t) = g(x)(t+\varepsilon)^{-2}.$ In this case, the kernel of symmetry operators is expanded by the operator

$$X_1 = y\frac{\partial}{\partial y} + 2(t+\varepsilon)\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

Thus, all cases of expansion of the kernel of the symmetry operators of Equation (5) are listed for all possible functions $r(x)$ and $f(x,t)$.

Proposition 1. *For arbitrary functions $r(x)$ and $f(x,t)$, Equation (5) admits an infinite-dimensional kernel of symmetry operators of the following form:*

$$X = a(x,t)\frac{\partial}{\partial y} + b(x,t)\frac{\partial}{\partial u},$$

where the functions $a(x,t)$, $b(x,t)$ satisfy the relation $r(x)a_t(x,t) + b_x(x,t) = 0$. The largest kernel expansion is allowed in case 2.1 when

$$r(x) = \alpha(x+\beta)^{-4/3}, \qquad f(x,t) = q(t)(x+\beta)^{-5/3},$$

where $\alpha \neq 0$ and the function $q(t)$ is arbitrary. In this case, the kernel of symmetry operators is expanded by a four-dimensional Lie algebra.

4. Non-Existence of the Unsteady Analogue of Stepanov–Mangler Transformation

Consider the equation of an unsteady flat boundary layer [2]

$$u_{ty} + u_y u_{xy} - u_x u_{yy} - u_{yyy} - F(x,t) = 0. \tag{14}$$

Equation (14) is a particular case of Equation (5). It coincides with it when $r(x) = 1$, $f(x,t) = F(x,t)$.

The components of the allowed symmetry operators of Equation (14) should also be searched for in the form (8). The system of determining equations can also be obtained by substitution in the system of determining Equation (9) $r(x) = 1$, $f(x,t) = F(x,t)$

$$\begin{aligned}
&d_2(t) - a_1(t) + b_1(t) = 0, \qquad a_1'(t)x + a_2'(t) - d_1(x,t) = 0,\\
&b_1'(t) + d_{1x}(x,t) = 0, \qquad d_2'(t) + d_{1x}(x,t) - b_1'(t) = 0,\\
&c_1'(t) - 2b_1(t) = 0,\\
&c_1(t)F_t(x,t) + (a_1(t)x + a_2(t))F_x(x,t)\\
&\qquad + (3b_1(t) - d_2(t))F(x,t) - d_{1t}(x,t) = 0.
\end{aligned} \tag{15}$$

From the system of Equation (15), it is not difficult to obtain that

$$b_1(t) = b_{10} = \text{const}, \qquad d_2(t) = d_{20} = \text{const}, \qquad a_1(t) = b_{10} + d_{10} = \text{const}.$$

Next, $d_1(x,t) = a_2'(t)$, $c_1(t) = 2b_{10}t + c_{20}$. Then, the classifying equation for the function $F(x,t)$ will take the following form:

$$\begin{aligned}
&(a_{10}x + a_2(t))F_x(x,t) - a_2''(t) + (2b_{10}t + c_{20})F_t(x,t)\\
&\qquad + (3b_{10} - d_{20})F(x,t) = 0.
\end{aligned} \tag{16}$$

Proposition 2. *The kernel of the symmetry operators of Equation (14) has the form (12)*

$$X = b_2(x,t)\frac{\partial}{\partial y} + d_3(x,t)\frac{\partial}{\partial u},$$

where the functions $b_2(x,t)$, $d_3(x,t)$ satisfy the relation

$$b_{2t}(x,t) + d_{3x}(x,t) = 0.$$

Theorem 1. *There is no analog of the Stepanov–Mangler transformation for unsteady equations of flat and axisymmetric boundary layers.*

Proof of Theorem 1. Consider the extension of the kernel of the symmetry operators of Equation (14), allowed for $F(x,t) = 0$. Substituting $F(x,t) = 0$ into Equation (16) leads to the relation $a_2''(t) = 0$ or $a_2(t) = a_{20}t + a_{30}$. Then the kernel of the symmetry operators of Equation (14) is expanded by a five-dimensional Lie subalgebra with the following basis operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \qquad X_2 = t\frac{\partial}{\partial x} + y\frac{\partial}{\partial u}, \qquad X_3 = \frac{\partial}{\partial t}, \\ X_4 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t}, \qquad X_5 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \end{aligned} \tag{17}$$

It is shown in Proposition 1 that the kernel of the symmetry operators of Equation (5) has the widest expansion with Lie subalgebra with dimensions equal to four. If there were an analogue to the Stepanov–Mangler transformation, then the algebras of symmetry operators of Equations (14) and (5) should be isomorphic. But subalgebras with different dimensions could not be isomorphic. This means that there is no analogue of the Stepanov–Mangler transformation for Equations (14) and (5). □

5. Conclusions

We have considered the unsteady equations of flat and axisymmetric boundary layers. For the unsteady axisymmetric boundary layer equation, we have solved the group classification problem. We have shown that the kernel of symmetry operators can be extended by no more than a four-dimensional Lie algebra. We have found the kernel of symmetry operators of the unsteady flat boundary layer equation and have shown that it can be extended by no more than a five-dimensional Lie algebra. We have proved the non-existence of the unsteady analogue of the Stepanov–Mangler transformation.

The results of group classification can be used to construct new exact solutions and reductions of the unsteady axisymmetric boundary layer equation.

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