mathematics

Editorial

Preface to: Submanifolds in Metric Manifolds

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Citation: Hretcanu, C.E. Preface to: Submanifolds in Metric Manifolds. *Mathematics* **2024**, *12*, 1102. https://doi.org/10.3390/math12071102

Received: 25 March 2024
Accepted: 2 April 2024
Published: 7 April 2024

The present editorial contains 11 research articles, published in the Special Issue entitled *"Submanifolds in metric manifolds"* of the MDPI *mathematics* journal, which cover a wide range of topics from differential geometry in relation to the theory and applications of the structure induced on submanifolds by the structure defined on various ambient manifolds. The geometry of some particular manifolds and their submanifolds, with examples and applications (characterization properties and inequality or equality cases), is studied.

The concept of Riemannian submersion between Riemannian manifolds is very popular in theoretical physics, as well as in differential geometry. In Contribution **1**, the authors consider a contact-complex Riemannian submersion, which puts the almost-contact-metric structure from the domain manifold to the almost-Hermitian structure of the target manifold. The authors provide several new results, showing mainly when the base manifold admits a Ricci soliton when it is Einstein, when the fibres are η-Ricci solitons, and when they are η-Einstein.

A classical challenge in Riemannian geometry is to discuss the geometrical and topological structures of submanifolds. In Contribution **2**, the authors obtain some topological characterizations for the warping function of a warped product pointwise semi-slant submanifold in a complex projective space (where the constant sectional curvature c = 4). Some important applications of this theory can be found for the singularity structure in liquid crystals, in the system in statistical mechanics with low dimensions, and physical phase transitions. Since paper **2** considers both the warped product manifold and the homotopy–homology theory, its results can be used as part of physical applications.

In Contribution **3**, the authors prove the non-existence of stable integral currents in a compact oriented warped product pointwise semi-slant submanifold of a complex space form, under extrinsic conditions, which involve the Laplacian, the squared norm gradient of the warped function, and pointwise slant functions. The authors investigate the curvature and topology of submanifolds in a Riemannian manifold and the usual sphere theorems in Riemannian geometry. The second target of paper **3** is to establish topological sphere theorems from the viewpoint of warped product submanifold geometry with positive constant sectional curvature and pinching conditions in terms of the squared norm of the warping function and the Laplacian of the warped function as extrinsic invariants.

The importance of spaces with constant curvature is well understood in cosmology. A cosmological model of the universe is obtained by assuming that the universe is isotropic and homogeneous. In Contribution **4**, the authors focuse on the *Z-symmetric manifold* admitting certain types of Schouten tensor and they find some properties of Z-symmetric spacetime admitting Codazzi-type Schouten tensor.

The Kähler manifold is the subject of symplectic geometry. Contact geometry appears as the odd dimensional counterpart of symplectic geometry, in which the almost-contact manifold corresponds to the almost-complex manifold. In Contribution **5**, the authors study the $*$-Ricci operator on Hopf real hypersurfaces in the complex quadric. As the correspondence to the semi-symmetric Ricci tensor, the authors give a classification of real hypersurfaces in the complex quadric with the semi-symmetric $*$-Ricci tensor.

In Contribution **6**, the author studies the geometry of the almost-paracontact and almost-paracomplex Riemannian manifolds. The author defines the first natural connection and constructs a relation between this connection and the Levi–Civita connection. Moreover, some properties of curvature tensors, torsion tensors, Ricci tensors and scalar curvatures using these connections are given, along with an explicit five-dimensional example.

A statistical manifold can be considered as an expanse of a Riemannian manifold, such that the compatibility of the Riemannian metric is developed to a general condition. In Contribution **7**, the authors present nearly Sasakian and nearly Kähler structures on statistical manifolds and show the relation between these geometric notions. Moreover, the authors study some properties of (anti-)invariant statistical submanifolds of nearly Sasakian statistical manifolds.

The geometry of submanifolds in Golden or metallic Riemannian manifolds was widely studied by many geometers. The notion of a Golden structure was introduced 15 years ago and has been a constant interest of several geometers. In Contribution **8**, the authors propose a new generalization (apart from that called the metallic structure), named the almost (α, p)-Golden structure. By adding a compatible Riemannian metric, the authors focus on the study of the structure induced on submanifolds in this setting and its properties.

In Contribution **9**, the authors study a new holonomic Riemannian geometric model, associated with the Gibbs–Helmholtz equation from classical thermodynamics, in a canonical way. Using a specific coordinate system, the authors define a parameterized hypersurface in R^4 as the graph of the entropy function. The main geometric invariants of this hypersurface are determined and some of their properties are derived. Using this geometrization, the authors characterize the equivalence between the Gibbs–Helmholtz entropy and the Boltzmann–Gibbs–Shannon, Tsallis and Kaniadakis entropies, respectively, by means of three stochastic integral equations. They prove that some specific (infinite) families of normal probability distributions are solutions for these equations. This particular case offers a glimpse of the more general equivalence problem between classical entropy and statistical entropy.

The background of Contribution **10** is the total space TM of the tangent bundle of a Riemannian manifold (M, g), endowed with a metric G, constructed as a general natural lift of the metric from the base manifold. The author studies the conditions under which the (pseudo-)Riemannian manifold (TM, G), endowed with the Schouten–Van Kampen connection and associated with the Levi Civita connection of G, is a statistical manifold admitting torsion. The results obtained in this work lead to new examples of (quasi-)statistical structures on the tangent bundle of a Riemann manifold.

In Contribution **11**, the authors investigate the properties of the induced structure on a light-like hypersurface by a meta-Golden semi-Riemannian structure. Moreover, the authors study the properties of the invariant and anti-invariant light-like hypersurfaces, and screen semi-invariant light-like hypersurfaces of almost-meta-Golden semi-Riemannian manifolds. They also provide some useful examples.

As the Guest Editor of this Special Issue, I would like to thank the MDPI publishing editorial team, who gave me the opportunity to undertake the role of Guest Editor for the Special Issue “Submanifolds in metric manifolds”. I am very grateful to all the authors who have contributed through their research articles. Also, I would like to express my gratitude to all the reviewers for their valuable suggestions and critical comments, which have improved the quality of the papers in this Issue.

We hope that the selected research studies will have a positive impact on the international scientific community, inspiring other researchers to study and expand on the topics covered in this book.

Conflicts of Interest: The author declares no conflicts of interest.

List of Contributions

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4. Ali, M.; Haseeb, A.; Mofarreh, F.; Vasiulla, M. Z-Symmetric Manifolds Admitting Schouten Tensor. *Mathematics* **2022**, *10*, 4293.
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7. Uddin, S.; Peyghan, E.; Nourmohammadifar, L.; Bossly, R. On Nearly Sasakian and Nearly Kähler Statistical Manifolds. *Mathematics* **2023**, *11*, 12644.
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10. Druta-Romaniuc, S.L. Quasi-Statistical Schouten–van Kampen Connections on the Tangent Bundle. *Mathematics* **2023**, *11*, 4614.
11. Erdogan, F.E.; Perktas, S.Y.; Bozdag, S.N.; Acet, B.E. Lightlike Hypersurfaces of Meta-Golden Semi-Riemannian Manifolds. *Mathematics* **2023**, *11*, 4798.