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Analyticity of the Cauchy Problem for a Three-Component Generalization of Camassa–Holm Equation

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Abstract: In this paper, we investigate the Cauchy problem for a three-component generalization of Camassa–Holm equation (G3CH equation henceforth) with analytic initial data. The analyticity of its solutions is proved in both variables, globally in space and locally in time.

Keywords: G3CH equation; analyticity; abstract Cauchy–Kowalevski theorem

MSC: 35B30; 35G25; 35L05

1. Introduction

In this paper, we investigate the Cauchy problem for a G3CH equation given in Ref. [1]. as follows:

$$\begin{cases} u_t = -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\ v_t = 2vq_x + v_x q, \\ w_t = vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr), \\ u = p - p_{xx}, w = r_{xx} - r, \\ v = \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x - 3pr_x) \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), t > 0,\ x \in \mathbb{R}. \end{cases} \tag{1}$$

Based on the following spectral problem

$$\phi_x = U\phi,\ \ \phi = \begin{pmatrix} \phi_1 \\ \phi_1 \\ \phi_3 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 & 0 \\ 1+\lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix},$$

recently, Geng and Xue [1] proposed a new three-component Camassa–Holm system with N-peakon solutions (1). Here u, v, w are three potentials and λ is a constant spectral parameter. It was shown in [1] that the N-peakon solitons of the system (1) have the form

$$p(x,t) = \sum_{j=1}^{N} p_j(t)e^{-|x-x_j(t)|},$$
$$q(x,t) = \sum_{j=1}^{N} q_j(t)e^{-2|x-x_j(t)|},$$
$$r(x,t) = \sum_{j=1}^{N} r_j(t)e^{-|x-x_j(t)|},$$

where p_j, q_j, r_j, and x_j evolve according to a dynamical system (1). In Ref. [1], with the aid of the zero-curvature equation, they derived a hierarchy of new nonlinear evolution equations and established their Hamiltonian structures. Also, they demonstrated that (1) is

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exactly a negative flow in the hierarchy and admits exact solutions with N-peakons and an infinite sequence of conserved quantities.

When $p = r = 0$, the system (1) becomes

$$u_t = 2uq_x + u_x q, \;\; u = \frac{1}{2}(q_{xx} - 4q). \tag{2}$$

Using an appropriate scaling $\tilde{u}(x,t) = u(\frac{x}{2}, \frac{t}{2})$, $\tilde{q}(x,t) = -q(\frac{x}{2}, t)$ and using (2), one has

$$\tilde{u}_t + 2\tilde{u}\tilde{q} + \tilde{u}\tilde{q}_x = 0, \;\; \tilde{u} = \tilde{q} - \tilde{q}_{xx}, \tag{3}$$

which is nothing but the famous Camassa–Holm (CH) equation [2]. (3) models a one-dimensional unidirectional propagation of shallow water waves over a flat bottom under the influence of gravity, and $\tilde{u}(t,x)$ represents the fluid velocity at time t in the horizontal direction x. It is a well-known integrable equation describing the velocity dynamics of shallow water waves. Dropping the symbol ˜, (3) reduces

$$u_t + 2uq + uq_x = 0, \;\; u = q - q_{xx}, \tag{4}$$

In the case of the CH Equation (4), there are two local Hamilton structures [3] given by

$$q_t = B_0 \frac{\delta H_2}{\delta q} = B_1 \frac{\delta H_1}{\delta q},$$

$$B_0 = -\partial_x + \partial_x^3 = -\mathcal{L}, B_1 = -(q\partial_x + \partial_x q),$$
$$H_1 = \frac{1}{2}\int (u^3 + uu_x^2)dx, H_2 = \frac{1}{2}\int (u^2 + u_x^2)dx,$$

with $q = u - u_{xx}$, whose compatibility was known in [4]. In [2], Camassa and Holm showed that (4) has peaked solitary wave solutions (peakons) $u(t,x) = ce^{-|x-ct|}$, which have a discontinuous first derivative at the wave peak, in contrast to the smoothness of most previously known specious of solitary waves, and thus are called peakons. Also, (4) has the multi-peakon solutions

$$u(t,x) = \sum_{i=1}^{n} p_i(t)e^{-|x-q_i(t)|},$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system

$$\begin{cases} \frac{dp_i}{dt} = \sum\limits_{j\neq i} p_i p_j sgn(q_i - q_j)e^{-|q_i-q_j|} = -\frac{\partial H}{\partial q_i}, \\ \frac{dq_i}{dt} = \sum\limits_{j} p_j e^{-|q_i-q_j|-|q_i-q_k|} = \frac{\partial H}{\partial p_i}, \end{cases}$$

with the Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|}.$$

The CH Equation (4) is integrable in the sense of an infinite-dimensional Hamiltonian system and arises as a model for shallow water waves [2,5]. Indeed, the CH Equation (4) and its bi-Hamiltonian structure were earlier established by Fokas and Fuchssteiner [6]. Moreover, it also used to describe small amplitude radial deformation waves in cylindrical compressible hyper-elastic rods [7]. The local and global well-posedness of the Cauchy problem of the CH Equation (4) have been extensively investigated in [8]. It was shown that there are strong solutions to the CH Equation (4) [8] and finite time blow-up strong solutions to the CH Equation (4) [8,9]. The existence and uniqueness of global weak solutions to the CH Equation (4) were studied in [10,11]. Recently, the well-posedness, the scattering problem, and some qualitative properties for the CH Equation (4) were studied in Refs. [12–19] and the references therein.

Due to the singularity of strong solutions in finite time, we are forced to study weak solutions. In particular, in order to go beyond the breaking wave (i.e, the wave profile remains bounded but its slope becomes unbounded in finite time) [8,9], if one considers a global weak solution, it is natural to consider Hölder a continuous solution, for instance, the H^1 solution for CH (4). It is well known that there are two methods to deal with the global existence of weak solutions to CH (4). One method is the vanishing viscosity technique, see Refs. [10,11]. The other method is to introduce a new semi-linear system on new characteristic coordinates, see Refs. [8,9]. It is worth mentioning that Zhang et al. [12] investigated the global energy conservation solution for (4). More precisely, using both the lower order and the higher order energy conservation laws, as well as the characteristic method, they established the global existence and uniqueness of the Hölder continuous energy weak solution to (4) in the energy space $H^1(\mathbb{R}) \times W^{1,2N}(\mathbb{R})$. Also, they demonstrated that a very natural and interesting problem is to study how the regularity of solution changes with respect to N. Namely, they established Hölder continuous energy weak solutions with the exponent $1 - \frac{1}{2N}$. This improves the contributions in the literature in [20,21].

Quite recently, using the Littlewood–Paley theory and transport equations theory, Luo and Yin [22] proved the local well-posedness of the G3CH Equation (1) in Besov spaces $B^s_{p,r}$ with $p, r \in [1, \infty]$, $s > \max\{\frac{1}{p}, \frac{1}{2}\}$. Also, they established two blow-up criteria which, along with the conservation laws, enable us to study global existence. Moreover, when the initial data satisfy some certain sign conditions, they obtained a global existence result. Finally, they verified that the system possesses peakon solutions.

On the other hand, one of the most popular generalized systems is the following integrable two-component Camassa–Holm shallow water system (2CH) [13]:

$$\begin{cases} m_t + um_x + 2u_x m + k\rho\rho_x = 0, \ \ m = (1 - \partial_x^2)u, \\ \rho_t + (u\rho)_x = 0, \\ m(x,0) = m_0(x), \rho(x,0) = \rho_0(x). \end{cases} \tag{5}$$

There are many research results with respect to 2CH (5) [14,23–27], such as the blow-up of solutions, the well-posedness of solutions, the existence of weak solutions, the global solutions of the Cauchy problem, and wave-breaking criteria. In [14], Yan and Yin first proved the local well-posedness of the 2CH (5) in the Besov spaces. Then, under certain conditions on the initial data, they established the global existence and the finite lifespan. Also, in the case of finite time singularities, they demonstrated the precise blow-up scenario for breaking waves. In [23], they established the local well-posedness and precise blow-up scenarios for strong solutions for the 2CH (5). In [24], Guan and Yin investigated the global existence and blow-up phenomena for 2CH (5). Also, a new global existence result and several new blow-up results of strong solutions to the system are presented. Their obtained results for the system are sharp and considerably improve earlier results, such as [13]. In [25], in the sense of weakness, they proved the existence of a global weak solution to 2CH (5) given the initial data satisfying some certain conditions, which improves the earlier result, such as [24]. It is worth mentioning that Gui et al. [27] established the local well-posedness for the two-component Camassa–Holm system in a range of the Besov spaces and derived a wave-breaking mechanism for strong solutions. In addition, they determined the exact blow-up rate of such solutions to 2CH (5), which improves the earlier result, such as [26].

Another one is the modified two-component Camassa–Holm system (M2CH) [28]

$$\begin{cases} m_t + um_x + 2u_x m + k\rho\overline{\rho}_x = 0, \ \ m = (1 - \partial_x^2)u, \\ \rho_t + (u\rho)_x = 0, \ \ \rho = (1 - \partial_x^2)(\overline{\rho} - \overline{\rho}_0), \\ m(x,0) = m_0(x), \rho(x,0) = \rho_0(x). \end{cases} \tag{6}$$

There are lots of contributions concerning well-posedness, including local and global well-posedness as well as blow-up phenomena results for M2CH (6) (see, for instance, Refs. [29–33]). In [29], using the viscous approximation technique, they established the existence of global-in-time weak solutions for the Cauchy problem of M2CH (6). The key elements in their analysis are the Helly theorem and some a priori one-sided supernorm and space-time higher integrability estimates on the first-order derivatives of approximation solutions. In [30], Tan and Yin first proved the existence of global conservative solutions to the Cauchy problem for M2CH (6). Then, they demonstrated that these global solutions, which depend continuously on the initial date, construct a semigroup. This improves the contributions in the literature in [20]. In [32], introducing a new set of independent variables, they transformed M2CH (6) into a semilinear system. To obtain a dissipative solution, they modified the corresponding system into a discontinuous system. Then, they mapped the solution of system to the dissipative solution of original equation. Furthermore, they proved that these global dissipative solutions construct a semigroup. This improves the contribution in the literature in [31].

The analyticity of solutions to Euler equations of hydrodynamics has been studied extensively. It was initiated by [33,34] and later further developed in [35–39]. As mentioned before, we know that the approach is based on contraction-type arguments in a suitable scale of Banach spaces. It is well known that KdV solutions are analytic in the space variable for all time (see Trubowitz [40]) but are not analytic in the time variable (see Kato and Masuda [41], and Byers and Himonas [42]). However, the analytical properties of solutions to the CH (4) are quite different from those of the KdV, studied, for instance, in Bona and Smith [43]; Bourgain [44]; and Kenig, Ponce, and Vega [45,46]. For example, the Cauchy problem for CH (4) is not globally well-posed since for certain initial data the first derivative of the solution becomes unbounded in L^∞ norm in finite time, see [2,47].

Concerning the analyticity of solutions, it is worth mentioning that Himonas and Misiolek [48] considered the periodic Cauchy problem for the CH (4) with analytic initial data and proved that its solutions are analytic in both variables, globally in space and locally in time. Later on, Yan and Yin [49] investigated the periodic Cauchy problem for the M2CH (5) with analytic initial data and proved that its solutions are analytic in both variables, globally in space and locally in time. Quite recently, Yan and Yin [50] investigated the higher dimensional Camassa–Holm equations

$$\begin{cases} m_t + um_x + u \cdot \nabla m + \nabla u^T \cdot m + m(divu) = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{7}$$

where the vector fields $u = u(x,t)$ and $m = m(x,t)$ are defined from $\mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}^d$ such that $m = (I - \Delta)u$ or $u = G * m$ with the Green function G for the Helmholtz operator $I - \Delta$. Also, the analyticity of solutions for (7) is proved in both variables, globally in space and locally in time. A natural idea is to extend such a study to the G3CH systems (1). We observe that the classical Cauchy–Kowalevski theorem does not apply to (1) since the initial line $t = 0$ is characteristic. Thus, the novel result can be viewed as an extended Cauchy–Kowalevski theorem for the nonlinear situation (1). To our best knowledge, the analyticity of the Cauchy problem for (1) has not been studied yet. In this paper, we will prove the analyticity of solutions to system (1) in both variables, with x on the real line $\mathbb{R}$ and t in a neighborhood of zero, provided that the initial data are analytic $\mathbb{R}$.

The rest of this paper is organized as follows. In Section 2, we obtain the analytic solutions to the system (1) on the line $\mathbb{R}$, and we present the proof of our results (i.e., Theorem 2). In Section 3, we present the conclusions for our paper.

2. Analytic Solutions to the System (1)

In this section, we shall show the existence and uniqueness of analytic solutions to the system (1) on the line $\mathbb{R}$. Before proceeding to our analysis, we present some notations that will be used throughout our paper.

Let

$$P_1 = (1-\partial_x^2)^{-1},\ P_2 = (4-\partial_x^2)^{-1},\ P_3 = \partial_x,$$
$$p = P_1 u,\ r = P_1 u,$$

$$q = P_2(-w \cdot P_1P_3u + u \cdot P_1P_3w) + 2P_2(-P_1P_3u \cdot P_1w + P_1u \cdot P_1P_3w) - 2P_2v = B(u,w) - 2P_2v.$$

To further facilitate our analysis, we need to rewrite the system (1) in the following non-local form:

$$\begin{cases} U_t = F(U), \\ U(x,0) = U_0. \end{cases} \tag{8}$$

Here, $U = (u,v,w)^T$, $U_0 = (u_0,v_0,w_0)^T$, and

$$F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \\ F_3(U) \end{pmatrix} \tag{9}$$

where

$$\begin{aligned} F_1(U) = &-v \cdot P_1P_3u + P_3u(B(u,w) - 2P_2v) + \tfrac{3}{2}u(P_3B(u,w) - 2P_2P_3v) \\ &-\tfrac{3}{2}u(-P_1P_3u \cdot P_1P_3w + P_1u \cdot P_1w), \end{aligned}$$

$$F_2(U) = 2v \cdot (P_3B(u,w) - 2P_2P_3v) + P_3v \cdot (B(u,w) - 2P_2v),$$

$$\begin{aligned} F_3(U) = &-v \cdot P_1P_3w + P_3w(B(u,w) - 2P_2P_3v) + \tfrac{3}{2}w(P_3B(u,w) - 2P_2v) \\ &+\tfrac{3}{2}w(-P_1P_3u \cdot P_1P_3w + P_1u \cdot P_1w). \end{aligned}$$

Next, before stating the main result of this section, we first introduce the following theorem.

Theorem 1 ([36,37])**.** *Let $(X_s, |||\cdot|||_s), s > 0$ be a scale of decreasing Banach spaces, such that for any $0 < s' \le s$, we have $X_s \subset X_{s'}$ with $|||\cdot|||_{s'} \le |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases} u_t = F(t,u(t)), \\ u(0) = 0. \end{cases}$$

Let T, R, and C be positive numbers and suppose that F satisfies the following conditions:

(i) If for any $0 < s' < s < 1$, the function $t \mapsto u(t)$ is holomorphic in $|t| < T$ and continuous on $|t| \le T$ with values in X_s and $\sup\limits_{|t|\le T} |||u(t)|||_s < R$, then $t \mapsto F(t,u(t))$ is a holomorphic functionon $|t| < T$ with values in $X_{s'}$.

(ii) For any $0 < s' < s \le 1$, and any $u,v \in B(0,R) \subset X_s$, that is, $|||u|||_s < R$, $|||v|||_s < R$, we have

$$\sup_{|t|\le T} |||F(t,u) - F(t,v)|||_{s'} \le \frac{C}{s-s'}|||u-v|||_s.$$

(iii) $M > 0$ exists such that for any $0 < s < 1$,

$$\sup_{|t|\le T} |||F(t,0)|||_s \le \frac{M}{1-s}.$$

Then, $T_0 \in (0,T)$ exists, along with a unique function $u(t)$, which is holomorphic in $t < (1-s)T_0$ with values in X_s for every $s \in (0,1)$ and is a solution to the above IVP.

We are now in position to state our main theorem.

Theorem 2. *Let $(u_0,v_0,w_0)^T$ be a real analytic function on $\mathbb{R}$. Then, $\varepsilon > 0$ exists, along with a unique solution $(u,v,w)^T$ of the IVP (1) that is analytic on $(-\varepsilon,\varepsilon) \times \mathbb{R}$.*

Now, we use a contraction argument to analytic solutions to the system (1). For that purpose, we will need a suitable scale of Banach spaces. For any $s > 0$, we define the spaces

$$E_s = \{u \in C^\infty(\mathbb{R}) : |||u|||_s = \sup_{k\in\mathbb{N}} \frac{s^k \|\partial^k\|_{H^r(\mathbb{R})}}{k!/(k+1)^2} < \infty\},$$

where $r > \frac{1}{2}$ is any fixed real number. It is obvious that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space, and for any $0 < s' < s$, $E_s \subset E_{s'}$ with $|||u|||_{s'} \le |||u|||_s$. Note that any u in E_s is real analytic on $\mathbb{R}$.

Let

$$|||U|||_s = |||u|||_s + |||v|||_s + |||w|||_s,$$
$$|||F(U)|||_s = |||F_1(U)|||_s + |||F_2(U)|||_s + |||F_3(U)|||_s.$$

We will use Theorem 1 to prove Theorem 2. The conditions (i) and (iii) in Theorem 1 can be easily verified once our system (1) is transformed into a new system with zero initial data as in (8). In order to verify Theorem 2, it suffices to let the system (8) satisfy the following condition.

Proposition 1. *Let $R > 0$; there is a constant $C > 0$ such that for any $0 < s' < s \le 1$, we have*

$$|||F(U_1) - F(U_2)|||_{s'} \le \frac{C}{s - s'}|||U_1 - U_2|||_s.$$

for any U_1 and U_2 in the ball $B(0, R) \subset E_s$.

To establish Proposition 1, we need the following two lemmas:

Lemma 1 ([49])**.** *Let $0 < s < 1$; there is a constant $C > 0$, independent of s, such that for any f and g in E_s, we have*

$$|||fg|||_s \le C|||f|||_s|||g|||_s,$$

where $C = C(r)$ depends only on r.

Lemma 2. *For any $0 < s' < s < 1$, we have*

$$|||P_1 f|||_s \le |||f|||_s, \tag{10}$$

$$|||P_2 f|||_s \le \frac{1}{4}|||f|||_s, \tag{11}$$

$$|||P_3 f|||_{s'} \le \frac{1}{s - s'}|||f|||_s, \tag{12}$$

$$|||P_1 P_3 f|||_s \le |||f|||_s, \tag{13}$$

$$|||P_2 P_1 f|||_s \le \frac{1}{4}|||f|||_s. \tag{14}$$

Proof. As for (10), (12), and (13), we can see Ref. [49] (p. 1119). Thus, we only prove (11) and (14). For the proof of (11), it suffices to show that

$$\|\partial^k P_2 f\|_{H^r(\mathbb{R})} \le \frac{1}{4}\|\partial^k f\|_{H^r(\mathbb{R})}.$$

Indeed,

$$\begin{aligned}\|\partial^k P_2 f\|^2_{H^r(\mathbb{R})} &= \int_{\mathbb{R}} (1+\xi^2)^r |\widehat{\partial^k P_2 f}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} (1+\xi^2)^r |\tfrac{\xi^k}{4+\xi^2} \widehat{f}(\xi)|^2 d\xi \\ &\le \tfrac{1}{16} \int_{\mathbb{R}} (1+\xi^2)^r |\xi^k \widehat{f}(\xi)|^2 d\xi \\ &= \tfrac{1}{16} \|\partial^k f\|^2_{H^r(\mathbb{R})}.\end{aligned}$$

For the proof of (14), it suffices to show that

$$\|\partial^k P_2 P_1 f\|_{H^r(\mathbb{R})} \le \frac{1}{4} \|\partial^k f\|_{H^r(\mathbb{R})}.$$

In a similar way, we have

$$\begin{aligned}\|\partial^k P_2 P_1 f\|^2_{H^r(\mathbb{R})} &= \int_{\mathbb{R}} (1+\xi^2)^r |\widehat{\partial^k P_2 P_1} f(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}} (1+\xi^2)^r |\tfrac{\xi^k}{(1+\xi^2)(4+\xi^2)} \widehat{f}(\xi)|^2 d\xi \\ &\le \tfrac{1}{16} \int_{\mathbb{R}} (1+\xi^2)^r |\xi^k \widehat{f}(\xi)|^2 d\xi \\ &= \tfrac{1}{16} \|\partial^k f\|^2_{H^r(\mathbb{R})}.\end{aligned}$$

This completes the proof of Lemma 2. □

Next, by Lemmas 1 and 2, we prove Proposition 1.

Proof. It follows from (9) that

$$\begin{aligned}F_1(U_1) - F_1(U_2) = &-v_1 \cdot P_1 P_3 u_1 + P_3 u_1 (B(u_1, w_1) - 2P_2 v_1) \\ &+ \tfrac{3}{2} u_1 (P_3 B(u_1, w) - 2P_2 P_3 v_1) - \tfrac{3}{2} u_1 (-P_1 P_3 u_1 \cdot P_1 P_3 w_1 + P_1 u_1 \cdot P_1 w_1) \\ &+ v_2 \cdot P_1 P_3 u_2 - P_3 u_2 (B(u_2, w_2) - 2P_2 v_2) \\ &- \tfrac{3}{2} u_2 (P_3 B(u_2, w_2) - 2P_2 P_3 v_2) + \tfrac{3}{2} u_2 (-P_1 P_3 u_2 \cdot P_1 P_3 w_2 + P_1 u_2 \cdot P_1 w_2),\end{aligned} \tag{15}$$

$$\begin{aligned}F_2(U_1) - F_2(U_2) = &\, 2v_1 \cdot (P_3 B(u_1, w_1) - 2P_2 P_3 v_1) + P_3 v_1 \cdot (B(u_1, w_1) - 2P_2 v_1) \\ &- 2v_2 \cdot (P_3 B(u_2, w_2) - 2P_2 P_3 v_2) - P_3 v_2 \cdot (B(u_2, w_2) - 2P_2 v_2),\end{aligned} \tag{16}$$

$$\begin{aligned}F_3(U_1) - F_3(U_2) = &-v_1 \cdot P_1 P_3 w_1 + P_3 w_1 (B(u_1, w_1) - 2P_2 P_3 v_1) \\ &+ \tfrac{3}{2} w_1 (P_3 B(u_1, w_1) - 2P_2 v_1) + \tfrac{3}{2} w_1 (-P_1 P_3 u_1 \cdot P_1 P_3 w_1 + P_1 u_1 \cdot P_1 w_1) \\ &+ v_2 \cdot P_1 P_3 w_2 - P_3 w_2 (B(u_2, w_2) - 2P_2 P_3 v_2) \\ &- \tfrac{3}{2} w_2 (P_3 B(u_2, w_2) - 2P_2 v_2) - \tfrac{3}{2} w_2 (-P_1 P_3 u_2 \cdot P_1 P_3 w_2 + P_1 u_2 \cdot P_1 w_2).\end{aligned} \tag{17}$$

From (15), we have

$$\begin{aligned}|||F_1(U_1) - F_1(U_2)|||_{s'} \le &\, |||v_1 \cdot P_1 P_3 u_1 - v_2 \cdot P_1 P_3 u_2|||_{s'} + |||P_3 u_1 B(u_1, w_1) - P_3 u_2 (B(u_2, w_2)|||_{s'} \\ &+ 2|||P_3 u_1 P_2 v_1 - P_3 u_2 P_2 v_2)|||_{s'} + \tfrac{3}{2} |||u_1 P_3 B(u_1, w_1) - u_2 P_3 B(u_2, w_2)|||_{s'} \\ &+ 3|||u_1 P_2 P_3 v_1 - u_2 P_2 P_3 v_2|||_{s'} \\ &+ \tfrac{3}{2} |||u_1 P_1 P_3 u_1 \cdot P_1 P_3 w_1 - u_2 P_1 P_3 u_2 \cdot P_1 P_3 w_2|||_{s'} \\ &+ \tfrac{3}{2} |||u_1 P_1 u_1 \cdot P_1 w_1 - u_2 P_1 u_2 \cdot P_1 w_2|||_{s'} \\ \triangleq &\, I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.\end{aligned} \tag{18}$$

In what follows, using Lemmas 1 and 2, we shall to estimate I_i in the right-hand side of (18), respectively. □

Estimate I_1. It follows from (13) and (18) that

$$\begin{aligned}|||I_1|||_{s'} &= |||v_1 \cdot P_1 P_3 u_1 - v_2 \cdot P_1 P_3 u_2|||_{s'} \\ &\le |||(v_1 - v_2) P_1 P_3 u_1|||_{s'} + |||v_2 P_1 P_3 (u_1 - u_2)|||_{s'} \\ &\le |||v_1 - v_2|||_s |||P_1 P_3 u_1|||_s + |||v_2|||_s |||P_1 P_3 (u_1 - u_2)|||_s \\ &\le |||v_1 - v_2|||_s |||u_1|||_s + |||v_2|||_s |||u_1 - u_2|||_s \\ &\le C_1(r, R) |||U_1 - U_2|||_s \le \tfrac{C_1(r,R)}{s - s'} |||U_1 - U_2|||_s.\end{aligned}$$

Observing

$$B(u,w) = P_2(-w \cdot P_1P_3u + u \cdot P_1P_3w) + 2P_2(-P_1P_3u \cdot P_1w + P_1u \cdot P_1P_3w),$$

and using Lemma 2 (i.e., (10), (11), and (13)), we arrive at

$$\begin{aligned} |||B(u_1,w_1)|||_s &= |||P_2(-w_1 \cdot P_1P_3u_1 + u_1 \cdot P_1P_3w_1) + 2P_2(-P_1P_3u_1 \cdot P_1w_1 + P_1u_1 \cdot P_1P_3w_1)|||_s \\ &\leq \tfrac{1}{4}[||| - w_1 \cdot P_1P_3u_1 + u_1 \cdot P_1P_3w_1|||_s + 2||| - P_1P_3u_1 \cdot P_1w_1 + P_1u_1 \cdot P_1P_3w_1|||_s] \\ &\leq \tfrac{1}{4}[|||w_1|||_s|||P_1P_3u_1|||_s + |||u_1|||_s|||P_1P_3w_1|||_s \\ &\quad +2(|||P_1P_3u_1|||_s|||P_1w_1|||_s + |||P_1u_1|||_s|||P_1P_3w_1|||_s)] \\ &\leq \tfrac{1}{4}[|||w_1|||_s|||u_1|||_s + |||u_1|||_s|||w_1|||_s + 2(|||u_1|||_s|||w_1|||_s + |||u_1|||_s|||w_1|||_s)] \\ &= \tfrac{3}{2}|||u_1|||_s|||w_1|||_s, \end{aligned} \tag{19}$$

$$\begin{aligned} B(u_1,w_1) - B(u_2,w_2) &= P_2(-w_1 \cdot P_1P_3u_1 + u_1 \cdot P_1P_3w_1) \\ &\quad +2P_2(-P_1P_3u_1 \cdot P_1w_1 + P_1u_1 \cdot P_1P_3w_1) \\ &\quad +P_2(-w_2 \cdot P_1P_3u_2 + u_2 \cdot P_1P_3w_2) \\ &\quad +2P_2(-P_1P_3u_2 \cdot P_1w_2 + P_1u_2 \cdot P_1P_3w_2) \\ &= -P_2(w_1 \cdot P_1P_3u_1 - w_2 \cdot P_1P_3u_2) + P_2(u_1 \cdot P_1P_3w_1 - u_2 \cdot P_1P_3w_2) \\ &\quad -2P_2(P_1P_3u_1 \cdot P_1w_1 - P_1P_3u_2 \cdot P_1w_2) \\ &\quad +2P_2(P_1u_1 \cdot P_1P_3w_1 - P_1u_2 \cdot P_1P_3w_2) \\ &\triangleq J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{20}$$

For J_1, it follows from (20), Lemmas 1 and 2 (i.e., (11) and (13)) that

$$\begin{aligned} |||J_1|||_s &= ||| - P_2(w_1 \cdot P_1P_3u_1 - w_2 \cdot P_1P_3u_2)|||_s \\ &\leq \tfrac{1}{4}|||w_1 \cdot P_1P_3u_1 - w_2 \cdot P_1P_3u_2|||_s \\ &\leq \tfrac{1}{4}[|||(w_1 - w_2) \cdot P_1P_3u_1|||_s + |||w_2 \cdot P_1P_3(u_1 - u_2)|||_s] \\ &\leq \tfrac{1}{4}[|||w_1 - w_2|||_s|||P_1P_3u_1|||_s + |||w_2|||_s|||P_1P_3(u_1 - u_2)|||_s] \\ &\leq \tfrac{1}{4}(|||w_1 - w_2|||_s|||u_1|||_s + |||w_2|||_s|||u_1 - u_2|||_s) \\ &\leq \tilde{C}_1(r,R)|||U_1 - U_2|||_s. \end{aligned} \tag{21}$$

Similarly, we have

$$\begin{aligned} |||J_2|||_s &= |||P_2(u_1 \cdot P_1P_3w_1 - u_2 \cdot P_1P_3w_2)|||_s \\ &\leq \tilde{C}_2(r,R)|||U_1 - U_2|||_s, \end{aligned} \tag{22}$$

$$\begin{aligned} |||J_3|||_s &= ||| - 2P_2(P_1P_3u_1 \cdot P_1w_1 - P_1P_3u_2 \cdot P_1w_2)|||_s \\ &\leq \tilde{C}_3(r,R)|||U_1 - U_2|||_s, \end{aligned} \tag{23}$$

$$\begin{aligned} |||J_4|||_s &= |||2P_2(P_1u_1 \cdot P_1P_3w_1 - P_1u_2 \cdot P_1P_3w_2)|||_s \\ &\leq \tilde{C}_4(r,R)|||U_1 - U_2|||_s. \end{aligned} \tag{24}$$

Thus, it follows from (20)–(24) that

$$|||B(u_1,w_1) - B(u_2,w_2)|||_s \leq \tilde{C}_5(r,R)|||U_1 - U_2|||_s, \tag{25}$$

where

$$\tilde{C}_5(r,R) = \tilde{C}_1(r,R) + \tilde{C}_2(r,R) + \tilde{C}_3(r,R) + \tilde{C}_4(r,R).$$

Estimate I_2. It follows from (12), (18), (19), and (25) that

$$\begin{aligned} |||I_2|||_{s'} &= |||P_3u_1B(u_1,w_1) - P_3u_2(B(u_2,w_2)|||_{s'} \\ &\leq |||P_3(u_1 - u_2)|||_s|||B(u_1,w_1)|||_s + |||P_3u_2|||_s|||B(u_1,w_1) - B(u_2,w_2)|||_s \\ &\leq \tfrac{3}{2}\tfrac{1}{s-s'}|||u_1|||_s|||w_1|||_s|||u_1 - u_2|||_s + \tfrac{1}{s-s'}|||u_2|||_s\tilde{C}_5(r,R)|||U_1 - U_2|||_s \\ &\leq \tfrac{C_2(r,R)}{s-s'}|||U_1 - U_2|||_s. \end{aligned}$$

Estimate I_3. Using (11), (12), and (18), we obtain

$$\begin{aligned}|||I_3|||_{s'} &= 2|||P_3u_1P_2v_1 - P_3u_2P_2v_2)|||_{s'} \\ &\leq 2|||P_3(u_1-u_2)|||_s|||P_2v_1|||_s + |||P_3u_2|||_s|||P_2(v_1-v_2)|||_s \\ &\leq 2[\tfrac{1}{s-s'}|||u_1-u_2|||_s\tfrac{1}{4}|||v_1|||_s + \tfrac{1}{s-s'}|||u_2|||_s\tfrac{1}{4}|||v_1-v_2|||_s] \\ &\leq \tfrac{C_3(r,R)}{s-s'}|||U_1-U_2|||_s.\end{aligned}$$

Estimate I_4. Using (12), (18), (19), and (25), we obtain

$$\begin{aligned}|||I_4|||_{s'} &= \tfrac{3}{2}|||u_1P_3B(u_1,w_1) - u_2P_3B(u_2,w_2)|||_{s'} \\ &\leq \tfrac{3}{2}[|||(u_1-u_2)P_3B(u_1,w_1)|||_s + |||u_2P_3(B(u_1,w_1)-B(u_2,w_2))|||_s] \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||P_3B(u_1,w_1)|||_s + |||u_2|||_s|||P_3(B(u_1,w_1)-B(u_2,w_2))|||_s \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s\tfrac{1}{s-s'}|||B(u_1,w_1)|||_s + |||u_2|||_s\tfrac{1}{s-s'}|||B(u_1,w_1)-B(u_2,w_2)|||_s \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s\tfrac{1}{s-s'}\tfrac{3}{2}|||u_1|||_s|||w_1|||_s + |||u_2|||_s\tfrac{1}{s-s'}\tilde{C}_5(r,R)|||U_1-U_2|||_s \\ &\leq \tfrac{C_4(r,R)}{s-s'}|||U_1-U_2|||_s.\end{aligned}$$

Estimate I_5. Using (11), (12), and (18), we obtain

$$\begin{aligned}|||I_5|||_{s'} &= 3|||u_1P_2P_3v_1 - u_2P_2P_3v_2|||_{s'} \\ &\leq 3[|||P_2(u_1-u_2)P_3v_1|||_s + |||P_2u_2P_3(v_1-v_2)|||_s] \\ &\leq 3[|||P_2(u_1-u_2)|||_s|||P_3v_1|||_s + |||P_2u_2|||_s|||P_3(v_1-v_2)|||_s] \\ &\leq 3[\tfrac{1}{4}|||u_1-u_2|||_s\tfrac{1}{s-s'}|||v_1|||_s + \tfrac{1}{4}|||u_2|||_s\tfrac{1}{s-s'}|||v_1-v_2|||_s \\ &\leq \tfrac{C_5(r,R)}{s-s'}|||U_1-U_2|||_s.\end{aligned}$$

Estimate I_6. It follows from (13) and (18) that

$$\begin{aligned}|||I_6|||_{s'} &= \tfrac{3}{2}|||u_1P_1P_3u_1\cdot P_1P_3w_1 - u_2P_1P_3u_2\cdot P_1P_3w_2|||_{s'} \\ &= \tfrac{3}{2}|||[(u_1-u_2)P_1P_3u_1\cdot P_1P_3w_1 + u_2(P_1P_3u_1\cdot P_1P_3w_1 - P_1P_3u_2\cdot P_1P_3w_2)|||_s \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||P_1P_3u_1|||_s|||P_1P_3w_1|||_s + |||u_2|||_s|||P_1P_3u_1\cdot P_1P_3w_1 - P_1P_3u_2\cdot P_1P_3w_2|||_s] \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||u_1|||_s|||w_1|||_s + |||u_2|||_s|||P_1P_3u_1\cdot P_1P_3w_1 - P_1P_3u_2\cdot P_1P_3w_2|||_s].\end{aligned}$$

On the other hand, using (13), we have

$$\begin{aligned}&|||P_1P_3u_1\cdot P_1P_3w_1 - P_1P_3u_2\cdot P_1P_3w_2|||_s \\ &\quad= |||P_1P_3(u_1-u_2)\cdot P_1P_3w_1 + P_1P_3u_2\cdot P_1P_3(w_1-w_2)|||_s \\ &\quad\leq |||P_1P_3(u_1-u_2)|||_s|||P_1P_3w_1|||_s + |||P_1P_3u_2|||_s|||P_1P_3(w_1-w_2)|||_s \\ &\quad\leq |||u_1-u_2|||_s|||w_1|||_s + |||u_2|||_s|||w_1-w_2|||_s.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|||I_6|||_{s'} &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||u_1|||_s|||w_1|||_s \\ &\qquad+|||u_2|||_s(|||u_1-u_2|||_s|||w_1|||_s + |||u_2|||_s|||w_1-w_2|||_s)] \\ &\leq C_6(r,R)|||U_1-U_2|||_s \leq \tfrac{C_6(r,R)}{s-s'}|||U_1-U_2|||_s.\end{aligned}$$

Estimate I_7. It follows from (10) and (18) that

$$\begin{aligned}|||I_7|||_{s'} &= \tfrac{3}{2}|||u_1P_1u_1\cdot P_1w_1 - u_2P_1u_2\cdot P_1w_2|||_{s'} \\ &\leq \tfrac{3}{2}[|||(u_1-u_2)P_1u_1\cdot P_1w_1|||_s + |||u_2(P_1u_1\cdot P_1w_1 - P_1u_2\cdot P_1w_2)|||_s] \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||P_1u_1|||_s|||P_1w_1|||_s + |||u_2|||_s|||P_1u_1\cdot P_1w_1 - P_1u_2\cdot P_1w_2|||_s] \\ &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||u_1|||_s|||w_1|||_s + |||u_2|||_s|||P_1u_1\cdot P_1w_1 - P_1u_2\cdot P_1w_2|||_s].\end{aligned}$$

On the other hand, using (10), we have

$$\begin{aligned}|||P_1u_1\cdot P_1w_1 - P_1u_2\cdot P_1w_2|||_s &= |||P_1(u_1-u_2)\cdot P_1w_1 + P_1u_2\cdot P_1(w_1-w_2)|||_s \\ &\leq |||P_1(u_1-u_2)\cdot P_1w_1|||_s + |||P_1u_2\cdot P_1(w_1-w_2)|||_s \\ &\leq |||P_1(u_1-u_2)|||_s|||P_1w_1|||_s + |||P_1u_2|||_s|||P_1(w_1-w_2)|||_s \\ &\leq |||u_1-u_2|||_s|||w_1|||_s + |||u_2|||_s|||w_1-w_2|||_s.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}|||I_7|||_{s'} &\leq \tfrac{3}{2}[|||u_1-u_2|||_s|||u_1|||_s|||w_1|||_s+|||u_2|||_s(|||u_1-u_2|||_s|||w_1|||_s+|||u_2|||_s|||w_1-w_2|||_s)]\\ &\leq C_7(r,R)|||U_1-U_2|||_s \leq \tfrac{C_7(r,R)}{s-s'}|||U_1-U_2|||_s.\end{aligned}$$

Hence, using estimate $I_i (i = 1, 2, \dots, 7)$, we obtain

$$|||F_1(U_1)-F_1(U_2)|||_{s'} \leq \frac{C_8(r,R)}{s-s'}|||U_1-U_2|||_s,$$

where

$$\frac{C_8(r,R)}{s-s'} = \max\{\frac{C_i(r,R)}{s-s'}\},\ \ (i = 1, 2, \dots, 7).$$

Similarly, we have

$$|||F_2(U_1)-F_2(U_2)|||_{s'} \leq \frac{C_9(r,R)}{s-s'}|||U_1-U_2|||_s,$$

$$|||F_3(U_1)-F_3(U_2)|||_{s'} \leq \frac{C_{10}(r,R)}{s-s'}|||U_1-U_2|||_s.$$

Thus, we obtain

$$\begin{aligned}|||F(U_1)-F(U_2)|||_{s'} &= |||F_1(U_1)-F_1(U_2)|||_{s'}+|||F_2(U_1)-F_2(U_2)|||_{s'}+|||F_3(U_1)-F_3(U_2)|||_{s'}\\ &\leq \tfrac{C}{s-s'}|||U_1-U_2|||_s,\end{aligned}$$

where

$$\frac{C}{s-s'} = \max\{\frac{C_i(r,R)}{s-s'}\},\ \ (i = 8, 9, 10).$$

The proof of Proposition 1 is then completed.

By Theorem 1 and Proposition 1, thus we complete the proof of Theorem 2.

3. Conclusions

This paper is mainly interested with the Cauchy problem for a three-component generalization of the Camassa–Holm equation with analytic initial data. Based on contraction-type arguments in a suitable scale of Banach spaces, we establish that the analyticity of its solutions is proved in both variables, globally in space and locally in time.

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