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Using a Mix of Finite Difference Methods and Fractional Differential Transformations to Solve Modified Black–Scholes Fractional Equations

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Abstract: This paper discusses finding solutions to the modified Fractional Black–Scholes equation. As is well known, the options theory is beneficial in the stock market. Using call-and-pull options, investors can theoretically decide when to sell, hold, or buy shares for maximum profits. However, the process of forming the Black–Scholes model uses a normal distribution, where, in reality, the call option formula obtained is less realistic in the stock market. Therefore, it is necessary to modify the model to make the option values obtained more realistic. In this paper, the method used to determine the solution to the modified Fractional Black–Scholes equation is a combination of the finite difference method and the fractional differential transformation method. The results show that the combined method of finite difference and fractional differential transformation is a very good approximation for the solution of the Fractional Black–Scholes equation.

Keywords: modified fractional Black–Scholes; call option; put option; solution; finite difference method; fractional differential transformation method

MSC: 35A01; 35A02; 35R11

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1. Introduction

Nowadays, the development and application of mathematics have penetrated almost every area of life, including investment problems. Investment problems are one of the applications of mathematics in financial mathematics. Investors seek to buy or sell assets traded on financial markets to obtain maximum profits. Derivative assets are financial instruments whose value is determined by an underlying asset. One of the purposes of using derivative instruments is to reduce risk by hedging against possible adverse asset price movements. Options are a type of derivative product that is well known to many people.

In 1973, Fisher Black and Myron Scholes built a model for option values called the Black–Scholes model. The problem of determining the option value, which is determined by the value of the underlying asset at a particular time, is not only a problem in economics and finance but also in mathematics. The methods often used to solve the Black–Scholes equation include the Laplace transformation and the Ito integral [1,2]. By using the Stochastic Process, we finally obtain the formula for call-and-put options.

Mathematicians then developed the Black–Scholes equation model into a Fractional Black–Scholes equation. This Fractional Black–Scholes equation model is a generalized form of the Black–Scholes equation. Several methods for solving the Fractional Black–Scholes equation include a combination of homotopy perturbation methods, Sumudu Transformations, and He's polynomials [3]. The Fractional Black–Scholes equation can be solved using the series decomposition method. It confirms that Sumudu Transformation and fractional calculus are used to solve the Fractional Black–Scholes equation [4]. Meanwhile, the properties of Sumudu Transformation are used to solve partial differential equations [5,6]. The Fractional Black–Scholes equation can be solved using the series decomposition method, asserting that the Sumudu Transformation, combined with fractional calculus, is utilized to solve the Fractional Black–Scholes equation [7].

In the same year, Ref. [8] combined the Laplace transform and radial kernel methods to solve the Fractional Black–Scholes equation. According to several studies on analytical solutions, the Fractional Black–Scholes equation is an endless sequence of Mittag–Lefler functions. Ref. [9] researched the existence and uniqueness of solutions to the Fractional Black–Scholes equation. Banach's fixed point and Arzella Ascoli's fixed point have all been used to discuss the problem's existence and uniqueness. To discuss the numerical solution to the Fractional Black–Scholes issue, the Crank–Nickholson technique is used. Ref. [10] solved Burger's equation using the Modified Laplace Adomian decomposition technique in 2015. Burger's equation was represented using a partial differential equation. The Laplace Adomian decomposition method provides a precise approach for obtaining precise solutions and very speedy convergence of results. Various approaches can be used to solve partial differential equations, including homotopic perturbation, variational iteration, and Adomian decomposition methods. The embedding parameter in the homotopic perturbation method is quite small. It was assumed that the solution to the differential equation would be an infinite series. The Adomian decomposition approach makes use of Adomian polynomials. The absence of the discretization variable is the primary advantage of this strategy. Another advantage is the lack of the necessity for problem linearization, although these approaches were equivalent in terms of the rate of solution convergence.

The topic of European option pricing in the regime-switching model's FMLS (limited log-stable moment) was then investigated [11–13]. The Homotopic Analysis Method (HAM) was used in [14,15] to calculate the European Call Option (ECO) using the Time-Fractional Black–Scholes Equation (TFBSE), where stock prices are supposed to move according to geometric Brownian motion and do not pay dividends. The HAM has discovered a series of solutions for TFBSE. Furthermore, the ECO pricing calculation formula has been obtained. The efficacy, suitability, and correctness of the HAM were demonstratively investigated in the context of Crank Nicolson (CRN), Binomial Model (BM), and Black–Scholes Model (BSM) approaches, using two examples. Because it can converge to analytical results faster, the HAM is judged to be the best alternative instrument for determining ECO prices with fractional orders. Ref. [16] provided a numerical technique for the Time-Fractional Black–Scholes model, which is used in the fractional structural model within financial markets. This method uses an initial discretization based on time and a weighted finite difference spatial approach. Some spatial discretization characteristics are also investigated. A fundamental limitation of this technology is its inability to proceed in time layer by layer.

Common approaches for solving the Fractional Black–Scholes equation include the homotopic perturbation method, He's polynomials, and Adomian decomposition. The numerical operational transformation method is used. However, extreme vigilance is required when modifying time fractional derivatives. Mistakes can occur when using the differential operator after the time inversion techniques have changed. The constructed numerical model employs a variety of methodologies. The discrete, linear, and nonlinear characteristics of European Black–Scholes option pricing models are then captured by [17,18]. To achieve this, this article combines the third-order strong stability of the Runge–Kutta method with a sixth-order finite difference scheme. The findings from the current literature and its precise answers were examined and contrasted. The finite–difference method is a more popular common technique. The key challenge will be to find a more sophisticated model solution with an approach that aligns with precise results. Therefore, the strong stability approach of the third-order Runge–Kutta and the sixth-order finite–difference method must be combined to produce an efficient numerical solution. Asymptotic convergence has been demonstrated through convergence using the norm definition. Ref. [19]

provided a numerical technique for the Time-Fractional Black–Scholes model, which is used for the fractional structural model in financial markets. This method uses initial discretization based on time and a weighted finite difference spatial approach. In 2023, Ref. [20] used a simpler method, the Daftardar–Geijji method, to solve the Fractional Black–Scholes equation. It also discusses the problem of existence and the uniqueness of solutions of the Fractional Black–Scholes equation.

2. Formation of the Modified Fractional Black–Scholes Equation

Suppose $V(S,\tau)$ is the option value, while S is the value of the underlying asset, and τ is time. The total flux rate of option value $\overline{Y}(s,\tau)$ per unit of time from time τ to the expiration date T and the option value $V(S,\tau)$ must satisfy:

$$\int_\tau^T \overline{Y}(s,\tau')d\tau' = S^{d_f-1}\int_\tau^T H(\tau'-\tau)[V(S,\tau')-V(S,T)]d\tau'$$

where $H(\tau)$ is the transmission function, and d_f is the Haudorff dimension of the fractal transmission system. The equation above is called the conservation equation for the diffusion process of option values in a fractal structure. The transmission function $H(\tau)$ is defined as follows:

$$H(\tau) = m\frac{A_{2\gamma}}{\Gamma(1-2\gamma)t^{2\gamma}} + (1+mk)\frac{B_\gamma}{\Gamma(1-\gamma)t^\gamma}$$

So, the transmission function $H(\tau)$ is a linear combination of two other transmission functions. In this case, $A_{2\gamma}$ and B_γ are constants, while 2γ and γ are transmission exponents. By differentiating the conservation equation above with respect to τ, we obtain:

$$-\overline{Y}(s,\tau') = S^{d_f-1}\frac{d}{dt}\left(\int_\tau^T H(\tau'-\tau)\left[V(S,\tau')-V(S,T)\right]d\tau'\right).$$

Based on the modified Black–Scholes equation, we obtain:

$$\overline{Y}(s,\tau) = \frac{\partial^2 v}{\partial S^2} + (k-1)\frac{\partial v}{\partial S} - kv \tag{1}$$

Combined with Equation (1), we obtain:

$$A_{2\gamma}S^{d_f-1}m\,\frac{\partial^{2\gamma}v}{\partial\tau^{2\gamma}} + B_\gamma S^{d_f-1}(1+mk)\frac{\partial^\gamma v}{\partial\tau^\gamma} + \frac{\partial^2 v}{\partial S^2} + (k-1)\frac{\partial v}{\partial S} - kv = 0$$

where:

$$\frac{\partial^\gamma f}{\partial\tau^\gamma} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial\tau^n}\int_t^T \frac{f(S,\tau')-f(S,\tau)}{(\tau'-\tau)^{\alpha+1-n}}d\tau'$$

and $\frac{\partial^{2\gamma}f}{\partial\tau^{2\gamma}} = \frac{\partial^\gamma}{\partial\tau^\gamma}\left(\frac{\partial^\gamma f}{\partial\tau^\gamma}\right)$. If $A_{2\gamma}S^{d_f-1} = 1$ and $B_\gamma S^{d_f-1} = 1$, we obtain the modified Fractional Black–Scholes equation as follows:

$$\boldsymbol{m\frac{\partial^{2\gamma}v}{\partial\tau^{2\gamma}}+(1+mk_1)\frac{\partial^\gamma v}{\partial\gamma}=\frac{\partial^2 v}{\partial S^2}+(k_1-1)\frac{\partial v}{\partial S}-k_1 v} \tag{2}$$

with $\boldsymbol{m}$:: constant
$\boldsymbol{k_1}$: risk-free interest
$\boldsymbol{S}$: asset value
Equation (2) is called the modified Fractional Black–Scholes equation.
Based on Equation (2), we obtained:
For $\boldsymbol{m=0}$, Equation (1) becomes the Fractional Black–Scholes equation.
For $\boldsymbol{m=0}$ and $\boldsymbol{\gamma=1}$, Equation (1) becomes the Black–Scholes equation.

So, the Fractional Black–Scholes equation is a special case of the modified Fractional Black–Scholes equation.

3. Fractional Differential Transformation Method

The Fractional Differential Transformation Method is a generalization of the differential transformation method based on the Fractional Taylor formula. The Fractional Taylor series expansion with order α of the function $u(t)$ around the point $t = t_0$ is defined as:

$$u(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} \left(\frac{d^\alpha}{dt^\alpha}\right)^k u(t);$$

where $\frac{d^\alpha}{dt^\alpha}$ is the Caputo fractional derivative with order α, and $\left(\frac{d^\alpha}{dt^\alpha}\right)^k = \frac{d^\alpha}{dt^\alpha} \cdots \frac{d^\alpha}{dt^\alpha}$ consists of k terms.

The Fractional Differential Transformation with order α of the function $u(t)$ in the neighborhood $t = t_0$ is defined as $U^\alpha(k) = \frac{1}{\Gamma(k\alpha+1)}\left(\frac{d^\alpha}{dt^\alpha}\right)^k u(t)$, and the inverse transformation is $u(t) = \sum_{k=0}^{\infty} U^\alpha(k)(t-t_0)^{k\alpha}$. So, for $t_0 = 0$, we obtain:

$$u(t) = \sum_{k=0}^{\infty} U^\alpha(k) t^{k\alpha} = \sum_{k=0}^{\infty} \varphi_k(t).$$

Theorem 1 ([21])**.** *If $F^\alpha(k)$ and $G^\alpha(k)$ and $H^\alpha(k)$ are Fractional Differential Transformations of the functions $f(t), g(t)$, and $h(t)$, then this applies:*

(a) *If $f(t) = g(t) \pm h(t)$, then $F^\alpha(k) = G^\alpha(k) \pm H^\alpha(k)$,*

(b) *If $f(t) = (t-t_0)^q$, then $F^\alpha(k) = \delta\left(k - \frac{q}{\alpha}\right)$ where $\delta = \begin{cases} 1\,, & if\ k = 0 \\ 0, & if\ k \neq 0 \end{cases}$*

(c) *If $f(t) = g(t)h(t)$, then $F^\alpha(k) = \sum_{l=0}^{k} G^\alpha(k) H^\alpha(k-l)$.*

Theorem 2 ([22])**.** *If $f(t) = t^\lambda g(t)$ where $\lambda > -1$ and $g(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^{n\alpha}$ with a convergence radius $R > 0$ and $0 < \alpha \leq 1$, then:*

$$D_\alpha^\gamma D_\alpha^\beta = D_\alpha^{\gamma+\beta} f(t).$$

Theorem 3 ([22])**.** *Suppose $f(t) = D_{t_0}^\gamma g(t),\ m-1 < \gamma \leq m$, and the function $g(t)$ satisfies the conditions in the theorem above, then:*

$$F^\alpha(k) = \frac{\Gamma(k\alpha+\gamma+1)}{\Gamma(k\alpha+1)} G^\alpha\left(k + \frac{\gamma}{\alpha}\right)$$

for each $t \in (0, R)$, if:

(a) *$\beta < \lambda + 1$, for any α or*

(b) *$\beta \geq \lambda + 1$, for any γ, and $a_k = 0$ for $k = 0, 1, \ldots, m-1$, where $m - 1 < \beta \leq m$.*

The following is the convergence theorem to solve the modified Fractional Black–Scholes equation.

Theorem 4. *If, for any $k \in \mathbb{N}_0$ and for each $i \geq k_0$, there exists $0 < \partial_i < 1$ such that $\|\varphi_{i+1}\| < \delta_{i+1}\|\varphi_i\|$, then the series $\sum_{k=0}^{\infty} \varphi_k(t)$ converges to u_t.*

Proof. Suppose there is a Cauchy sequence $u_1, u_2, \ldots$, where $u_n = \sum_{k=0}^{\infty} \varphi_k(t)$. It will be shown that u_n is a Cauchy sequence. For $0 < \partial_i < 1$, it implies:

$$\|u_i - u_{i-1}\| = \|\varphi_i\| \le \delta_i \|\varphi_{j-1}\| < \delta_i \delta_{i-1} \ldots \delta_{k_0} \|\varphi_{k_0}\|$$

For $n \ge m \ge k_0$, this is obtained:

$$\|u_n - u_m\| = \sum_{i=m+1}^{n} (s_i - s_{i-1}) \le \sum_{i=m+1}^{n} \delta_i \delta_{i-1} \ldots \delta_{k_0} \|\varphi_{k_0}\|.$$

Let $\delta = max\left\{\delta_{k_0}, \delta_{k_{0+1}}, \ldots, \delta_n\right\}$. So, it satisfies $\|u_n - u_m\| \le \frac{1-\delta^{n-m}}{1-\delta} \delta^{m-k_0} \|\varphi_{k_0}\|$. Because for $0 < \partial_i < 1$, it implies $\|u_n - u_m\| \to 0$. Hence, $\{u_n\}$ is a Cauchy sequence. □

a Fractional Differential Transformation Method for solving BSFM

If the second partial derivative of (S, τ) is substituted by $\frac{1}{h^2}(v(S-h,\tau) - 2v(S,\tau) + v(S+h,\tau)) + O\left(h^2\right)$ and the first partial derivative on (S,τ) is substituted by $\frac{1}{h}(v(S+h,\tau) - V(S,\tau)$, this is obtained:

$$m\frac{\partial^{2\gamma} v}{\partial \tau^{2\gamma}} + (1+mk)\frac{\partial^{\gamma} v}{\partial \gamma} = \frac{1}{h^2}(v(S-h,\tau) - 2v(S,\tau) + v(S+h,\tau)) + O\left(h^2\right) + \frac{(k-1)}{h}(v(S+h,\tau) - V(S-h,\tau) + O(h) - kv,$$

Then, the interval $[a, b]$ is divided into n subintervals of the same length, denoted by $h = \frac{b-a}{n}$.

So, we obtain *mesh points* $S_i = a + ih$, $i = 1, 2, \ldots, n-1$. If the truncation error is removed and $u_i(t)$ is an approximate solution of $v_i(\tau) = v(S_i, \tau)$, then we will obtain a system of ordinary differential equations:

$$\begin{aligned} & m\tfrac{d^{2\gamma} u_i(\tau)}{d\tau^{2\gamma}} + (1+mk_1)\tfrac{d^{\gamma} u_i(\tau)}{d\gamma} \\ & = \tfrac{1}{h^2}(u_{i-1}(\tau) - 2u_i(\tau) + u_{i+1}(\tau)) + \tfrac{(k_1-1)}{h}((u_{i+1}(\tau) - u_i(\tau)) \\ & -k_1 u_i(\tau)),\ i = 1, 2, \ldots, n-1. \end{aligned} \tag{3}$$

The system of ordinary differential equations above will be solved using the Fractional Differential Transformation method. Suppose the solution to the system of differential equations above is:

$$u_i(t) = \sum_{k=0}^{\infty} U_i^{\alpha}(k) t^{k\alpha}, \tag{4}$$

where U_i^{α} is the unknown coefficient, i.e., the Fractional Differential Transformation of $u_i(t)$.

Based on Theorem 3, Equation (3) can be written as:

$$\begin{aligned} m\tfrac{\Gamma(k\alpha+2\gamma+1)}{\Gamma(k\alpha+1)} \quad & U_i^{\alpha}\left(k + \tfrac{2\gamma}{\alpha}\right) + (1+mk_1)\tfrac{\Gamma(k\alpha+\gamma+1)}{\Gamma(k\alpha+1)} U_i^{\alpha}\left(k + \tfrac{\gamma}{\alpha}\right) \\ & = \tfrac{1}{h^2}\left(U_{i-1}^{\alpha}(k) - 2U_i^{\alpha}(k) + U_{i+1}^{\alpha}(k)\right) + \tfrac{(k_1-1)}{h}\left(U_{i+1}^{\alpha}(k) - U_i^{\alpha}(k)\right) \\ & -k_1 U_i^{\alpha}(k)) \end{aligned} \tag{5}$$

with initial conditions:

$$U_i^{\alpha}(0) = f_1(x_1) \tag{6}$$

$$U_i^{\alpha}\left(\frac{1}{\alpha}\right) = f_2(x_1) \tag{7}$$

and with boundary conditions:

$$U_0^{\alpha}(k) = G_1^{\alpha}(k) \tag{8}$$

$$U_N^{\alpha}(k) = G_2^{\alpha}(k) \tag{9}$$

So, Equations (5)–(9) can be written as the new equation, as follows:

$$\begin{aligned} U_i^{\alpha}\left(k+\frac{2\gamma}{\alpha}\right) &= \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+2+1)}\left(\frac{U_{i-1}^{\alpha}(k)-2U_i^{\alpha}(k)+U_{i+1}^{\alpha}(k)}{mh^2}\right. \\ &\left.+\frac{(k_1-1)}{mh}\left(\left(U_{i+1}^{\alpha}(k)-U_i^{\alpha}(k)\right)-k_1U_i^{\alpha}(k)\right)\right) \\ &-(1+mk_1)\frac{\Gamma(k\alpha+\gamma+1)}{\Gamma(k\alpha+1)}U_i^{\alpha}\left(k+\frac{\gamma}{\alpha}\right) \end{aligned} \tag{10}$$

with initial conditions:

$$U_i^{\alpha}(0) = f_1(x_1),$$

$$U_i^{\alpha}\left(\frac{1}{\alpha}\right) = f_2(x_1),$$

satisfying the boundary conditions:

$$U_0^{\alpha}(k) = G_1^{\alpha}(k),$$

$$U_N^{\alpha}(k) = G_2^{\alpha}(k).$$

Based on Equation (5) with unknown coefficients $U_i^{\alpha}(1),\ U_i^{\alpha}(2),\ldots,U_i^{\alpha}\left(\frac{2\gamma}{\alpha}-1\right)$, we can satisfy the following equation:

$$U_i^{\alpha}(k) = \begin{cases} \frac{1}{\Gamma(k\alpha+1)} & \left[\frac{d^{k\alpha}}{dt^{k\alpha}}\right]_{t=0} \quad if\ k\alpha \in \mathbb{Z}^+, \\ 0 & \quad if\ k\alpha \notin \mathbb{Z}^+. \end{cases} \tag{11}$$

4. BSFM Solution Using the Fractional Differential Transformation Method

The following will show that the Fractional Differential Transformation method can be used to find solutions to the modified Fractional Black–Scholes equation.

Example 1. *Solve the following modified Fractional Black–Scholes equation:*

$$\frac{\partial^{2\gamma}v}{\partial\tau^{2\gamma}} + 2\frac{\partial^{\gamma}v}{\partial\tau^{\gamma}} = \frac{\partial^2 v}{\partial S^2} + \frac{\partial v}{\partial S} - 2v;\ 0<\gamma<1,\ 0<S<1$$

with initial conditions:

$$v(0,\tau) = e^{-2\tau},\ v(1,\tau) = e^{1-2\tau},\ v(S,0) = \max(e^S-1,0),\ v_\tau(S,0) = 2e^S.$$

Solution:

Given that $\gamma = 0.75$, $\alpha = 0.25$, and $h = 0.1$,

$\frac{\partial^{1.5}v(S,\tau)}{\partial\tau^{1.5}}$ is fractionally differentially transformed into $\frac{\Gamma(k\alpha+2\gamma+1)}{\Gamma(k\alpha+1)}U_i^{0.25}(k+6)$ $= \frac{\Gamma(0.25k+2.5)}{\Gamma(0.25k+1)}U_i^{0.25}(k+6)$,

$\frac{\partial^{0.75}v(S,\tau)}{\partial\tau^{0.75}}$ is fractionally differentially transformed into $\frac{\Gamma(k\alpha+2\gamma+1)}{\Gamma(k\alpha+1)}U_i^{0.25}(k+3)$ $= \frac{\Gamma(0.25k+1.75)}{\Gamma(0.25k+1)}U_i^{0.25}(k+3)$,

$$v(S,\tau) \to U_i^{0.25}(k),$$

$\frac{\partial^2 v}{\partial S^2}$ is fractionally differentially transformed into $\frac{U_{i-1}^{0.25}(k)-2U_i^{0.25}(k)+U_{i+1}^{0.25}(k)}{h^2}$,

$\frac{\partial v}{\partial S}$ is fractionally differentially transformed into $\frac{(U_{i+1}^{0.25}(k)-U_{i-1}^{0.25}(k)}{2h}$,

with initial conditions:

$v(S,0) = \max(e^S-1,0)$ is fractionally differentially transformed into $U_i^{0.25}(0) = \max(e^{S_i}-1,0)$, for each $i = 0,\ 1,\ 2,\ldots,10$.

Based on Equations (4), (5) and (8) above, we obtain:

$U_i^{0.25}(1) = U_i^{0.25}(2) = U_i^{0.25}(3) = U_i^{0.25}(5) = 0$; for each $i = 0,\ 1,\ 2, \ldots,\ 10$.

Based on Equations (4), (5) and (9), the following system of equations is obtained:

$$v(0,\tau) = e^{-2\tau} \text{ is fractionally differentially transformed into } U_0^{0.25}(k) = \begin{cases} \frac{2.(-2)^{\frac{k}{4}}}{\Gamma\left(\frac{k}{4}+1\right)},\ if\ \frac{k}{4} \in \mathbb{Z}^+ \\ 0,\ if\ \frac{k}{4} \notin \mathbb{Z}^+ \end{cases},$$

$$v(1,\tau) = e^{1-2\tau} \text{ is fractionally differentially transformed into } U_{10}^{0.25}(k) = \begin{cases} \frac{2.(-2)^{\frac{k}{4}.}e}{\Gamma\left(\frac{k}{4}+1\right)},\ if\ \frac{k}{4} \in \mathbb{Z}^+ \\ 0,\ if\ \frac{k}{4} \notin \mathbb{Z}^+ \end{cases},$$

$$U_i^{0.25}(k+6) = \frac{1}{m}\left\{\frac{U_{i-1}^{0.25}(k) - 2U_i^{0.25}(k) + U_{i+1}^{0.25}(k)}{h^2} + (k_1 - 1)\frac{\left(U_{i+1}^{0.25}(k) - U_{i-1}^{0.25}(k)\right.}{2h} - k_1 U_i^{0.25}(k)\right\} - (1 + mk_1)\frac{\Gamma(0.25k + 1.75)}{\Gamma(0.25k + 1)}U_i^{0.25}(k+3),$$

$$S_0 = 0, \text{ we obtain } u_0(t) = \textstyle\sum_{k=0}^{\infty} U_0^{0.25}(k)t^{0.25k} = m\sum_{k=0}^{\infty} U_0^{0.25}(k)t^{0.25k} = 2\sum_{k=0}^{\infty} U_0^{0.25}(k)t^{0.25k} = U_0^{0.25}(0) + U_0^{0.25}(1)t^{0.25} + U_0^{0.25}(2)t^{0.5} + U_0^{0.25}(3)t^{0.75} + \cdots = 2 + 0t^{0.25} + 0t^{0.5} + 0t^{0.75} - 4t + 0t^{5.0.25} + 0t^{6.0.25} + 0t^{7.0.25} + 2t^2 - \tfrac{4}{3}t^3 + \tfrac{4}{3}t^4 - \tfrac{32}{120}t^5 + \cdots = \textstyle\sum_{4m}^{\infty} \frac{(-2)^{\frac{k}{4}}}{\Gamma\left(\frac{k}{4}+1\right)} t^{0.25.k}, m = 0,1,2,\ldots$$

$$S_0 = 0 - 2t = -2t$$

$$S_1 = 0.1, \text{ we obtain } u_1(t) = \textstyle\sum_{k=0}^{\infty} U_1^{0.25}(k)t^{0.25k} = U_1^{0.25}(0) + U_1^{0.25}(1)t^{0.25} + U_1^{0.25}(2)t^{0.5} + U_1^{0.25}(3)t^{0.75} + \cdots = max\left\{e^{0.1} - 1, 0\right\} + 0t^{0.25} + 0t^{0.5} + \cdots - 2e^{0.1}t + \cdots = e^{0.1} - 1 - 2e^{0.1}t + \cdots$$

$$S_2 = 0.2 \text{ we obtain } u_2(t) = \textstyle\sum_{k=0}^{\infty} U_2^{0.25}(k)t^{0.25k} = U_2^{0.25}(0) + U_2^{0.25}(1)t^{0.25} + U_2^{0.25}(2)t^{0.5} + U_2^{0.25}(3)t^{0.75} + \cdots = \max\left(e^{0.2} - 1, 0\right) + 0t^{0.5} + \cdots - 2e^{0.2}t^{0.75} + \cdots = e^{0.2} - 1 - 2e^{0.2}t + \cdots$$

$$S_3 = 0.3 \to u_3(t) = e^{0.3} - 1 - 2e^{0.3}t + \cdots$$

$$S_4 = 0.4 \to u_4(t) = e^{0.4} - 1 - 2e^{0.4}t + \cdots$$

$$S_5 = 0.5 \to u_5(t) = e^{0.5} - 1 - 2e^{0.5}t + \cdots$$

$$S_6 = 0.6 \to u_5(t) = e^{0.6} - 1 - 2e^{0.6}t + \cdots$$

$$S_7 = 0.7 \to u_6(t) = e^{0.7} - 1 - 2e^{0.7}t + \cdots$$

$$S_8 = 0.8 \to u_8(t) = e^{0.8} - 1 - 2e^{0.8}t + \cdots$$

$$S_9 = 0.9 \to u_9(t) = e^{0.9} - 1 - 2e^{0.9}t + \cdots$$

$$S_{10} = 1, \text{ we obtain } u_{10}(t) = \textstyle\sum_{k=0}^{\infty} U_{10}^{0.25}(k)t^{0.25k} = U_{10}^{0.25}(0) + U_{10}^{0.25}(1)t^{0.25} + U_{10}^{0.25}(2)t^{0.5} + U_{10}^{0.25}(3)t^{0.75} + \cdots S_{10} = 1 \to u_{10}(t) = e - 1 - 2et + \cdots$$

The following is a simulation using the Python program to illustrate the solutions obtained:

Figure 1 shows the graph for solving the modified Black–Scholes fractional equation with $n = 5$ and $t = 0.01$. The resulting graph increases monotonically, but it seems not to be smooth due to a wide interval partition and a few number of sampled points taken. The minimum value is obtained when $S_i = 0$, so $u_i = 0$, while the maximum value is obtained when $S_i = 1.0$, so $u_i = 1.6$. and $t = 0.025$.

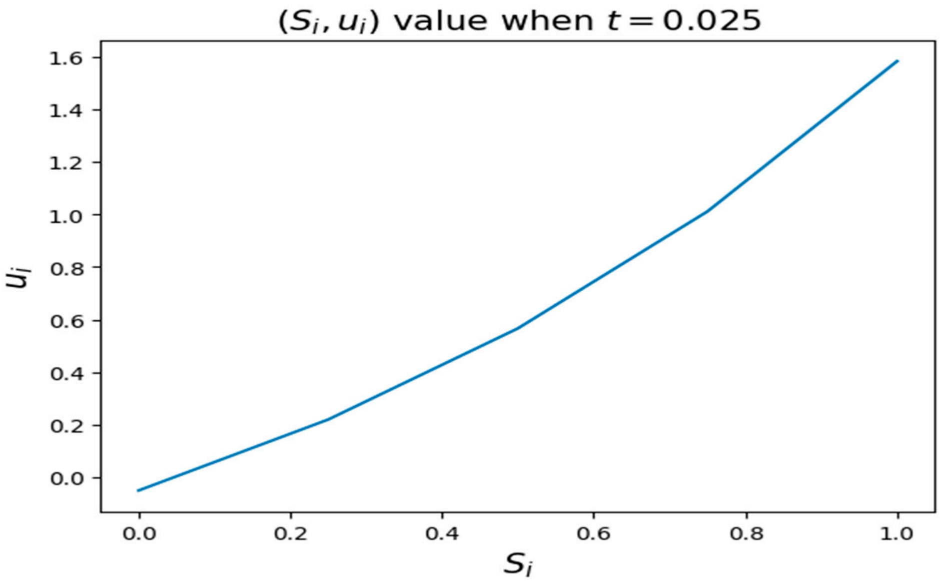

Figure 1. Graph of the solution to the Modified Fractional Black–Scholes equation with $n = 5$ and t = 0.025.

Figure 2 shows a graph of the solution to the modified Fractional Black–Scholes equation, with $n = 25$ and $t = 0.025$. The solution graph obtained is a monotonically increasing function, but it is relatively smooth compared to the graph in Figure 1.

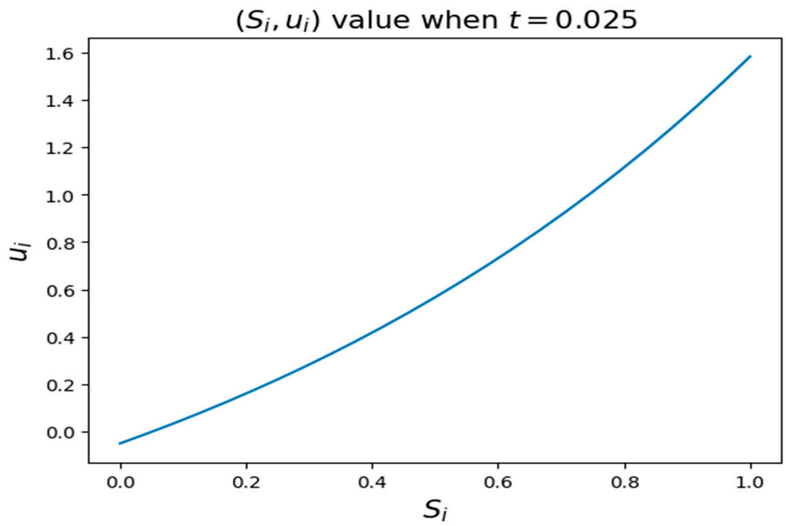

Figure 2. Graph of the solution to the modified Fractional Black–Scholes equation with $n = 25$ and $t = 0.025$.

Example 2. *The following is the procedure for determining the solution of the modified Fractional Black–Scholes equation:*

$$m\frac{\partial^{2\gamma} v}{\partial \tau^{2\gamma}} + (1 + mk_1)\frac{\partial^{\gamma} v}{\partial \tau^{\gamma}} = \frac{\partial^2 v}{\partial S^2} + (k_1 - 1)\frac{\partial v}{\partial S} - k_1 v;$$

with initial conditions:

$$v(S, 0) = \frac{e^x - 1}{(1 + mk_1)}$$

$$v_i(S, 0) = \frac{k_1 e^x}{(1 + mk_1)}$$

$$v(0,\tau) = \frac{\left(1 - e^{-k_1\tau}\right)}{(1+mk_1)}$$

$$v(1,\tau) = \frac{\left(2e - e^{1-k_1\tau} - 1\right)}{(1+mk_1)}.$$

Suppose we take $\gamma = 1.0$ and $\alpha = 0.25$, then we obtain the following:
$\frac{\partial^2 v(S,\tau)}{\partial\tau^2}$ is transformed into $\frac{\Gamma(0.25k+3)}{\Gamma(0.25k+1)}U^{0.25}(k+8)$,
$\frac{\partial v}{\partial\tau}$ is transformed into $\frac{\Gamma(0.25k+2)}{\Gamma(0.25k+1)}U^{0.25}(k+4)$,
$\frac{\partial v}{\partial\tau}$ is transformed into $\frac{\left(U_i^{0.25}(k) - U_{i-1}^{0.25}(k)\right)}{2h}$,
$v(S,\tau)$ is transformed into $U_i^{0.25}(k)$. Then, we successively obtain:

$$v(S,0) = \frac{e^S - 1}{(1+mk_1)} = U_i^{0.25}(0),$$

for each $i = 0,\ 1,\ 2, \ldots,\ 10$:

$$v_i(S,0) = \frac{k_1 e^S}{(1+mk_1)} = U_i^{0.25}(4),$$

for each $i = 0,\ 1,\ 2, \ldots,\ 10$. Meanwhile, based on Equation (11), the following is obtained:

$$U_i^{0.25}(1) = U_i^{0.25}(2) = U_i^{0.25}(3) = U_i^{0.25}(5) = \cdots = 0, v(0,\tau) = \frac{1 - e^{-k_1\tau}}{(1+mk_1)},$$

transformed into:

$$U_0^{0.25}(k) = \begin{cases} \frac{1}{1+mk_1} \quad \frac{k_1^{\frac{k}{4}}}{\Gamma\left(\frac{k}{4}+1\right)}, & if\ \frac{k}{4} \in \mathbb{Z}^+ \\ 0, & if\ \frac{k}{4} \notin \mathbb{Z}^+ \end{cases}.$$

For $S_0 = 0$, we obtain:

$$\begin{aligned} &u_0(\tau) = \textstyle\sum_{k=0}^{\infty} U_0^{0.25}(k)\tau^{0.25k} \\ &= U_0^{0.25}(0) + U_0^{0.25}(1)\tau^{0.25} + U_0^{0.25}(2)\tau^{0.5} + U_0^{0.25}(3)\tau^{0.75} + U_0^1(4)\tau^1 + \cdots \\ &= 0 + 0\tau^{0.25} + 0\tau^{0.5} + 0\tau^{0.75} + \left(\tfrac{k_1}{1+mk_1}\right)\tau = \tfrac{k_1\tau}{1+mk_1} \end{aligned}$$

For $S_1 = 0.1$, we obtain:

$$\begin{aligned} &u_1(\tau) = \textstyle\sum_{k=0}^{\infty} U_1^{0.25}(k)\tau^{0.25k} \\ &= U_1^{0.25}(0) + U_1^{0.25}(1)\tau^{0.25} + U_1^{0.25}(2)\tau^{0.5} + U_1^{0.25}(3)\tau^{0.75} + U_1^1(4)\tau^1 + \cdots \\ &= \tfrac{e^{0.1}-1}{1+mk_1} + 0\tau^{0.25} + 0\tau^{0.5} + 0\tau^{0.75} + \tfrac{k_1 e^{0.1}}{1+mk_1}\tau \\ &= \tfrac{e^{0.1}-1}{1+mk_1} + \tfrac{k_1 e^{0.1}}{1+mk_1}\tau \end{aligned}$$

For $S_2 = 0.2$, we obtain:

$$\begin{aligned} &u(\tau) = \textstyle\sum_{k=0}^{\infty} U_2^{0.25}(k)\tau^{0.25k} \\ &= U_2^{0.25}(0) + U_2^{0.25}(1)\tau^{0.25} + U_2^{0.25}(2)\tau^{0.5} + U_2^{0.25}(3)\tau^{0.75} + U_2^1(4)\tau^1 + \cdots \\ &= \tfrac{e^{0.2}-1}{1+mk_1} + \tfrac{k_1 e^{0.2}}{1+mk_1}\tau \end{aligned}$$

For $S_3 = 0.3$, we obtain

$$
\begin{aligned}
u_3(\tau) &= \textstyle\sum_{k=0}^{\infty} U_3^{0.25}(k)\tau^{0.25k} \\
&= U_3^{0.25}(0) + U_3^{0.25}(1)\tau^{0.25} + U_3^{0.25}(2)\tau^{0.5} + U_3^{0.25}(3)\tau^{0.75} + U_3^{1}(4)\tau^{1} + \cdots \\
&= \tfrac{e^{0.3}-1}{1+mk_1} + \tfrac{k_1 e^{0.3}}{1+mk_1}\tau \\
&\ldots
\end{aligned}
$$

For $S_{10} = 1$, we obtain $u_{10}(\tau) = \frac{e-1}{1+mk_1} + \frac{k_1 e}{1+mk_1}\tau$. So, we obtain the points (S_i, u_i) for $i = 0,\ 1,\ 2,\ 3, \ldots,\ 10$. Accordingly, the graph of the solution is as follows.

According to Figure 3, a graph of the solution to the Fractional Black–Scholes equation is obtained, modified with the values $0 \le m \le 1$ and $k_1 = 0.05$ in the form of a family of exponential functions. When $m = 0$, the graph is at the bottom. Meanwhile, when $m = 1$, the graph is at the top.

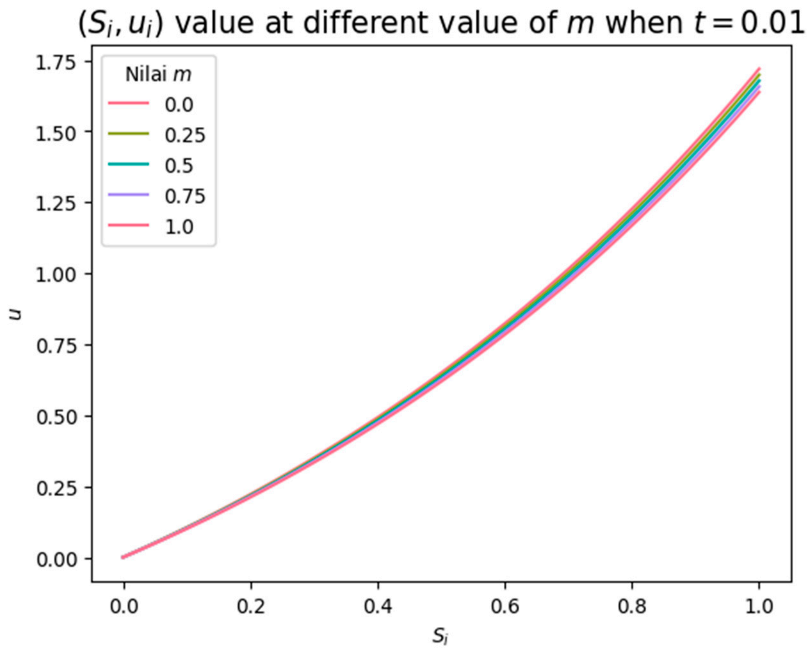

Figure 3. Graph of the solution to the Fractional Black–Scholes equation with $0 \le m \le 1$ and $k_1 = 0.05$.

Figure 4 shows that the solution to the Fractional Black–Scholes equation is modified in the form of an exponential function. The minimum value is obtained when $S_i = 0$, with a value of $u_i = 0$. Meanwhile, the maximum value is obtained when $S_i = 1$, with a value of $u_i = 1.75$. Furthermore, based on Equation (2), the Fractional Black–Scholes equation is a special case of the Fractional Black–Scholes equation when the value of $m = 0$.

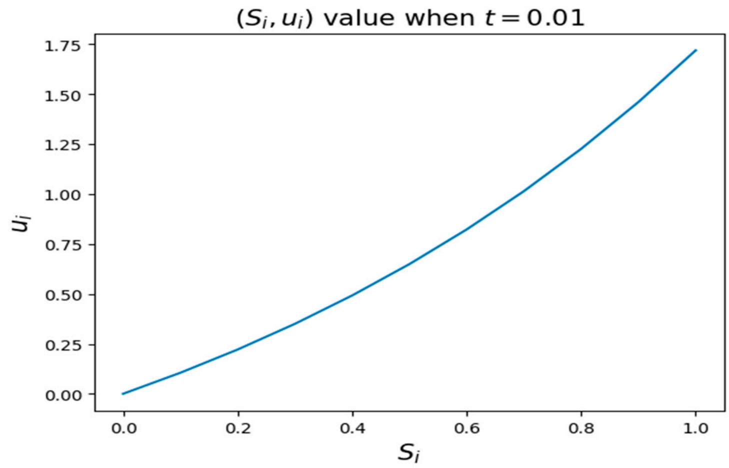

Figure 4. Graph of the solution to the Fractional Black–Scholes equation. modified with $t = 0.01$, $\gamma = 1.0$, $k = 0.05$, $n = 10$, $m = 0$, and $0 \leq S_i \leq 1$.

The general form of the Fractional Black–Scholes equation with a value of $\gamma = 1.0$ is:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial S^2} + (k-1)\frac{\partial v}{\partial S} - kv$$

with the initial condition $v(S,0) = \max\{e^x - 1,\ 0\}$. Using the Daftardar–Gejji method, the general solution is:

$v(S,\tau) = \sum_{n=0}^{\infty} v_n(S,\tau) = \max\{e^S - 1,\ 0\}E_{1.0}(-k_1\tau) + \max\{e^x,\ 0\}(1 - E_{1.0}(-k_1\tau)$. If the graph is plotted, it is obtained as follows:

Figure 5 shows that the solution to the Fractional Black–Scholes equation can be approximated using the Fractional Black–Scholes equation solution by taking the value of $m = 0$. This is because, in Figure 5, the pink and blue graphs almost coincide. Thus, the solution error is guaranteed to be very small. The following is the error calculation between the solution of the Fractional Black–Scholes equation and the solution of the modified Fractional Black–Scholes equation:

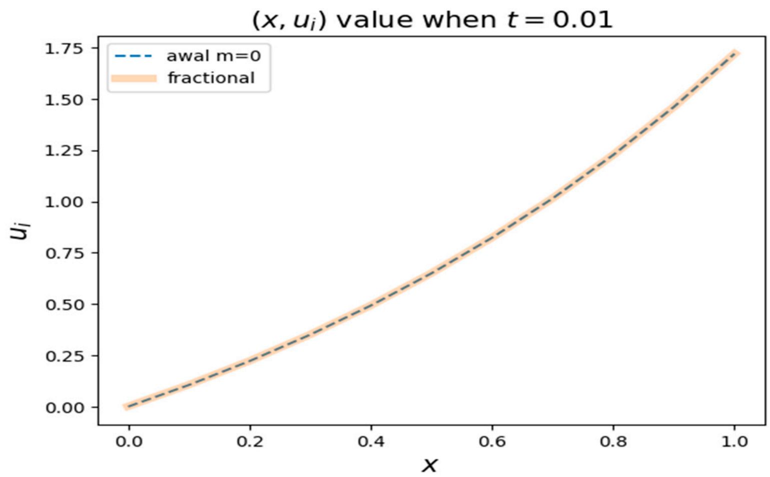

Figure 5. Graph of the solution to the Fractional Black–Scholes equation and the solution to the modified Fractional Black–Scholes equation by taking $k_1 = 0.05$.

Symbol v_i represents the solution to the Fractional Black–Scholes equation, while the symbol u_i represents the solution to the modified Fractional Black–Scholes equation. The values are $k_1 = 0.05$, $\gamma = 1.0$, and $\tau = 0.01$. Table 1 uses the absolute error formula

$= \frac{|v_i - u_i|}{v_i} \times 100\%$ and squared error $= \sum_{i=1}^{n} (v_i - u_i)^2$. Table 1 shows that the squared error and the absolute error for each point is very small. Using Phyton 3.7 software, it is obtained that the mean squared error is $2.1214603575846715 \times 10^{-7}$, and the mean absolute error is 0.049973388247889494%. This means that the solution to the modified Fractional Black–Scholes equation, taking the value s of $m = 0$ and $\gamma = 1.0$, is a very good approximation to the solution to the Fractional Black–Scholes equation with $\gamma = 1.0$. In other words, the Fractional Black–Scholes equation is a special case of the modified Fractional Black–Scholes equation when the value of $m = 0$.

Table 1. Error between v_i and u_i.

No	x_i	v_i	u_i	Squared Error	Abs Error (%)
1	0.100	0.105671	0.105724	2.778390×10^{-9}	0.049882
2	0.109	0.115662	0.115720	3.330000×10^{-9}	0.049892
3	0.118	0.125744	0.125807	3.937190×10^{-9}	0.049901
4	0.127	0.135917	0.135985	4.601382×10^{-9}	0.049908
...	...	...	...	...	...
96	0.964	1.622664	1.623475	6.580569×10^{-7}	0.049992
97	0.973	1.646370	1.647193	6.774279×10^{-7}	0.049992
98	0.982	1.670290	1.671125	6.972588×10^{-7}	0.049993
99	0.991	1.694427	1.695274	7.175589×10^{-7}	0.049993
100	1.000	1.718782	1.719641	7.383379×10^{-7}	0.049993

Moreover, v_i and u_i are compared using the 4th order Rungge–Kutta method. As is known, the Runge–Kutta method is a very accurate method for solving ordinary differential equations numerically. In this paper, the solution to the Fractional Black–Scholes equation, with the value of $\alpha = 1$, will be approached using the 4th order Runge–Kutta method.

Using Python 3.7 software, Figure 6 shows a graph of the solution to the Fractional Black–Scholes equation with $\gamma = 1.0$ using the 4th order Runge–Kutta method in three dimensions with $0 \leq S \leq 1$ and $0 \leq \tau \leq 1$. If the graph in Figure 6 is cut by $\tau = 0.01$, it will obtain Figure 7.

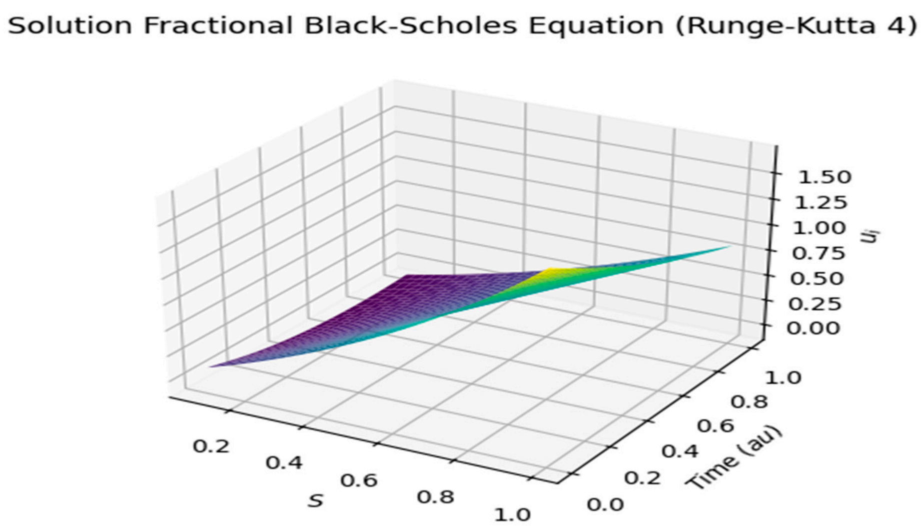

Figure 6. Graph of the solution to the Fractional Black–Scholes Equation with $\gamma = 1.0$ using the 4th order Runge–Kutta method.

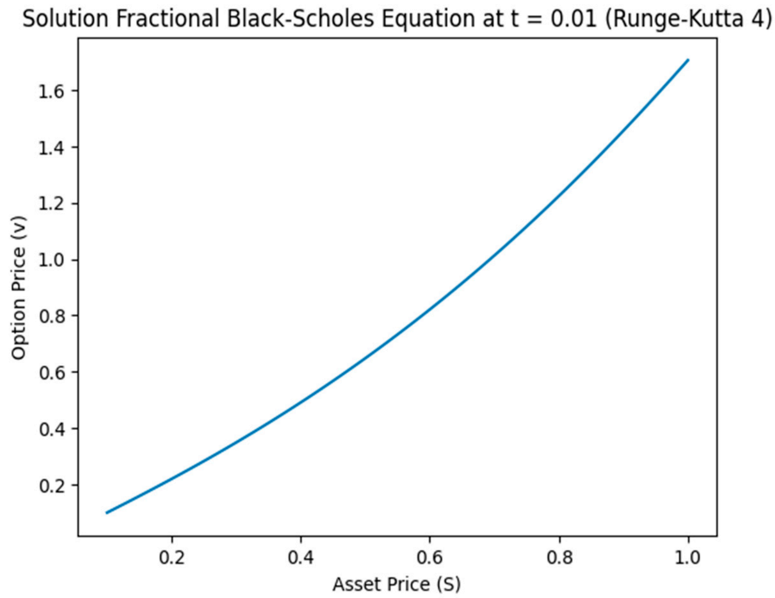

Figure 7. Graph of the solution to the Fractional Black–Scholes equation with $\gamma = 1.0$ using the Runge–Kutta method when $\tau = 0.01$.

Using Python 3.7 software, Figure 7 shows a graph of the solution to the Fractional Black–Scholes equation with $\gamma = 1.0$ when $\tau = 0.01$. The resulting graph is an increasing function graph. Then, if the solution graph for the Fractional Black–Scholes equation with $\gamma = 1.0$, obtained using the combined method of finite difference and fractional differential transformation, along with the graph using the 4th order Runge–Kutta method, are combined into one graph, Figure 8 will be obtained, as shown below.

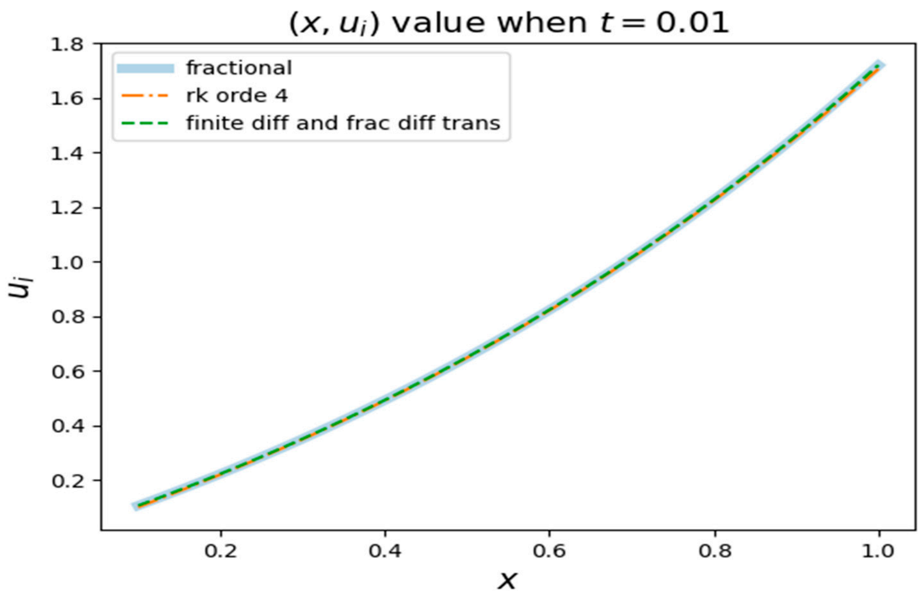

Figure 8. Graph of the combined solution method to the Fractional Black–Scholes equation with $\gamma = 1.0$ when $t = 0.01$.

The graph in blue is a solution to the Fractional Black–Scholes equation, the graph in yellow is an approximate graph of the solution to the Fractional Black–Scholes equation using the 4th order Runge–Kutta method, while the graph in green is an approximation using a combined method of finite difference and fractional differential transformation. Visually, the combined graph between these methods almost coincides. This means that both the 4th order Runge–Kutta method and the combined method of finite difference and fractional differential transformation are good approximations for graphing solutions to the Fractional Black–Scholes equation with $\gamma = 1.0$. Therefore, the error obtained by the 4th order Runge–Kutta method and the combination of finite difference and fractional differential transformation are compared to the solution of the Fractional Black–Scholes equation with $\gamma = 1.0$, as shown in Table 2 below:

Table 2. Error between v_i and RK4.

No	x_i	v_i	RK4	Squared Error	Abs Error (%)
1	0.100	0.105671	0.099972	0.000032	5.700230
2	0.109	0.115662	0.110417	0.000028	4.750148
3	0.118	0.125744	0.120956	0.000023	3.958334
4	0.127	0.135917	0.131507	0.000019	3.353538
...	...	...	...	...	...
96	0.964	1.622664	1.613694	0.000080	0.555898
97	0.973	1.646370	1.636458	0.000098	0.605712
98	0.982	1.670290	1.659431	0.000118	0.654394
99	0.991	1.694427	1.682425	0.000144	0.713348
100	1.000	1.718782	1.705629	0.000173	0.771142

Table 2 shows that the squared error and absolute error for each point are very small. Using Python 3.7 software, the mean squared error was $1.3939683876496377 \times 10^{-5}$, and the mean absolute error was 0.6089656268506086%. Based on the mean squared error and mean absolute error, it can be said that the resulting error is very small, being less than 5%. Therefore, it can be concluded that the 4th order Runge–Kutta method is a very good approximation. When comparing with Table 1, the mean absolute error and mean squared error caused by the combination of the finite difference method and fractional differential transformation are smaller than those of the 4th order Runge–Kutta method. However, both methods are said to be very good for approaching the solution to the Fractional Black–Scholes equation with $\gamma = 1.0$.

5. Conclusions

The combined method of finite difference and fractional differential transformation can be used to solve the modified Fractional Black–Scholes equation. In real financial market conditions, the Black–Scholes equation is more realistic to use for modeling option values compared to the Fractional Black–Scholes equation. This is because the fractional order of the modified Fractional Black–Scholes equation is greater than the order of the Fractional Black–Scholes equation and can vary the value of m.

6. Further Research

There is a lot of research that can be done on the modified Fractional Black–Scholes equation; for example, looking for guarantees of existence and unique solutions. Then, the analytical solution of the Fractional Black–Scholes equation can be modified. Next, conducting error comparisons between the numerical and analytical solutions would be valuable. Other interesting things can also be developed for the Fractional Black–Scholes equation with multiple assets.

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