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Some Non-Linear Evolution Equations and Their Explicit Smooth Solutions with Exponential Growth Written into Integral Form

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Abstract: In this paper, exact solutions of semilinear equations having exponential growth in the space variable x are found. Semilinear Schrödinger equation with logarithmic nonlinearity and third-order evolution equations arising in optics with logarithmic and power-logarithmic nonlinearities are investigated. In the parabolic case, the solution u is written as $u = be^{-ax^2}$, $a < 0$, a, b being real-valued functions. We are looking for the solutions u of Schrödinger-type equation of the form $u = be^{-a\frac{x^2}{2}}$, respectively, for the third-order PDE, $u = Ae^{i\Phi}$, where the amplitude b and the phase function a are complex-valued functions, $A > 0$, and Φ is real-valued. In our proofs, the method of the first integral is used, not Hirota's approach or the method of simplest equation.

Keywords: semilinear parabolic equation; semilinear Schrödinger equation; logarithmic nonlinearity; parabolic equations with solutions of exponential growth; solutions into explicit form; special functions of Jacobi type; hyperbolic functions; Radhakrishnan–Kundu–Lakshmanan optic equation

MSC: 35K58; 35Q51; 35Q55; 35K91; 35Q99

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1. Introduction

This paper deals with exact (explicitly written) solutions of several semilinear evolution equations of mathematical physics. It concerns parabolic equations for which solutions with exponential growth in the space variable x are found, semilinear Schrödinger equation with logarithmic nonlinearity, and third-order evolution equations arising in optics with power and logarithmic-power nonlinearities. The latter are generalizations of the standard semilinear Schrödinger equation, which is of second order. We are looking for the solutions u of Schrödinger-type equations of the form Ae^{Φ}, where A is the amplitude and Φ is the phase function. A, B are complex-valued. That ansatz is often used in mathematical physics and we shall mention only the classical paper [1] that stimulated in the middle of last century different applications of the asymptotical solutions in analysis, PDE, and certainly in physics. For the parabolic equation $u = be^{-ax^2}$ with $a < 0$, a and b are real-valued. Depending on the sign of some parameter λ, we can have dispersive and non-dispersive cases for the classical Schrödinger equation. Therefore, two different cases appear. In the first one, the phase Φ is periodic in t, while in the general case, the amplitude is only bounded but not periodic in t. Its modulus is periodic. In the second case, the amplitude is bounded and tends to zero as a spiral for $|t| \to \infty$, while for fixed t, the phase is exponentially increasing in x. The "third order" Schrödinger-type equation possesses solutions of the type $\varphi(\xi)e^{i\psi}$, ξ, ψ being linear functions of t, x. Under many restrictions, three types of solutions are constructed, namely solutions for which $\varphi(\xi)$ forms one-parametric family of periodic solutions, φ is a soliton, φ blows up at some $\xi = \tilde{\xi}_0$, but $\varphi(\pm\infty) = 0$ and φ blows up at some point $\xi = \tilde{\xi}_0$ being periodic in t.

In proving our results, we use the method of the first integral from the theory of mechanical systems having one degree of freedom. For the sake of completeness, we shall say several words about the method of the first integral that possesses many applications to the theory of autonomous systems of ODE and to first-order quasilinear PDE [2,3]. The first integral appeared for the first time in the investigations of Newton and the proof of Kepler's law and it relies on the construction into explicit form of the appropriate first integral. Assume that $\varphi(t)$ stands for the trajectory of some particle in the one-dimensional space with initial position $\varphi(t_0)$ and initial velocity $\dot{\varphi}(t_0)$. According to Newton's second law, $\varphi(t_0)$ satisfies the ODE $m\ddot{\varphi} = f(\varphi)$, where $f(\varphi)$ is the corresponding acting force. Denote by $U(\varphi) = -\int f(\varphi)d\varphi$ the kinetic energy of the particle. Then the full energy $E(t) = \frac{m\dot{\varphi}^2}{2} + U(\varphi)$ satisfies the relation $\dot{E}(t) = 0 \Rightarrow E(t) \equiv E_0 = const$. This is the classical energy conservation law. From a mathematical point of view, the function $E(y, \varphi) = \frac{my^2}{2} + U(\varphi)$ is the first integral of Newton's second law, i.e., the phase trajectory $\varphi(t)$ is located on surface level $E \equiv E_0$, $y = \dot{\varphi}$. Therefore, $\varphi(t)$ satisfies the first-order ODE with separate variables $\dot{\varphi}^2 = 2(E_0 - U(\varphi))$, $E_0 \geq U(\varphi(t))$ and $E \not\equiv U(\varphi)$. If $\dot{\varphi}(t_0) > 0$, we have that $t - t_0 = \int_{\varphi(t_0)}^{\varphi(t)} \frac{d\lambda}{\sqrt{2(E-U(\lambda))}}$. By using different reference books on analysis as [4–6], we can express in some cases $\xi = \int_{z_0}^{z} \frac{d\lambda}{\sqrt{2(E-U(\lambda))}} = G(z)$ via elementary functions or some special functions (hyperbolic, Jacobi elliptic, Legendre elliptic function, etc.). As $G^{'}(z) > 0$, the smooth mapping $\xi = G(z)$ is invertible, i.e., there exists uniquely determined $z = G^{-1}(\xi)$. So $\varphi(t) = G^{-1}(t - t_0)$ with $z_0 = \varphi(t_0)$. The above-mentioned results are usually local but it could happen that solutions global in t exist. The well-known approach of Hirota [7] and the method of the simplest equation [8] are not used here. Several historical notes are proposed below. The logarithmic Schrödinger equation was introduced in [9]. Applications of that equation in quantum optics, nuclear physics, transport and diffusion phenomena, theory of super fluidity, and Bose–Einstein condensation can be found, respectively, in the following papers: [10–13]. Numerical experiments in [14] show that the dynamical properties of the solutions in the logarithmic case are rather different from that for power-like nonlinearity. The strong superposition of two or finitely many Gaussians was studied in [15]. We rely here on [16–18] generalizing the dispersive case from [16]. In the last 10 years, many papers appeared on the cubic-quartic Fokas–Lenells equation with perturbation terms. The corresponding PDE of fourth order occurs in different systems in fluid mechanics, solid state physics and condensed matter, nonlinear optic and plasma physics. One can see [19] on the subject and the references therein.

Our aim here is to find out explicitly written solutions by using purely mathematical tools. So there are no numerical simulations. We give here a detailed study of the dispersive case of the logarithmic Schrödinger equation looking for solution $u = b(t)e^{-a(t)\frac{x^2}{2}}$, $a(t)$, $b(t)$ being smooth complex-valued functions. $b(t)$ is not periodic in general but we find a necessary and sufficient condition for its periodicity. It is interesting to mention that $|b(t)|$ is always periodic. Our ODE are studied under Cauchy initial conditions. The situation is delicate if $\ddot{\varphi} = sh\varphi$, $\varphi(\frac{\pi}{2}) = 0$, $\dot{\varphi}(\frac{\pi}{2}) = 2$. The approach sketched above shows that its unique solution $\varphi = 2lntg\frac{t}{2}$, $t \in (0, \pi)$. Certainly, $|\varphi| = \infty$ at $t = 0$, $t = \pi$. The Troesh boundary value problem $\ddot{\varphi} = \lambda sh(\lambda\varphi)$, $\varphi(0) = 0$, $\varphi(0) = 1$ is out of the scope of this paper but some aspect was studied in 2014 by H. Temini and H. Kurkcu (precise numerical solution for $\lambda \geq 10$).

The paper is organized as follows. In Section 2, we formulate our results, in Section 3, the proof of Theorem 1 is given, in Section 4, the detailed proof of Theorem 2 is proposed, geometrically illustrated, and mechanically interpreted, and Section 5 contains the proof of Theorem 3. References are given at the end of the paper. We point out that for logarithmic nonlinearities, technically the things differ from those for power nonlinearities.

2. Formulation of the Main Results

1. We shall begin with the following nonlinear Cauchy problem:

$$\left| \begin{array}{l} u_t = u_{xx} + \lambda u ln u^2, x \in \mathbf{R}^1, 0 \le t \le T \\ u|_{t=0} = u_0(x) > 0, \lambda \in \mathbf{R}^1 \setminus 0. \end{array} \right. \tag{1}$$

The case of bounded real-valued solutions for parabolic equations is well-studied and we shall assume further on that $u > 0$ and $u_0(x) = b_0 e^{-a_0 x^2}$, $b_0 > 0$, $a_0 < 0$. Certainly, $u = 0$ is not a solution of (1). Let $M_{2,a} = \{u \in C^2(\mathbf{R}^1_+ \times \mathbf{R}^1_x), |u| \le C(T)e^{ax^2}, a > 0, x \in \mathbf{R}^1, 0 \le t \le T\}$. $M_{2,a}$ is a linear space. We shall look for a solution of (1) having the form

$$u(t,x) = b(t)e^{-a(t)x^2}, b(0) = b_0, a(0) = a_0, \tag{2}$$

if $u_0 = b_0 e^{-a_0 x^2}$, $b_0 > 0$, $a_0 < 0$.

This is our first result.

Theorem 1. ***Uniqueness***. *Consider (1) with positive solution u such that*

(i) $u \ge A_1 e^{-e^{bx^2}}$, $A_1 > 0, b > 0, b \le a$

(ii) $0 \le |u_x(t,x)| \le A_2(1+|x|)u$, $A_2 = A_2(T) > 0, \forall x \in \mathbf{R}^1, 0 \le t \le T$.

Then u is uniquely determined for appropriate $0 < a$ *in* $M_{2,a}$.

Existence *of solution for* $u_0 = b_0 e^{-a_0 x^2}$. *Then there are three possible cases depending on the sign of* λ, *i.e.,*

- *if* $\lambda < 0, 0 > a_0 > \frac{\lambda}{2} \Rightarrow$ *one can find*

$$a(t) \in C^\infty(\mathbf{R}^1), 0 > a(t) > \frac{\lambda}{2}, a' > 0, a(-\infty) = \frac{\lambda}{2}, a(+\infty) = 0, 0 > a(t) \ge a_0$$

- *if* $\frac{\lambda}{2} = a_0 < 0 \Rightarrow a(t) \equiv \frac{\lambda}{2}$
- *if* $a_0 < \frac{\lambda}{2} < 0$, *then* $a(t) \in C^\infty(t < \bar{t})$, $\bar{t} > 0$ *and a blows up for* $t = \bar{t}$; $a'(t) < 0$ *for* $t < \bar{t}$.

The function $b(t) \in C^\infty(\mathbf{R}^1)$ *in the first two cases.*

- *if* $\lambda > 0$, $a_0 < 0$ *again* $a(t) \in C^\infty(t < \bar{\bar{t}})$, $a'(t) < 0$ *but a blows up for* $t = \bar{t}$.

According to the Granwall lemma, (ii) implies that $|u| \le A_3 e^{A_4 x^2}$ for some $A_3, A_4 > 0$, i.e., $u \in M_{2,A_4}$. Certainly, $e^{-ax^2} \ge e^{-e^{ax^2}}$, $a > 0, \forall x \in \mathbf{R}^1$.

Our second model example is the Schrödinger equation with logarithmic nonlinearity:

$$iu_t + \frac{1}{2}u_{xx} = \lambda u ln|u|^2, t \in \mathbf{R}^1, x \in \mathbf{R}^1, \lambda \in \mathbf{R}^1 \setminus 0. \tag{3}$$

Evidently, $u = 0$ is not a solution of (3), while $u = e^{i\alpha}$ satisfies it $\forall \alpha \in \mathbf{R}^1$.

Again we shall find a solution of (3) having the form

$$u(t,x) = b(t)e^{-a(t)\frac{x^2}{2}}, u(0,x) = b(0)e^{-\frac{1}{2}a(0)x^2}, \tag{4}$$

$a(0) \neq 0, b(0) \neq 0, a(0) = \alpha_0 + i\beta_0$, $Rea(0) < 0$. Otherwise $|u_0|$ will be bounded in x. (4) is called the Gaussian solution of (3). Below we present some historical remarks.

(3) possesses standing wave solutions u_ω called Gaussons: $u_\omega(t,x) = e^{i\omega t}e^{1/2-\frac{\omega}{2\lambda}}e^{\lambda x^2}$, $\omega \in \mathbf{R}^1$. $u_\omega(t,x)$ are not asymptotically stable for $\lambda < 0$ due to the Galilean invariance of (3). The latter means that if u is a solution of (3), then for each $p \in \mathbf{R}$ $v(t,x) = u(t, x - pt)e^{ipx - \frac{i}{2}p^2 t}$ satisfies (3).

Introduce now the Sobolev-type space $W = \{u \in H^1(\mathbf{R}) \setminus 0 : |u|^2 ln|u|^2 \in L^1(\mathbf{R}^1)\}$. T. Cazenave and A. Haraux proved in [20] in 1980 that for $\lambda < 0$ and $u_0 \in W \setminus 0$ the

Cauchy problem for Equation (3) with initial data $u_0 = u|_{t=0}$ possesses a global solution $u \in C(\mathbf{R}^1_t : W_x \setminus 0)$ such that

$$||u(t)||^2_{L^2(\mathbf{R}^1_x)} = ||u_0||^2_{L^2},$$

$$||E(u(t))||_{L^2(\mathbf{R}^1_x)} = \frac{1}{2}||\nabla u(t)||^2_{L^2(\mathbf{R}^1_x)} + \lambda \int_{\mathbf{R}} |u(t,x)|^2 ln|u(t,x)|^2 dx = E(u_0)$$

(conservation laws).

Below we formulate the logarithmic Sobolev inequality in $W \setminus 0$ [21]: For each $\alpha > 0$ and $f \in W \setminus 0$

$$\int_{\mathbf{R}^1} |f(x)|^2 log|f(x)|^2 dx \leq \frac{\alpha^2}{\pi}||\nabla f||_{L^2(\mathbf{R}^1)} + [log||f||^2_{L^2} - (1 + log\alpha)]||f||^2_{L^2}.$$

There is equality in the previous estimate if and only if up to a translation $x \to x + q$, $q \in \mathbf{R}$ the function $f(x)$ is a multiple of $e^{-\pi x^2/2\alpha^2}$. Evidently, $e^{|x|^\alpha} \notin W$ for $\alpha \geq 0$.

This is our second result.

Theorem 2. *Consider (3) with $\alpha_0 < 0$, $\beta_0 \neq 0$. Then $a(t) \in C^\infty(\mathbf{R}^1)$ exists and $Rea(t) < 0$, $a(t)$ is a periodic function for $\lambda\alpha_0 < 0$ ($\iff \lambda > 0$). In the case $\lambda\alpha_0 > 0$ ($\iff \lambda < 0$), $t \to a(t) \in \mathbf{C}^1$ is a bounded smooth closed curve located in the half plane $Rea < 0$ tangential to the imaginary axes at the origin, $a(\pm\infty) = 0$. The function $b(t)$ is defined everywhere and is written explicitly.*

We point out that $b(t)$ is not periodic in general for $\lambda > 0$ and we discuss this problem during the proof of Theorem 2, finding NSC $b(t + T) = b(t), \forall t$. In the investigations for $a(t)$-periodic, the second Kepler law appears, i.e., $r^2(t)\Theta^{'}(t) = \alpha_0$ and then the curve $t \to \omega(t) = r(t)e^{\Theta(t)} \to \mathbf{C^1}$ is either periodic or dense in the ring in $C^1 : r_1 \leq |a| \leq r^2$, $0 < r_1 = min_{\mathbf{R}^1} r(t)$, $r_2 = max_{\mathbf{R}^1} r(t)$, $0 < r(t)$ being a periodic function.

A. H. Ardila proved in 2016 in [22] that the Gaussons are orbitally stable in $W(\mathbf{R}^1_x)$.

2. Our next step is to study two Schrödinger-type semilinear PDEs arising in optics, finding their special solutions that have appropriate physical interpretation. The first one is known as the Radhakrishnan–Kundu–Lakshmanan(RKL) equation (see [23]), namely

$$iu_t + au_{xx} + b|u|^{2n}u + i(\beta u_{xxx} + (\alpha|u|^{2n}u)_x) = 0, \tag{5}$$

where the constants a, b, α, β are real and nonzero, $n > 0$ (see, for example, [19]).
The second equation has power-logarithmic nonlinearity:

$$iu_t + au_{xx} + b|u|^{2n}uln|u| + i(\beta u_{xxx} + \alpha(|u|^{2n}uln|u|)_x) = 0. \tag{6}$$

The solutions are also known as dispersive optical ones. Concerning (5), $u(t,x)$ presents the wave profile, u_t describes the temporal evolution, a stands for the coefficient of chromatic dispersion, b is the coefficient of power law of self-phase modulation, β is the coefficient of third-order dispersion, α stands for the coefficient of self-steeping term of the short pulses, and n is the power law parameter (see [19]).

The solutions of (5) and (6) we are looking for will have the following form:

$$u(t,x) = \varphi(\xi)e^{i\psi(t,x)}, \tag{7}$$

where ψ is linear phase function:

$$\psi(t,x) = -kx + \omega t, k, \omega \in \mathbf{R}^1,$$

$\xi = x - Vt$, $V \in \mathbf{R}^1$ and amplitude $\varphi(\xi) > 0$. As usual, ω is the frequency, k is the wave number, and V is the velocity of the wave.

Putting (7) into (5) and (6) and separating the real and imaginary parts of the corresponding expressions, we come to the overdetermined system that should be satisfied by the real-valued positive function $\varphi(\xi)$:

$$\left| \begin{array}{l} (a+3k\beta)\varphi^{''} - \varphi(\omega + ak^2 + \beta k^3) + \varphi^{2n+1}(b+k\alpha) = 0 \\ \beta\varphi^{'''} - \varphi^{'}(V + 2ak + 3\beta k^2) + \alpha(\varphi^{2n+1})^{'} = 0 \end{array} \right. \tag{8}$$

for Equation (5) and

$$\left| \begin{array}{l} (a+3k\beta)\varphi^{''} - \varphi(\omega + ak^2 + \beta k^3) + \varphi^{2n+1} ln\varphi(b+k\alpha) = 0 \\ \beta\varphi^{'''} - \varphi^{'}(V + 2ak + 3\beta k^2) + \alpha(\varphi^{2n+1} ln\varphi)_x = 0 \end{array} \right. \tag{9}$$

for Equation (6).

We integrate the second equations in (8) and (9), and taking the constant of integration equal to 0, we obtain:

$$\left| \begin{array}{l} (a+3k\beta)\varphi^{''} - \varphi(\omega + ak^2 + \beta k^3) + \varphi^{2n+1}(b+k\alpha) = 0 \\ \beta\varphi^{''} - \varphi(V + 2ak + 3\beta k^2) + \alpha\varphi^{2n+1} = 0, \end{array} \right. \tag{10}$$

$$\left| \begin{array}{l} (a+3k\beta)\varphi^{''} - \varphi(\omega + ak^2 + \beta k^3) + \varphi^{2n+1} ln\varphi(b+k\alpha) = 0 \\ \beta\varphi^{''} - \varphi(V + 2ak + 3\beta k^2) + \alpha\varphi^{2n+1} ln\varphi = 0. \end{array} \right. \tag{11}$$

The first and second equations in (10) and (11) are identical iff

$$\frac{a+3k\beta}{\beta} = \frac{b+k\alpha}{\alpha} = \frac{\omega + ak^2 + \beta k^3}{V + 2ak + 3\beta k^2}, \tag{12}$$

i.e.,

$$k = \frac{b\beta - a\alpha}{2\alpha\beta}, V = \frac{(\omega + ak^2 + \beta k^3)\alpha - (b+k\alpha)(2ak + 3\beta k^2)}{b + k\alpha},$$

$b + k\alpha \neq 0$, $\alpha\beta \neq 0$.

This is our Theorem 3. Certainly we assume further on that condition (12) holds.

Theorem 3. *For appropriate values of the real coefficients $a, b, \alpha, \beta, k, \omega, V$, Equations (5) and (6) possess three different types of solutions having the form (7):*

- *solutions for which φ forms a one parametric family of periodic functions;*
- *φ is soliton, $\varphi \in C^\infty(\xi \neq \tilde{\xi})$, $\varphi(\pm\infty) = 0$;*
- *φ blows up at some $\xi = \tilde{\xi}$ and φ is periodic unbounded function ($|\varphi(\tilde{\xi} + kT)| = |\varphi(\tilde{\xi})| = \infty$ for each $k \in \mathbf{Z}$, $T = const > 0$).*

The restrictions on the coefficients will be imposed during the proof of Theorem 3. The unbounded solutions are of two "qualitative" types: $|cosech\xi|$ and $|sec\xi|$, $|cosec\xi|$. The non-periodic solutions φ of (5) will be found in explicit form when (10) under condition (12) is investigated. In fact, then they are $|sech\xi|$, $cosech\xi$, $|sec\xi|$ and $|cosec\xi|$. Otherwise (say in the periodic case) the solutions φ up to the inverse mapping theorem are written into integral form. The solutions φ of (10) and (11) are written into integral form, but in general, the latter cannot be expressed via elementary or special functions (Jacobi ones, etc.).

3. Proof of Theorem 1

Consider the Cauchy problem (1) with $u_0(x) = b_0 e^{-a_0 x^2}$, $b_0 > 0$, $a_0 < 0$. To prove the uniqueness of the classical solution u for arbitrary $u_0(x) \in M_{2,a}$ and in the same class, we take the second solution $u_1 > 0$ of the same equation admitting the initial condition u_0. Then put $z = lnu \Rightarrow u = e^z \Rightarrow z_x = \frac{u_x}{u}$,

$$z_t = z_{xx} + z_x^2 + 2\lambda z, \tag{13}$$

$z|_{t=0} = lnu|_{t=0} = lnu_0 = z_0$.

According to (i), (ii) $z \le D(T) + ax^2$, $z \ge lnA_1(T) - e^{bx^2}$, i.e., $|z| \le D_1(T)e^{ax^2} \Rightarrow z \in M_{2,a}$. Moreover, $|z_x| = |\frac{u_x}{u}| \le A_2(1+|x|)$. In a similar way, $z_1 = lnu_1$ satisfies (13) , $z_1 \in M_{2,a}$, $|z_{1x}| \le A_2(1+|x|)$. Define $w = z_1 - z$. Evidently, $w \in M_{2,a}$, $|w_x| \le 2A_2(1+|x|)$ and

$$\left| \begin{array}{l} w_t = w_{xx} + D(t,x)w_x + 2\lambda w, D(t,x) = z_1 + z \\ w|_{t=0} = 0. \end{array} \right. \tag{14}$$

Certainly, $|D| \le 2A_2(1+|x|)$, λ is a constant.

We apply to the linear equation with respect to w Theorem 10 from Chapter II of [24] and conclude that $w \equiv 0 \Rightarrow z \equiv z_1 \Rightarrow u_1 \equiv u$. For parabolic equations, see also [25].

To find a solution of (1) of the form (2), we find $\partial_t u = (b' - a'bx)e^{-a(t)x^2}$, $\partial_x u = -2axbe^{-ax^2}$, $\partial_x^2 u = (-2ab + 4a^2x^2b)e^{-ax^2}$ and from the Equation (1), we obtain

$$b' - a'bx^2 = -2ab + 4a^2x^2b + \lambda b(lnb^2 - 2ax^2),$$

i.e., the system of ODE

$$\left| \begin{array}{l} a' = -4a^2 + 2\lambda a = 2a(\lambda - 2a) \\ b' = -2ab + \lambda blnb^2, \end{array} \right.$$

$a(0) = a_0, b(0) = b_0; a_0 < 0$.

As $u > 0 \Rightarrow b(t) > 0$ everywhere in $[0,T]$. It is easy to see that if $0 > a_0 > \frac{\lambda}{2}$ there exists a solution global in $t \in \mathbf{R}^1$ of the ODE $a' = 2a(\lambda - 2a)$ such that $a'(t) > 0$, $\exists a(+\infty) = 0$, $\exists a(-\infty) = \frac{\lambda}{2}$ and $F(a) = \int \frac{da}{a(\lambda-2a)} = 2t - lnC$, while $F(a) = -\frac{1}{\lambda}ln|\frac{\lambda-2a}{a}|$. Thus, $-\frac{1}{\lambda}ln|\frac{\lambda-2a}{a}| + lnC = 2t \Rightarrow C|\frac{\lambda-2a}{a}|^{-1/\lambda} = e^{2t}$.

Having in mind that $a(0) = a_0 < 0$, we find $C = |\frac{\lambda-2a_c}{a_0}|^{1/\lambda} \Rightarrow$

$$\left|\frac{\frac{\lambda-2a_0}{a_0}}{\frac{\lambda-2a}{a}}\right| = e^{2\lambda t}. \tag{15}$$

As $0 > a(t) > \frac{\lambda}{2} \Rightarrow$

$$a(t) = \frac{\lambda}{2 + k_0e^{-2\lambda t}} \in (\frac{\lambda}{2}, 0), \tag{16}$$

where $k_0 = \frac{\lambda-2a_0}{a_0} > 0$.

In the case $a_0 = \frac{\lambda}{2} \Rightarrow a(t) = \frac{\lambda}{2}$ (uniqueness).

If $\lambda > 2a_0 \Rightarrow 0 > \lambda > 2a(t)$, $a'(t) < 0$ and $a(t) = \frac{\lambda}{2+k_0e^{-2\lambda t}}$ but $k_0 < 0$ and $a(t)$ blows up for $\bar{t} = \frac{1}{2|\lambda|}ln\frac{2}{|k_0|} > 0$.

If $\lambda > 0 \Rightarrow a'(t) < 0$ and the solution $a(t)$ blows up.

The equation

$$\left| \begin{array}{l} b' = -2ab + 2\lambda blnb, b(t) > 0 \\ b(0) = b_0 > 0 \end{array} \right.$$

can be rewritten as

$$\left| \begin{array}{l} \frac{d}{dt}lnb = -2a + 2\lambda ln\, b \\ lnb(0) = lnb_0 \end{array} \right.$$

and this is linear ODE with respect to $q = lnb$.

Thus,

$$lnb(t) = e^{2\lambda t}(C_2 - 2\int_0^t a(s)e^{-2\lambda s}ds), C_2 = lnb_0 \Rightarrow \tag{17}$$

$$b(t) = b_0^{e^{2\lambda t}} e^{-2e^{2\lambda t}} \int_0^t a(s)e^{-2\lambda s}ds. \tag{18}$$

Evidently (see (16)),

$$\int a(s)e^{-2\lambda s}ds = \lambda \int \frac{e^{-2\lambda s}ds}{2+k_0e^{-2\lambda s}} =$$

$$\frac{-1}{2k_0}\int \frac{d(2+k_0e^{-2\lambda s})}{2+k_0e^{-2\lambda s}} = -\frac{1}{2k_0}ln(2+k_0e^{-2\lambda s}).$$

So for $k_0 > 0$,

$$b(t) = b_0^{e^{2\lambda t}} e^{-\frac{1}{k_0}e^{2\lambda t}ln\frac{2+k_0}{2+k_0e^{-2\lambda t}}}. \tag{19}$$

After easy computations, we conclude that

$$b(t) = b_0^{2\lambda t}(\frac{2+k_0e^{-2\lambda t}}{2+k_0})^{\frac{1}{k_0}e^{2\lambda t}} = \tag{20}$$

$$[b_0(\frac{2+k_0e^{-2\lambda t}}{2+k^0})^{\frac{1}{k_0}}]^{e^{2\lambda t}}.$$

Remark 1. *$\lambda < 0$, $t \geq 0$, $k_0 > 0$ in (20), i.e., $e^{2\lambda t} \to_{t\to\infty} 0$ and $e^{-2\lambda t} \to +\infty$ for $t \to +\infty$.*

If $a_0 = \frac{\lambda}{2}$

$$b(t) = b_0^{e^{2\lambda t}} e^{\frac{1}{2}-\frac{1}{2}e^{2\lambda t}} = (\frac{b_0}{\sqrt{e}})^{e^{2\lambda t}} e^{1/2}. \tag{21}$$

Therefore, the solution of (1) for $a_0 = \frac{\lambda}{2} < 0$ takes the form

$$u(t,x) = (\frac{b_0}{\sqrt{e}})^{e^{2\lambda t}} \sqrt{e}e^{-\frac{\lambda}{2}x^2}. \tag{22}$$

4. Proof of Theorem 2

We repeat the same procedure as in the previous case. Things are more complicated, as the functions a, b are complex-valued and the corresponding system ODE is in $\mathbf{C}^2$, not in $\mathbf{R}^2$.

Having in mind that

$$u_t = (b' - a'b\frac{x^2}{2})e^{-ax^2/2}, u_x = -xabe^{-ax^2/2},$$

$$u_{xx} = (-ab + a^2x^2b)e^{-ax^2/2}$$

we obtain from (3) that

$$\left| \begin{array}{l} ib' - \frac{ab}{2} = \lambda bln|b|^2 \\ ia' - a^2 = 2\lambda Re\, a, a(0) = a_0 \neq 0, b(0) = b_0 \neq 0, a_0, b_0 \in \mathbf{C}^1. \end{array} \right. \tag{23}$$

It is simpler to express $b(t)$ from the first ODE. We are looking for

$$\left| \begin{array}{l} b(t) = b_0e^{i\Phi(t)} \\ \Phi(0) = 0, \end{array} \right. \tag{24}$$

i.e., $b(0) = b_0$.

Certainly, $\Phi(t)$ is a complex-valued function, $t \in \mathbf{R}^1$. Substituting (24) in the first equation of (23), we obtain

$$\Phi' + \frac{1}{2}a = -\lambda ln|b_0|^2 + 2\lambda Im\Phi. \quad (25)$$

Splitting the real and imaginary parts of (25), we come to the system of ODE in the plane $\mathbf{R}^2$:

$$\left| \begin{array}{l} Re\Phi' + \frac{1}{2}Re\ a = -\lambda ln|b_0|^2 + 2\lambda Im\Phi \\ Im\Phi' + \frac{1}{2}Im\ a = 0, Re\Phi(0) = 0, Im\Phi(0) = 0. \end{array} \right. \quad (26)$$

Thus,

$$Im\Phi(t) = -\frac{1}{2}\int_0^t Ima(s)ds \Rightarrow \quad (27)$$

$$Re\Phi' + \frac{1}{2}Re\ a = -\lambda ln|b_0|^2 - \lambda \int_0^t Ima(s)ds \Rightarrow$$

$$Re\Phi = -\frac{1}{2}\int_0^t Rea(s)ds - \lambda tln|b_0|^2 - \lambda \int_0^t (\int_0^\gamma Ima(s)ds)d\gamma. \quad (28)$$

One can rewrite $\int_0^t (\int_0^\gamma Ima(s)ds)d\gamma = \int_0^t Ima(s)(t-s)ds$ applying the Fubini theorem in the triangle $\{0 \le s \le \gamma, 0 \le \gamma \le t\} = \{0 \le s \le t, s \le \gamma \le t\}$. So

$$Re\Phi(t) = -\frac{1}{2}\int_0^t Rea(s)ds - \lambda tln|b_0|^2 - \lambda \int_0^t (t-s)Ima(s)ds. \quad (29)$$

(2), (27), and (29) imply that

$$\Phi = -\frac{1}{2}\int_0^t Rea(s)ds - \lambda tln|b_0|^2 - \lambda \int_0^t (t-s)Ima(s)ds - \frac{i}{2}\int_0^t Ima(s)ds, \quad (30)$$

i.e., with $A(t) = \int_0^t a(s)ds$, we have

$$b = b_0 e^{i\Phi} = b_0 e^{-i\lambda tln|b_0|^2 - \frac{i}{2}A(t)} e^{-i\lambda Im \int_0^t (t-s)a(s)ds}. \quad (31)$$

In the solvability of

$$\left| \begin{array}{l} ia' - a^2 = 2\lambda Re\ a \\ a(0) = \alpha_0 + i\beta_0, \alpha_0 \ne 0, \beta_0 \ne 0 \end{array} \right.$$

we follow [16]. $a(0) = 0 \Rightarrow a(t) = 0$.

We shall find $a(t) = -i\frac{\omega'}{\omega}$, $\omega(t) \ne 0$, i.e., $a^2 = -\frac{(\omega')^2}{\omega^2}$, $ia' = \frac{\omega''}{\omega} - \frac{(\omega')^2}{\omega^2} = \frac{\omega''}{\omega} + a^2 = a^2 + 2\lambda Im\frac{\omega'}{\omega}$ (according to the equation).

Consequently,

$$\left| \begin{array}{l} \frac{\omega''}{\omega} = 2\lambda Im\frac{\omega'}{\omega} \\ a(0) = \alpha_0 + i\beta_0 = -i\frac{\omega'(0)}{\omega(0)}. \end{array} \right. \quad (32)$$

Here we imitate the theory of Kepler's law via Newton equation, i.e., we look for ω in polar coordinates:

$$\omega = r(t)e^{i\Theta(t)}; r(t) \ge 0, \quad (33)$$

$\Theta(t)$-real-valued polar angle. As

$$\omega' = (r' + ir\Theta')e^{i\Theta}$$

$$\omega'' = e^{i\Theta}(r'' + ir\Theta'' + 2ir'\Theta' - r(\Theta')^2)$$

from (32), we obtain

$$r'' - r(\Theta')^2 + i(r\Theta'' + 2r'\Theta') = 2\lambda r\Theta', \tag{34}$$

i.e.,

$$\left| \begin{array}{l} r'' - r(\Theta')^2 = 2\lambda r\Theta' \\ r\Theta'' + 2r'\Theta' = 0 \Rightarrow (r^2\Theta') = 0 \Rightarrow \end{array} \right. \tag{35}$$

$r^2\Theta' = A = const$. This way, we come to the area first integral of Kepler's law. According to this law, the revolution of the Earth around the Sun is with constant area velocity. Then

$$a(t) = -i\frac{\omega'}{\omega} = -i\frac{(r' + ir\Theta')}{r} = -i\frac{r'}{r} + \Theta' \Rightarrow$$

$$a(t) = \frac{A}{r^2} - i\frac{r'}{r} \Rightarrow \tag{36}$$

$$a(0) = \alpha_0 + i\beta_0 = \frac{A}{(r(0))^2} - i\frac{r'(0)}{r(0)}, \Theta'(0) = \frac{A}{(r(0))^2}$$

$$\Rightarrow a(0) = \alpha_0 + i\beta_0 = -i\frac{r'(0)}{r(0)} + \Theta'(0).$$

To simplify things, we take

$$r(0) = 1 \Rightarrow \left| \begin{array}{l} r'(0) = -\beta_0 \\ \Theta'(0) = \alpha_0 = A \end{array} \right. \Rightarrow \tag{37}$$

$\Theta'(0) = A = \alpha_0$. Further on, $\beta_0 < 0$.

Therefore,

$$a(t) = \frac{\alpha_0}{r^2} - i\frac{r'}{r}. \tag{38}$$

Of course, $Re\ a = \frac{\alpha_0}{r^2}$, $\Theta(t) = \Theta(t_0) + \alpha_0 \int_0^t \frac{ds}{r^2(s)}$.

This is the main equation in our considerations here:

$$\left| \begin{array}{l} r'' = r(\Theta')^2 + 2\lambda r\Theta' = \frac{\alpha_0^2}{r^3} + 2\lambda\frac{\alpha_0}{r} \\ r(0) = 1, r'(0) = -\beta_0 > 0. \end{array} \right. \tag{39}$$

In a standard way, by multiplying (39) with r and integrating from 0 to t, we conclude that

$$\left| \begin{array}{l} (r')^2(t) = \alpha_0^2 + \beta_0^2 - \frac{\alpha_0^2}{r^2(t)} + 4\lambda\alpha_0 ln\, r, t > 0 \\ r(0) = 1. \end{array} \right. \tag{40}$$

Therefore, we shall concentrate on the following nonlinear ODE with separate variables:

$$\left| \begin{array}{l} r'(t) = \sqrt{\alpha_0^2 + \beta_0^2 - \frac{\alpha_0^2}{r^2(t)} + 4\lambda\alpha_0 ln r(t)} \\ r(0) = 1. \end{array} \right. \tag{41}$$

Put $U(r) = \beta_0^2 + \alpha_0^2(1 - \frac{1}{r^2}) + 4\lambda\alpha_0 ln\, r$. $U \in C^\infty(r > 0)$, $U(1) = \beta_0^2 > 0$, $U(+0) = -\infty$, $U(+\infty) = -\infty$ for $\lambda\alpha_0 < 0$ and $U(+\infty) = +\infty$ for $\lambda\alpha_0 > 0$. Because of this, we must consider two different cases:

(1) $\lambda\alpha_0 < 0$
(2) $\lambda\alpha_0 > 0$.

Put $B = 2\lambda\alpha_0$. Then

$$U^{'}(r) = \frac{2\alpha_0^2}{r^3} + \frac{4\lambda\alpha_0}{r} = \frac{2}{r^3}(\alpha_0^2 + Br^2).$$

In case (2), $U^{'}(r) > 0$, i.e., U is strictly monotonically increasing and there exists a unique point $r_0 > 0$ such that $U(r_0) = 0$, i.e., $U(r) > 0$ for $r > r_0$, $U(r_0) = 0$, $U^{'}(r_0) > 0$ and $U(r) < 0$ for $0 < r < r_0$. So $r_0 < 1$.

In case (1), $U^{'}(r) = 0 \iff r = r_1 = \sqrt{\frac{\alpha_0^2}{-2\alpha_0\lambda}} = \sqrt{\frac{\alpha_0^2}{-B}} > 0$, $U^{'}(r) > 0$ for $r > r_1$; $U^{'}(r) < 0$ for $r < r_1$, i.e., $U_{max} = U(r_1) > 0$ as $U(1) > 0$.

The graph of $U(r)$ is given in Figure 1.

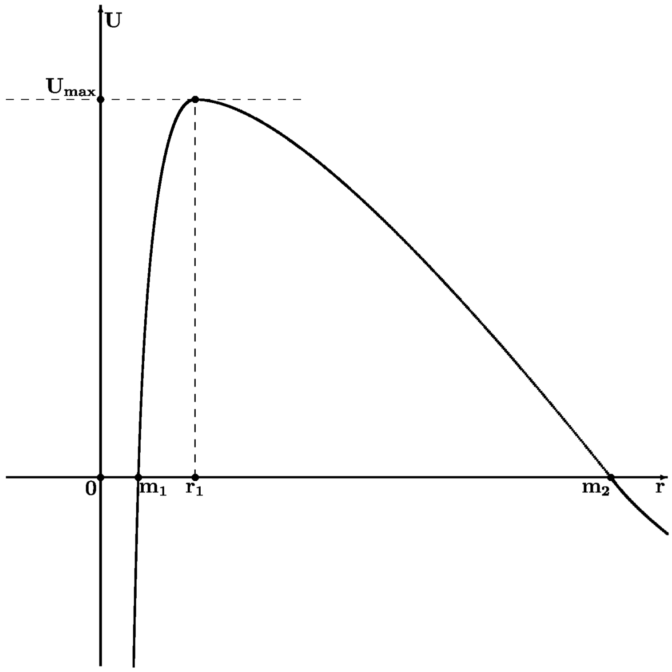

Figure 1. Graph of the potential $U(r)$, $B < 0$.

It is obvious that there exist uniquely determined points $0 < m_1 < m_2$ and $U(r) > 0$ on (m_1, m_2), $U(r) < 0$ for $0 < r < m_1$ and $r > m_2$, $U(m_1) = U(m_2) = 0$, $U^{'}(m_1) > 0$, $U^{'}(m_2) < 0$, $U(r)$ is strictly increasing for $0 < r < r_1$ and strictly decreasing for $r > m_2$. Evidently, $m_1 < 1 < m_2$.

In case (1), the unique solution of (41) is given by the formula

$$F(r(t)) = \int_1^{r(t)} \frac{ds}{\sqrt{U(s)}} = t \Rightarrow F(r(0)) = 0 \Rightarrow F(1) = 0. \tag{42}$$

This is the graph of

$$F(r) = \int_1^{r} \frac{ds}{\sqrt{U(s)}}, m_1 < r < m_2, F^{'}(r) > 0$$

for $m_1 < r < m_2$, $F(1) = 0$, $F^{'}(m_{1,2}) = +\infty$ and $0 > F(m_1), F(m_2) > 0$ are finite numbers as the integral is convergent at the end points m_1, m_2 (see Figure 2).

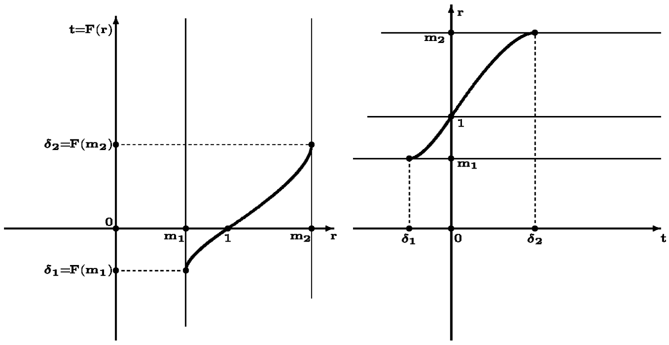

Figure 2. Graph of the function $F(r) = t$ for $B < 0$ and its inverse function $r = F^{-1}(t)$.

The inverse function $r = F^{-1}(t)$ is defined, and smooth in $[\delta_1, \delta_2]$, $r' > 0$ for $t \in (\delta_1, \delta_2)$, $r'(\delta_1) = r'(\delta_2) = 0$. We continue $r(t)$ smoothly in the interval $[\delta_2, 2\delta_2 - \delta_1]$ in an even way, i.e., $r(\delta_2 + t) = r(\delta_2 - t)$ for each $t \in [0, \delta_2 - \delta_1]$ and then periodically on $\mathbf{R}^1$ with period $T = 2(\delta_2 - \delta_1) = 2\int_{m_1}^{m_2} \frac{ds}{\sqrt{U(s)}} > 0$. Then $r(t)$ satisfies (41) for each t. Obviously, $r(t) \geq m_1$, $m_2 \geq r(t) \geq m_1$, $r(t+T) = r(t)$, $\forall t$.

Then $a(t) = \frac{\alpha_0}{r^2(t)} - i\frac{r'(t)}{r(t)}$ and $b(t)$ is given by (31). $a(t)$ is periodic with period $T > 0$ but we do not know anything about $b(t)$. To find a better expression for $b(t)$, we compute:

$$A(t) = \int_0^t a(s)ds = \alpha_0 \int_0^t \frac{ds}{r^2(s)} - i ln r(t) \tag{43}$$

$$\int_0^t (t-s) Im a(s)ds = -\int_0^t (t-s)\frac{d}{ds} ln r(s)ds = \tag{44}$$

$$-t ln r(t) + \int_0^t s\frac{d}{ds} ln r(s)ds = -\int_0^t ln r(s)ds.$$

Thus,

$$b(t) = b_0 e^{-i\lambda t ln|b_0|^2 - \frac{i\alpha_0}{2}\int_0^t \frac{ds}{r^2(s)} - ln\sqrt{r}} e^{i\lambda \int_0^t ln r(s)ds} = \tag{45}$$

$$\frac{b_0}{\sqrt{r}} e^{-i\lambda(t ln|b_0|^2 + \frac{\alpha_0}{2\lambda}\int_0^t \frac{ds}{r^2})} e^{i\lambda\int_0^t ln r(s)ds}.$$

On the other hand, if $f(t)$ is continuous periodic function with period T, its primitive $F(t) = \int_0^t f(s)ds$ is periodic with the same period if and only if $\int_0^T f(s)ds = 0$.

One can easily see that

$$F(t) = f_1(t) + \frac{t}{T}\int_0^T f(t)dt, \tag{46}$$

where the smooth periodic function $f_1(t)$ vanishes at $t = 0$.

In fact, $h(t) = f(t) - \frac{1}{T}\int_0^T f(s)ds$ is periodic with period T as $\int_0^T h(t)dt = 0$. Therefore, $f_1 = \int_0^t h(s)ds$ is periodic $\Rightarrow f_1(t) = \int_0^t f(s)ds - \frac{t}{T}\int_0^T f(s)ds$, i.e., (46) holds.

If $g(t) = e^{iKt}$, $K \neq 0$, K-real, then g is periodic with period T iff $\frac{KT}{2\pi} \in \mathbf{Z}$, $\mathbf{Z}$ being the set of the integers. The phase function of (45) can be written then as

$$-\lambda t ln|b_0|^2 - \frac{\alpha_0 t}{2T}\int_0^T \frac{ds}{r^2(s)} + g_1(t) + \frac{\lambda t}{T}\int_0^T ln r(s)ds + g_2(t),$$

$g_{1,2}(t)$ being real-valued periodic functions with period T.

Put $\mu = -\lambda ln|b_0|^2 - \frac{\alpha_0}{2T}\int_0^T \frac{ds}{r^2(s)} + \frac{\lambda}{T}\int_0^T ln r(s)ds$, $\mu \in \mathbf{R}^1 \Rightarrow$

$$b(t) = \frac{1}{\sqrt{r}} b_0 e^{i\mu t} e^{i(g_1+g_2)}. \tag{47}$$

$b(t)$ is periodic with period T if and only if $\frac{\mu T}{2\pi} \in \mathbf{Z}$. In other words, it is very rare.

Certainly, $|b(t)| = |b_0| \frac{1}{\sqrt{r(t)}}$ is periodic as $r(t+T) \equiv r(t) > 0$.

Remark 2. *The curve in* $\mathbf{C}^1$ $\omega : t \to r(t)e^{i\Theta}$ *is rather interesting. In fact,* ω *is smooth,* $\omega = r(t)e^{i\Theta(t)}$, $\Theta(t) = \Theta(0) + \alpha_0 \int_0^t \frac{ds}{r^2(s)} = \Theta(0) + \frac{\alpha_0 t}{T} \int_0^T \frac{ds}{r^2(s)} + g_3(t)$, $g_3(t+T) \equiv g_3(t)$.

Consequently the curve ω *is located in the ring* $\{z \in \mathbf{C}^1 : m_1 \le |z| \le m_2\}$. *It is periodic there if* $\frac{\alpha_0}{2\pi} \int_0^T \frac{ds}{r^2(s)} \in \mathbf{Z}$. *If* $\alpha_0 \int_0^T \frac{ds}{r^2(s)} ds$ *is an irrational number, then the curve* ω *is dense in the same ring [2,3].* ω *can have infinitely many points of self-intersection.*

Case (2) is absolutely different, as then $r(t) > 0$ is unbounded and ω is located in an angle: $\Theta_- < argz < \Theta_+$.

In case (2), we shall study (41) for $r(t) > 0$ but the integral $t = F(r) = \int_1^r \frac{ds}{\sqrt{U(s)}}$ exists for $r > r_0$, $F'(r) > 0$, $r_0 < 1$, $F'(r_0) = \infty$, $F(r_0) = \delta_0 < 0$ is a real number as the integral is convergent for $r = r_0$.

On the other hand, $U(r) \sim 2Blnr$ for $r \to \infty$, $B > 0$; $lnr \le r^{2\alpha}$ for $r \to \infty$, $0 < \alpha < 1$ implies that $F(+\infty) = \infty$. This is the graph of $F(r)$ (Figure 3).

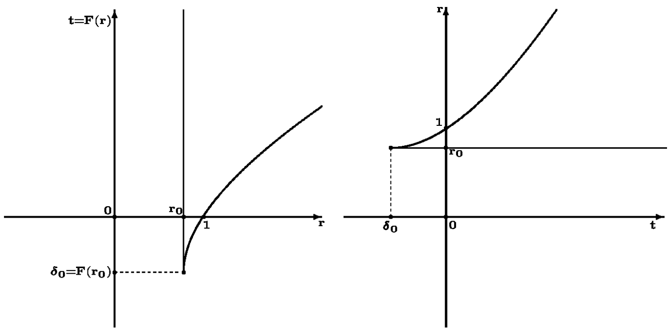

Figure 3. Graph of $t = F(r)$, $B > 0$ and its inverse function.

Its inverse function $r = F^{-1}(t)$ is such that $r'(t) > 0$ for $t > \delta_0$, $r'(\delta_0) = 0$, $r(\infty) = \infty$ (see Figure 3). Again we continue $r(t)$ in an even way with respect to $t = \delta$, i.e., $r(t+\delta) = r(\delta - t)$, $\forall t \ge 0 \Rightarrow r'(\delta) = 0$; $r(t)$ satisfies (41) for each t.

We are interested in the behavior of $F(r)$ at $+\infty$ (asymptote) as $F(\infty) = +\infty$. On the other hand, for $r \ge \tilde{r} \gg 1 : U(r) \sim 2Blnr$,

$$F(r) = \int_1^r \frac{ds}{\sqrt{U}} = \int_1^{\tilde{r}} \frac{ds}{\sqrt{U}} + \int_{\tilde{r}}^r \frac{ds}{\sqrt{U}}.$$

According to L'Hospital rule,

$$lim_{r\to\infty} \frac{\int_{\tilde{r}}^r ds/\sqrt{U(s)}}{\int_{\tilde{r}}^r \frac{ds}{\sqrt{E+2Blnr}}} = lim_{r\to\infty} \frac{\frac{1}{\sqrt{U(r)}}}{\frac{1}{\sqrt{E+2Blnr}}} = 1,$$

i.e., with $E > 0$, $E = const = \alpha_0^2 + \beta_0^2 > 0$, $F(r) \sim \frac{1}{\sqrt{2B}} \int_{\tilde{r}}^r \frac{ds}{\sqrt{E_1 + lns}}$, $r \to \infty$, $E_1 > 0$.

Obviously, after the change $lns = z^2$, $s = e^{z^2}$

$$\int_{\tilde{r}}^r \frac{ds}{\sqrt{E_1 + lns}} = \int_{\sqrt{ln\tilde{r}}}^{\sqrt{lnr}} \frac{2ze^{z^2}dz}{\sqrt{E_1 + z^2}} \sim 2\int_{\sqrt{ln\tilde{r}}}^{\sqrt{lnr}} e^{z^2} dz,$$

$r \to \infty$ (L'Hospital is again applied). Our last step is to show that $\int^{\mu} e^{z^2} dz \sim \frac{e^{\mu^2}}{2\mu}$ for $\mu \to \infty$.

This way we conclude that

$$F(r) \sim \frac{e^{lnr}}{\sqrt{2B}\sqrt{lnr}} = \sqrt{\frac{1}{2B}}\frac{r}{\sqrt{lnr}}, r \to \infty, \tag{48}$$

i.e.,

$$t = F(r) \sim \sqrt{\frac{1}{2B}}\frac{r}{\sqrt{lnr}} = \frac{1}{2\sqrt{\alpha_0\lambda}}\frac{r}{\sqrt{lnr}}, r \to \infty. \tag{49}$$

(49) can be asymptotically inverted looking for $r \sim 2\sqrt{\alpha_0\lambda}tln^{\beta}t$, $t \to \infty$, $\beta > 0$. As $lnr \sim lnt + \beta ln(lnt) \sim lnt$, $\to \infty$ we find $\beta = 1/2$, i.e.,

$$r(t) \sim 2\sqrt{\lambda\alpha_0}t\sqrt{lnt}, t \to \infty. \tag{50}$$

Due to the definition of $r(t)$ for $t \ll 0$

$$r(t) \sim 2\sqrt{\lambda\alpha_0}|t|\sqrt{ln|t|}, |t| \to \infty, \tag{51}$$

$r^{'} > 0$ for $r > \delta$, $r^{'} < 0$ for $r < \delta$,

$$r^{'}(t) \sim 2\sqrt{\lambda\alpha_0}sgnt\sqrt{ln|t|}, |t| \to +\infty. \tag{52}$$

From (38), (51), and (52), it follows that $Rea(t) = \frac{\alpha_0}{r^2} \neq 0, \forall t \Rightarrow a(t) \neq 0$ for each t,

$$a(t) \sim \frac{1}{4\lambda t^2 ln|t|} - \frac{i}{t}, |t| \to \infty,$$

i.e., $|t| \to \infty \Rightarrow a(t) \to 0$.

As it concerns $\Theta(t) = \Theta(0) + \alpha_0 \int_0^t \frac{ds}{r^2(s)}$ we know that $\frac{1}{r^2(t)} \sim M\frac{const}{t^2 ln|t|}$, $|t| \to \infty, M > 0$. The integral is convergent for $t \to \infty$ and $t \to -\infty \Rightarrow \Theta_+ = \Theta(+\infty)$, $\Theta_- = \Theta(-\infty)$ are finite numbers and the angle $\frac{1}{\alpha_0}\Theta \in (\Theta_-, \Theta_+)$. This is the graph of $\omega = r(t)e^{i\Theta}$, $min_t r = r_0 > 0$ (see Figure 4).

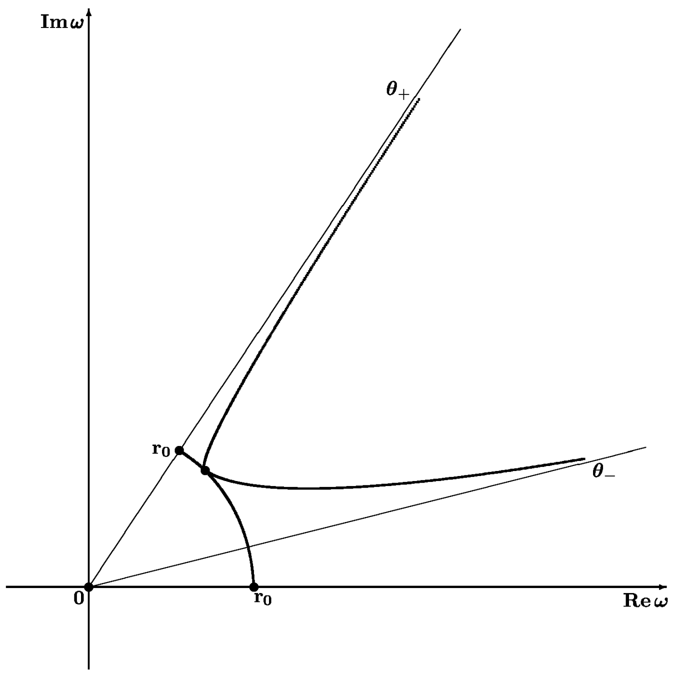

Figure 4. Graph of $\omega = r(t)e^{i\Theta(t)}$.

Our last step is to illustrate geometrically the curve $t \to a(t) \in \mathbf{C}^1$. As $Rea = \frac{\alpha_0}{r^2(t)}$, $\alpha_0 < 0 \iff \lambda < 0 \Rightarrow Rea(0) = \alpha_0 < 0 \Rightarrow |e^{-a_0x^2}| = e^{-\alpha_0 x^2}$, i.e., $|u|$ is rapidly increasing in function x for fixed t. Moreover, $|\frac{Ima}{Rea}| \sim const|t||ln|t||$, $|t| \to \infty$, $Rea(t) < 0$, $Rea \geq \frac{\alpha_0}{r_0^2}$, $r(\delta_0) = r_0$, $r^{'}(\delta_0) = 0$, $r(\delta_0) > 0$ and $a(\pm\infty) = 0$, a is tangential to the imaginary axes at the origin, $Ima < 0$ for $t \gg 1$, $Ima > 0$ for $t \ll -1$, $Ima = 0 \iff r^{'}(t) = 0 \iff t = \delta_0$. $a(t)$ is given in Figure 5.

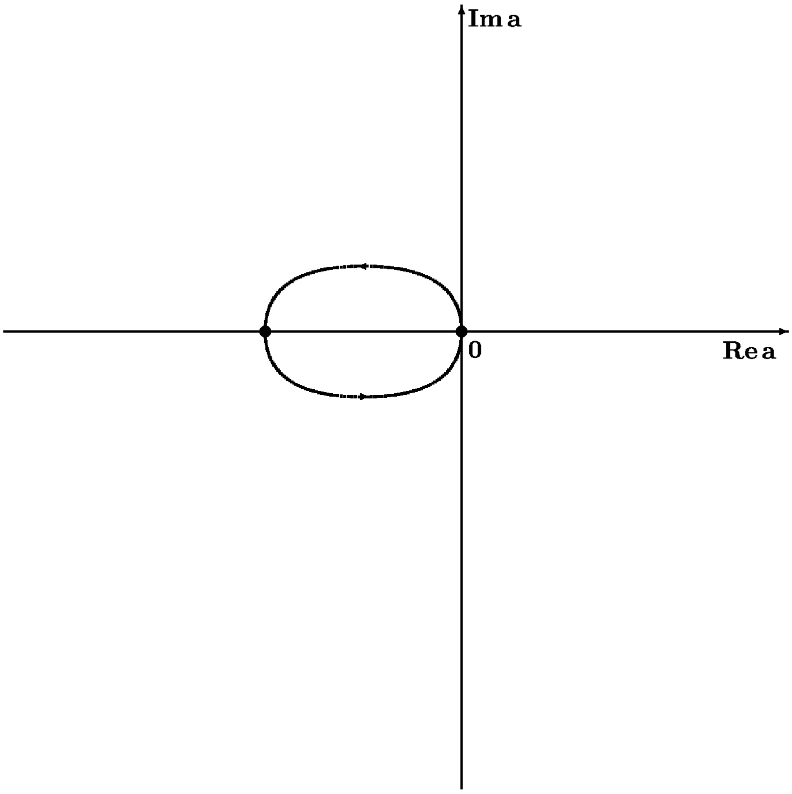

Figure 5. Graph of $a(t) = \frac{\alpha_0}{r^2} - i\frac{r^{'}}{r}$.

Formula (45) shows that $|b(t)| = \frac{|b_0|}{\sqrt{r}} \to 0$ for $|t| \to \infty$. The phase function in t: $tln|b_0|^2 + \frac{\alpha_0}{2\lambda}\int_0^t \frac{ds}{r^2(s)} + \lambda\int_0^t lnr(s)ds$ is such that $\int_0^{+\infty} \frac{ds}{r^2(s)} = \Theta_+ > 0$, $\int_0^{-\infty} \frac{ds}{r^2(s)} = \Theta_0 < 0$ and $\lambda\int_{\tilde{r}}^r lnr(s)ds \sim \lambda\int_{\tilde{t}}^t lnsds \sim \lambda tlnt$, $t \to \infty$.

In other words, the phase function in $b(t)$ is superlinear for $t \to +\infty$ with growth $\lambda tlnt$. Last observation: $b(t)$ describes a spiral in $\mathbf{C}^1$ tending to 0, i.e., $t \to b(t) \in \mathbf{C}^1$ is a focus in the complex plane, $e^{-ax^2} = e^{-Reax^2 - iImax^2}$. For each fixed t, $e^{-Rea(t)x^2}$ is exponentially increasing in x. For fixed x: $-Ima(t) \sim \frac{r^{'}}{r} \sim \frac{1}{t}$ for $|t| \to \infty$:

The angle of rotation is bounded, $0 > Rea(t)$, $Rea(t) \to 0$, $|t| \to \infty$.

5. Proof of Theorem 3

1. We shall study first the ODE (11) under number

$$\varphi^{''} = -A\varphi^{2n+1}ln\varphi + B\varphi, \tag{53}$$

where $A = \frac{\alpha}{\beta}$, $B = \frac{V+2\alpha k+3\beta k^2}{\beta}$ with $A \neq 0$, $B \neq 0$.

At the beginning, we observe that the smooth function $h(\varphi) = \varphi^\sigma ln\varphi$, $\varphi > 0$, $\sigma \geq 1$ has the following properties: $h(0) = 0$, $h^{'}(0) = 0$ for $\sigma > 1$ and $h^{'}(0) = -\infty$ for $\sigma = 1$,

$h'(\varphi) > (<)0$ for $\varphi > (<)\varphi_\sigma$, $\varphi_\sigma = e^{-\frac{1}{\sigma}} < 1$, $h_{min} = -\frac{1}{e\sigma}$, $h(1) = 0$, $h(\varphi) < 0$ for $0 < \varphi < 1$; $h(\varphi) > 0$ for $\varphi > 1$. In a standard way, we obtain from (53) that

$$(\varphi')^2 = B\varphi^2 + \frac{A\varphi^{2n+2}}{n+1}(\frac{1}{2n+2} - ln\varphi) + 2C, C = const.$$

Certainly, $(\varphi')^2 \geq 0$ and $\varphi > 0$. So if we put

$$g(\varphi) = \varphi^2(B + \frac{A}{n+1}\varphi^{2n}(\frac{1}{2n+2} - ln\varphi)), \varphi > 0 \tag{54}$$

we come to the ODE

$$(\varphi')^2 = 2(C + \frac{g}{2}) = 2(C - U(\varphi)), U(\varphi) = -\frac{1}{2}g(\varphi).$$

Thus,

$$\sqrt{2}(\xi - \xi_0) = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\sqrt{C - U(\lambda)}}, U(\lambda) \geq C, \sigma = 2n+1. \tag{55}$$

We have four different cases for $g(\varphi)$: $A > 0$, $B > 0$; $A > 0$, $B < 0$; $A < 0$, $B > 0$; $A < 0$, $B < 0$. Evidently, $g(+0) = 0$, $g(\varphi) \sim B\varphi^2$, $\varphi \to 0$, $g(+\infty) = -sgnA(+\infty)$.

Moreover,

$$g'(\varphi) = 2\varphi(B - A\varphi^{2n+1}ln\varphi), \varphi > 0$$

$$g'(\varphi) > (<)0 \iff B - A\varphi^{2n+1}ln\varphi > 0(< 0),$$

$g'(\varphi) = 0 \iff \varphi_1^{2n+1}ln\varphi_1 = \frac{B}{A}$ for some $\varphi_1 > 0$.

Assume (a) $B > 0$, $A > 0$. Then $g'(\varphi) > (<)0 \iff \varphi^{2n+1}ln\varphi < (>)\frac{B}{A}$, $g'(\varphi) = 0 \iff h(\varphi_1) = \frac{B}{A} > 0$, i.e., there exists a unique point $\varphi_1 > 1$ with this property $\Rightarrow g'(\varphi_1) = 0$, $g'(\varphi) < 0$ for $\varphi > \varphi_1$ and $g'(\varphi) > 0$ for $0 < \varphi < \varphi_1$, $g_{max} = g(\varphi_1) > 0$ as $g(1) = B + \frac{A}{2(n+1)^2} > 0$ (see the graph of $g(\varphi)$ in Figure 6 and the graph of $U(\varphi)$ in Figure 7). φ_2 is the unique point $\varphi_2 > \varphi_1$ such that $g(\varphi_2) = 0$, $g'(\varphi_2) < 0$.

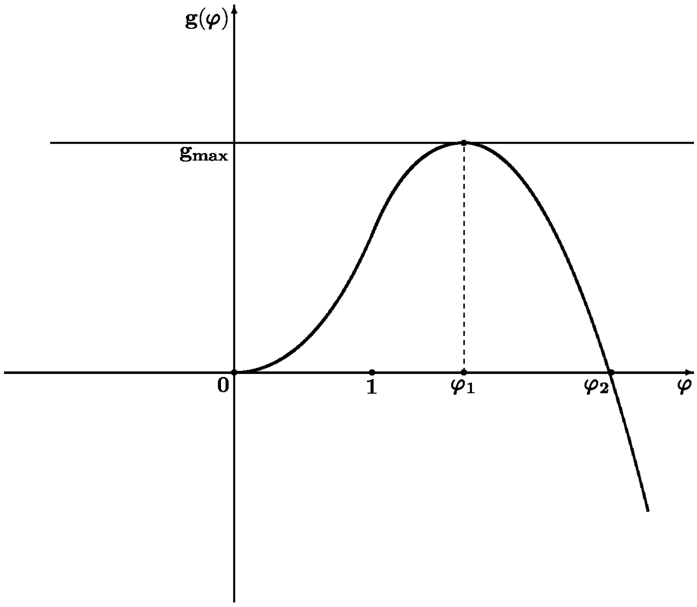

Figure 6. Graph of $g(\varphi)$, $B > 0$, $A > 0$.

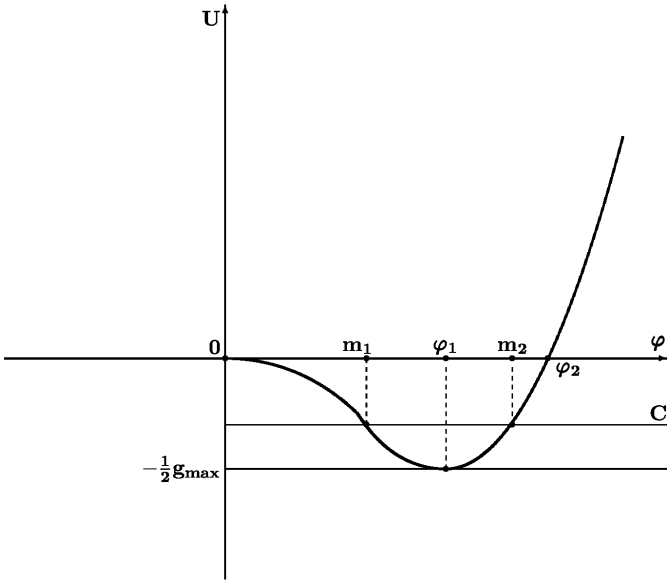

Figure 7. Graph of $U(\varphi) = -\frac{1}{2}g(\varphi)$.

Take $0 > C > -\frac{1}{2}g_{max}$ and let $0 < m_1 < m_2 < \varphi_2$ be the unique points $m_1 < \varphi_1$, $m_2 > \varphi_1$ with the properties $U(m_{1,2}) = C$, $U^{'}(m_1) < 0$, $U^{'}(m_2) > 0$. The initial data $\varphi_0 \in (m_1, m_2)$. As we know, formula (55) defines a periodic solution, $\varphi(\xi_0) = \varphi_0$ (see Proof of Theorem 2 , Figure 2 there, etc.). If $C = 0$, $m_1 = 0$, $m_2 = \varphi_2$, $0 < \varphi_0 < m_2$ and the integral $\xi = F(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\sqrt{g(\lambda)}}$, $0 < \varphi < \varphi_2$ has the following properties: $F(\varphi_0) = 0$, $F^{'}(\varphi) > 0$ on $(0, \varphi_2)$, $F(+0) = -\infty$, $F(\varphi_2) > 0$ is some number and $F^{'}(\varphi_2) = +\infty$. The inverse function $\varphi = F^{-1}(\xi)$ is defined on $\{-\infty < \xi \leq A\}$, $\varphi > 0$ there, $\varphi^{'}(\xi) > 0$, $\varphi^{'}(A) = 0$ for some A. We continue $\varphi(\xi)$ in an even way with respect to A, i.e., $\varphi(\xi + A) = \varphi(A - \xi)$, $\forall \xi \geq 0$ obtaining this way a soliton solution of (11).

(b) Suppose that $A < 0$, $B > 0$, $|\frac{B}{A}| < \frac{1}{(1+2n)e}$, i.e., $0 > \frac{B}{A} > -\frac{1}{(1+2n)e} = h_{min}$. It follows that $g^{'}(\varphi) = 0 \iff h(\varphi_1) = \varphi_1^{2n+1} ln\varphi_1 = \frac{B}{A} \in (0, -\frac{1}{(2n+1)e})$, $g(1) = B > 0$; $g^{'}(\varphi) > 0 \iff h_{min} < \frac{B}{A} < h(\varphi)$, $0 < F(\infty) < \infty$. We take the point $1 > \varphi_1 > e^{-\frac{1}{2n+1}}$ as there are two numbers satisfying $h(\varphi) = \frac{B}{A} < 0 \to$ one of them less that $e^{-\frac{1}{2n+1}}$ and the other is φ_1.

Having in mind that $h^{'}(\varphi) > 0$ for $\varphi > e^{-\frac{1}{2n+1}}$, we conclude that for $\varphi > \varphi_1 \Rightarrow h(\varphi) > h(\varphi_1) = \frac{B}{A} \Rightarrow g^{'}(\varphi) > 0$, i.e., g is strictly monotonically increasing and $g(\varphi) \geq g(\varphi_1)$ for $\varphi \geq \varphi_1$,

$$g(\varphi_1) = \varphi_1^2 (B + \frac{A}{n+1} \varphi_1^{2n} (\frac{1}{2n+2} - ln\varphi_1)),$$

i.e., $g(\varphi_1) \geq (<) 0$ if and only if $1 + \frac{1}{2(n+1)^2 \varphi_1 ln\varphi_1} \geq (<0) \frac{1}{(n+1)\varphi_1}$. Assuming $g(\varphi_1) = 0$ and having in mind that $g^{'}(\varphi_1) = 0$, we can construct a solution of (53) of the type $|cosech\xi|$, as for $\varphi \geq \varphi_1$, $\varphi_0 > \varphi_1$ the integral $F(\varphi)$ has the following properties: $F^{'}(\varphi) > 0$, $F(\varphi_0) = 0$, $F(\varphi_1) = -\infty$, $0 < F(+\infty) < \infty$. $\varphi(t)$ has a blow up for finite time, $\varphi(t) > 0$ everywhere and $\varphi(\pm\infty) = 0$. The latter result is rather implicit. So it is better to study the function $p(\varphi) = B + \frac{A}{n+1}\varphi^{2n}(\frac{1}{2n+2} - ln\varphi)$, $\varphi > 0$, $B > 0$, $A < 0$.

Evidently, $g(\varphi) = \varphi^2 p(\varphi)$, $p^{'}(\varphi) = \frac{-A}{n+1}\varphi^{2n-1}(\frac{1}{n+1} + 2nln\varphi)$, $p^{'}(\varphi) > (<) 0 \iff \varphi > (<) \varphi_3 = e^{-\frac{1}{2(n+1)n}}$, $p_{min} = p(\varphi_3) = B + \frac{A}{2n(n+1)}e^{-\frac{1}{n+1}}$, $p(0) = B$, $p(+\infty) = \infty$, $p^{'}(\varphi_3) = 0$.

Thus, $g(t) \geq \varphi^2 p_{min}$. We are interested in the subcases of (b) $p(\varphi_3) > 0$; $p(\varphi_3) = 0 \iff B = -\frac{A}{2n(n+1)}e^{-\frac{1}{n+1}} > 0 \Rightarrow p(\varphi) > 0$ for $\varphi > \varphi_3$, $p(\varphi_3) = p^{'}(\varphi_3) = 0$. The integral $\xi = F(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\lambda\sqrt{p(\lambda)}}$ is investigated for $\varphi > \varphi_3$ and the initial data $\varphi_0 > \varphi_3$. Then

$0 < F(+\infty) < \infty$, $F^{'}(\varphi) > 0$ for $\varphi > \varphi_3$, $F(\varphi_3) = -\infty$, $F(\varphi_0) = 0$. The inverse function of $\xi = F(\varphi)$, $\varphi = F^{-1}(\xi)$ is positive in some interval $\{\xi < A\}$, $\varphi(-\infty) = 0$, $\varphi(A) = \infty$, $\varphi^{'} > 0$. We continue in an even way with respect to A the function φ and obtain a $|cosech\xi|$-type solution of (53).

The third subcase is $p_{min} < 0$ and consequently there exists unique $\varphi_4 > \varphi_3$ such that $p(\varphi_4) = 0$, $p(\varphi) > 0$ for $\varphi > \varphi_4$, $p^{'}(\varphi_4) > 0$. (55) will be considered on the interval $\varphi \geq \varphi_4 > 0$, $\xi = F(\varphi) = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\lambda\sqrt{p(\lambda)}}$, $\varphi \geq \varphi_4$, $\varphi_0 > \varphi_4$. Obviously, $F^{'}(\varphi) > 0$, $F(\varphi_4) < 0$, $F^{'}(\varphi_4) = -\infty$, $F(+\infty) < \infty$. The inverse function F^{-1} has the behavior of $|cosec\xi|$ (Figure 8).

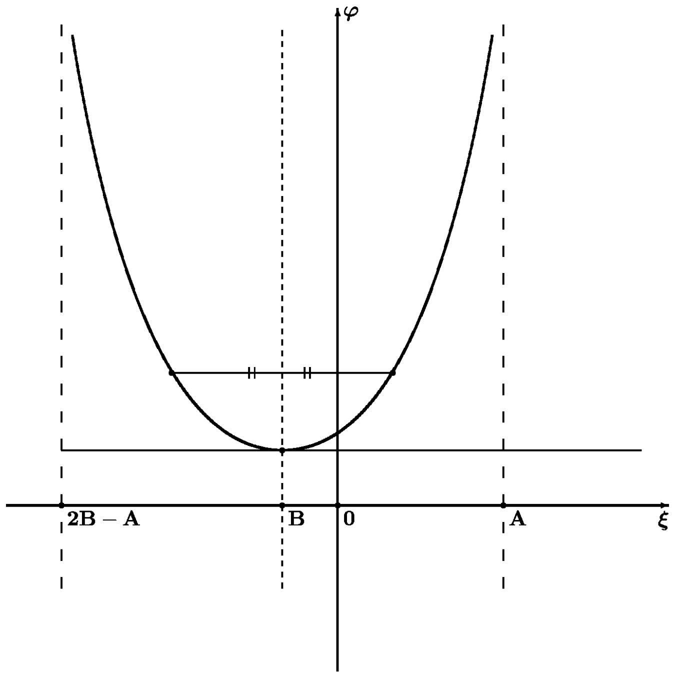

Figure 8. Graph of the inverse function φ, $p_{min} < 0$, $\varphi(B - \xi) = \varphi(B + \xi)$, $0 \leq \xi \leq A - B$, $\varphi(\xi + T) = \varphi(\xi)$, $T = \frac{(A-B)}{2}$.

The other two cases $A < 0$, $B > 0$ or $B < 0$ are omitted as they can be studied similarly to the previous cases (a) and (b).

2. We shall study now Equation (10) under condition (12), i.e.,

$$\varphi^{''} = B\varphi - A\varphi^{2n+1}, A = \frac{\alpha}{\beta}, B = \frac{V + 2ak + 3\beta k^2}{\beta}. \tag{56}$$

In a standard way, we come to the first-order ODE

$$(\varphi^{'})^2 = B\varphi^2 - \frac{A}{n+1}\varphi^{2n+2} + 2C = 2C + g, C = const,$$

where

$$g(\varphi) = \varphi^2(B - \frac{A}{n+1}\varphi^{2n}). \tag{57}$$

So $(\varphi^{'})^2 = 2(C - U)$, $U = -\frac{1}{2}g(\varphi)$.

In the special case $C = 0$, we conclude that

$$\xi - \xi_0 = G(\varphi) - G(\varphi_0) = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\sqrt{g(\lambda)}}, \tag{58}$$

i.e., $G(\varphi)$ is any primitive of $\frac{1}{\sqrt{g(\varphi)}}$, $\varphi, \varphi_0 > 0$, $g(\varphi) > 0$.

There are four cases for the signs of A, B in $g(\varphi)$: $A > 0$, $B > 0$; $A > 0$, $B < 0$ - impossible; $A < 0$, $B > 0$ and $A < 0$, $B < 0$.

Thus,

$$\sqrt{2}(\xi - \xi_0) = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\sqrt{C - U(\lambda)}}.$$

If $B > 0$, $A < 0$ (56) possesses a one parametric family of periodic solutions. The proof repeats the proof of the similar results for (41), (1) and we omit it.

Suppose that

$$\xi - \xi_0 = \int_{\varphi_0}^{\varphi} \frac{d\lambda}{\varphi\sqrt{B - \frac{A}{n+1}\lambda^{2n}}} = F(\varphi) \tag{59}$$

and $A > 0, B > 0$; $A < 0, B > 0$; $A < 0, B < 0$. Certainly, $\varphi > 0$, $\varphi_0 > 0$.

The change $\lambda^n = z$ in (59) gives us that

$$F(\varphi) = \frac{1}{n}\int_{\varphi_0^n}^{\varphi^n} \frac{dz}{z\sqrt{B - \frac{A}{n+1}z^2}} = \frac{1}{n}\sqrt{\frac{n+1}{|A|}}\int_{\varphi_0^n}^{\varphi^n} \frac{dz}{z\sqrt{\frac{B(n+1)}{|A|} - sgnAz^2}}.$$

We shall study the cases

$$\left| \begin{array}{l} B > 0, A > 0 \ \text{ with } \ \alpha^2 = \frac{B(n+1)}{A}, \alpha > 0 \\ B < 0, A > 0 \ \text{ with } \ \frac{B(n+1)}{A} = -\alpha^2, \alpha > 0. \end{array} \right.$$

The case $A < 0$, $B < 0$ is treated in a similar way as $A > 0$, $B > 0$ and we omit the proof.

According to formulae 341.01, 281.01 from [4], we have that with $\mu = \varphi^n$, $\mu_0 = \varphi_0^n$

$$F(\varphi) = \frac{1}{n}\sqrt{\frac{n+1}{|A|}} \begin{cases} -\frac{1}{\alpha}ln(\frac{\alpha+\sqrt{\alpha^2-\mu^2}}{\alpha+\sqrt{\alpha^2-\mu_0^2}}\frac{\mu_0}{\mu}), 0 < \mu < \alpha, A > 0, B > 0 \\ \frac{1}{\alpha}(arccos\frac{\alpha}{\mu} - arccos\frac{\alpha}{\mu_0}), 0 < \alpha < \mu, 0 < \alpha < \mu_0, \\ B < 0, A > 0. \end{cases}$$

Therefore, (59) implies that

$$\mu e^{-D(\xi-\xi_0)}E - \alpha = \sqrt{\alpha^2 - \mu^2}, A > 0, B > 0 \tag{60}$$

and $0 < D = n\sqrt{\frac{|A|}{n+1}}\alpha$, $E = \frac{\alpha+\sqrt{\alpha^2-\mu_0^2}}{\mu_0} > 0$.

From (60), we obtain that

$$\varphi^n = \mu = \frac{2\alpha E}{e^{D(\xi-\xi_0)} + E^2e^{-D(\xi-\xi_0)}}, \tag{61}$$

where $\alpha = \sqrt{\frac{(n+1)|B|}{|A|}}$, $E = \frac{\alpha+\sqrt{\alpha^2-\varphi_0^{2n}}}{\varphi_0^n}$, $D = n\sqrt{|B|}$.

(61) is a soliton-type solution, of course, and can be expressed via $sech(C_1\xi + C_2)$.

In the case $B < 0$, $A > 0$ we have that

$$D(\xi - \xi_0) = arccos\frac{\alpha}{\mu} - arccos\frac{\alpha}{\mu_0}, 0 < \alpha < \mu_0, 0 < \alpha < \mu.$$

From trigonometry, it is known that for $0 < x < 1$, $0 < y < 1$:

$$arccosx - arccosy = arctg\frac{\sqrt{\frac{1}{x^2} - 1} - \sqrt{\frac{1}{y^2} - 1}}{1 + \sqrt{\frac{1}{x^2} - 1}\sqrt{\frac{1}{y^2} - 1}}. \tag{62}$$

Put $R = \sqrt{\frac{1}{x^2} - 1}$, $S = \sqrt{\frac{1}{y^2} - 1} \Rightarrow x = \frac{1}{\sqrt{1+R^2}}$, $y = \frac{1}{\sqrt{1+S^2}}$. Put $x = \frac{\alpha}{\mu} \in (0,1)$; $y = \frac{\alpha}{\mu_0} \in (0,1)$. So

$$tgD(\xi - \xi_0) = \frac{R - S}{1 + RS} \Rightarrow R = \frac{tgD(\xi - \xi_0) + S}{1 - StgD(\xi - \xi_0)}$$

Thus, $x = \frac{1}{\sqrt{1+R^2}} = y|cosD(\xi - \xi_0) - SsinD(\xi - \xi_0)| \Rightarrow$

$$\frac{\varphi^n}{\alpha} = \frac{1}{x} = \frac{1}{y|cosD(\xi - \xi_0) - SsinD(\xi - \xi_0)|} \iff \tag{63}$$

$$\varphi^n = \frac{\varphi_0^n}{|cosD(\xi - \xi_0) - SsinD(\xi - \xi_0)|}, \tag{64}$$

where $D = n\sqrt{|B|}$, $S = \frac{\sqrt{\mu_0^2 - \alpha^2}}{\alpha} = \frac{\sqrt{\varphi_0^{2n} - \alpha^2}}{\alpha}$, $\alpha = \sqrt{\frac{(n+1)|B|}{|A|}}$. In other words,

$$\varphi = \frac{\varphi_0}{|cosD(\xi - \xi_0) - SsinD(\xi - \xi_0)|^{\frac{1}{n}}}. \tag{65}$$

The expression in the denominator can be expressed as $cosec(A_1\xi + B_1)$.

Below we propose several useful identities from classical and hyperbolic trigonometries.

$$\text{For each } P, Q \text{ real}: \quad Pcosx + Qsinx = \sqrt{P^2 + Q^2}sin(x + \alpha), \tag{66}$$

$$\text{where } sin\alpha = \frac{P}{\sqrt{P^2 + Q^2}}, cos\alpha = \frac{Q}{\sqrt{P^2 + Q^2}},$$

$$\text{For each } P > 0, Q \text{ such that } P > |Q| > 0: Pchx + Qshx = \sqrt{P^2 - Q^2}ch(x + \alpha), \tag{67}$$

$$\text{where } ch\alpha = \frac{P}{\sqrt{P^2 - Q^2}}, sh\alpha = \frac{Q}{\sqrt{P^2 - Q^2}}.$$

If $|P| = |Q|$, i.e., $P = \pm Q \Rightarrow Pchx + Qshx = P(chx \pm shx) = Pe^{\pm x}$.

Assume that $P > 0$, $Q > 0$. Then

$$Pe^x + Qe^{-x} = 2\sqrt{PQ}ch(x + \alpha), \tag{68}$$

where $ch\alpha = \frac{1}{2}(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}})$, $sh\alpha = \frac{1}{2}(\sqrt{\frac{P}{Q}} - \sqrt{\frac{Q}{P}})$.

Suppose that $P, Q > 0$. Then

$$Pe^x - Qe^{-x} = 2\sqrt{PQ}sh(x + \alpha), \tag{69}$$

where $sh\alpha = \frac{P-Q}{2\sqrt{PQ}} = \frac{1}{2}(\sqrt{\frac{P}{Q}} - \sqrt{\frac{Q}{P}})$, $ch\alpha = \frac{1}{2}(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}})$. Therefore, $\frac{1}{Pe^x - Qe^{-x}} = \frac{1}{2\sqrt{PQ}}cosech(x + \alpha)$, $P, Q > 0$.

Conclusion: The solution (61) can be rewritten as

$$\varphi^n = \alpha sech(D(\xi - \xi_0) + \delta), \tag{70}$$

where $ch\delta = \frac{1}{2}(\frac{1}{E} + E)$, $sh\delta = \frac{1}{2}(\frac{1}{E} - E)$. Formula (64) takes the form:

$$\varphi^n = \frac{\varphi_0^n}{|sin(D(\xi - \xi_0) - \beta)|\sqrt{1 + S^2}} = \alpha|cosec(D(\xi - \xi_0) - \beta)|, \tag{71}$$

where $sin\beta = \frac{1}{\sqrt{1+S^2}} = \frac{\alpha}{\mu_0}$, $cos\beta = \frac{S}{\sqrt{1+S^2}} = \frac{\sqrt{\mu_0^2 - \alpha^2}}{\mu_0}$; $\mu_0 = \varphi_0^n$, $D = n\sqrt{|B|}$, $\alpha = \sqrt{(n+1)\frac{|B|}{|A|}}$.

The function (71) is strictly positive and, being unbounded, is periodic with period $\frac{2\pi}{D} = \frac{2\pi}{n\sqrt{|B|}}$.

Remark 3. *Suppose that* $n = \frac{1}{2}$ *in (56), i.e.,*

$$\varphi^{''} = B\varphi - A\varphi^2.$$

Therefore,

$$(\varphi^{'})^2 = -\frac{2A\varphi^3}{3} + B\varphi^2 + C. \tag{72}$$

If

$$n = 1 \Rightarrow \varphi^{''} = B\varphi - A\varphi^3 \Rightarrow (\varphi^{'})^2 = B\varphi^2 - \frac{A}{2}\varphi^4 + C. \tag{73}$$

Put $P_3(\varphi) = -\frac{2A\varphi^3}{3} + B\varphi^2 + C$, $A > 0$ and assume that $P_3(\varphi) = 0$ has three simple real roots $\varphi_1 > \varphi_2 > \varphi_3$. Then (72) possesses the special solution $\varphi(\xi) = \varphi_1 + (\varphi_2 - \varphi_1)sn^2(\frac{1}{2}\sqrt{\frac{2A}{3}(\varphi_1 - \varphi_3)}\xi, m)$ and sn stands for the Jacobi elliptic function with modulus $m = \sqrt{\frac{\varphi_1 - \varphi_2}{\varphi_1 - \varphi_3}}$ [17].

The cubic equation $Q_3(\varphi) = -A_1\varphi^3 + B_1\varphi^2 + C_1\varphi + D_1 = 0$, $A_1 > 0$ possesses three simple real roots if and only if $Q_3^{'}(\varphi) = 0$ has two simple real roots $z_1 < z_2$ such that $Q_3(z_1) < 0$, $Q_3(z_2) > 0$. In our case $(P_3(\varphi))$, the coefficients of Q_3 are $-\frac{2A}{3} < 0$, B, 0, C and the corresponding roots of $Q_3^{'}$ are $z_1 = 0$, $z_2 = \frac{B}{A}$. So $C < 0$ and $\frac{B^3}{A^2} > -3C$ guarantee the existence of three simple real roots of $P_3(\varphi) = 0$. Certainly, $\varphi_1\varphi_2\varphi_3 = \frac{3C}{2A} < 0$.

Assume that $n = 1$ in (56), i.e., the biquadratic polynomial

$$P_4(\varphi) = -\frac{A}{2}\varphi^4 + B\varphi^2 + C, A > 0. \tag{74}$$

According to [17], the equation possesses the solution

$$\varphi(\xi) = \varphi_1 - \frac{\varphi_1 - \varphi_4}{1 + \frac{\varphi_3 - \varphi_4}{\varphi_1 - \varphi_3}} sn^2(\tau, m),$$

where $\tau = \frac{1}{2}\sqrt{(\varphi_1 - \varphi_3)(\varphi_2 - \varphi_4)}(\xi - \xi_0)$, $m^2 = \frac{(\varphi_1 - \varphi_2)(\varphi_3 - \varphi_4)}{(\varphi_1 - \varphi_3)(\varphi_2 - \varphi_4)}$ and the simple real roots of $P_4 = 0$ are $\varphi_1 > \varphi_2 > \varphi_3 > \varphi_4$, $\varphi_4 = -\varphi_1$, $\varphi_3 = -\varphi_2$, the modulus $0 < m^2 < 1$, $\varphi(0) = \varphi_4$, $\varphi_4 \le \varphi(\xi) < \varphi_3$.

As it concerns (73), there is a table for the solutions of (73) expressed by Jacobi elliptic functions for special values of the coefficients A, B.

Example. Consider (73) with the following coefficients depending on the parameter $0 < m < 1$: $A = -2m^2$, $B = -(m^2 + 1)$, $C = 1$. Then it possesses the elliptic function solutions $\varphi = sn(\xi, m)$, $cd(\xi, m) = \frac{cn(\xi,m)}{dn(\xi,m)}$.

Formulas (61) and (70) can be found directly, using the fact that $\frac{d}{d\varphi}Arcsech\varphi^n = \mp\frac{n}{\varphi\sqrt{1-\varphi^{2n}}}$, $0 < \varphi < 1$. Similarly, $\frac{d}{d\varphi}Arccosech\varphi^n = \frac{-n}{\varphi\sqrt{1+\varphi^{2n}}}$. So $\xi - \xi_0 = G(\varphi) - G(\varphi_0) = \mp\frac{1}{n}(Arcsech\varphi^n - Arcsech\varphi_0^n) \Rightarrow sech(\mp n(\xi - \xi_0) + Arcsech\varphi_0^n) = \varphi^n$, $|cosech(-n(\xi - \xi_0) + Arccosech\varphi_0^n)| = \varphi^n$. The second solution blows up for $\bar{\xi} = \xi_0 + \frac{1}{n}Arccosech\varphi_0^n$ and $\varphi(\pm\infty) = 0$.

6. Appendix on the Solvability of the Cauchy Problem for the Schrödinger Operator (3), $u|_{t=0} = u_0(x)$ in the Space $W \setminus 0$, $\lambda < 0$

For the sake of completeness, we shall prove unicity of (3) under the additional condition $u \in C(\mathbf{R}_t; L^2(\mathbf{R}^n_x))$. Assume that u_1, u_2 are two solutions of (3) having the same initial data and denote $w = u_2 - u_1$. Then

$$i\partial_t w + \frac{1}{2}\Delta_x w = \lambda(u_2 ln|u_2|^2 - u_1 ln|u_1|^2) \tag{75}$$

and we multiply both sides of (75) by $\bar{w}$ and integrate in $\mathbf{R}^n_x$. Thus,

$$i(\partial_t w, w)_{L^2(\mathbf{R}^n)} + (\Delta_x w, w)_{L^2(\mathbf{R}^n)} = \lambda(u_2 ln|u_2|^2 - u_1 ln|u_1|^2, w)_{L^2(\mathbf{R}^n)}. \tag{76}$$

Taking the imaginary part of (76), we obtain that

$$\frac{1}{2}\frac{d}{dt}||w(t)||^2_{L^2(\mathbf{R}^n)} = \lambda \int_{\mathbf{R}^n} Im[(u_2 ln|u_2|^2 - u_1 ln|u_1|^2)\bar{w}]dx = \lambda \int_{\mathbf{R^n}} Idx. \tag{77}$$

In fact, $Imi(\partial_t w, w) = Re(\partial_t w, w) = \frac{1}{2}\frac{d}{dt}||w||^2_{L_2}$. Having in mind that $Imz = \frac{z-\bar{z}}{2i}$ and $|z_1| \geq |z_2| > 0$ implies that $0 \leq ln|z_1| - ln|z_2| = \int_{|z_2|}^{|z_1|} \frac{dp}{p} \leq \frac{1}{|z_2|}(|z_1| - |z_2|) \leq \frac{|z_1 - z_2|}{|z_2|}$ we conclude that for $u_1, u_2 \in \mathbf{C}^1 \setminus 0$:

$$I = 2Im[(u_2 ln|u_2| - u_1 ln|u_1|)(\bar{u}_2 - \bar{u}_1)] = -2Im(u_2\bar{u}_1 ln|u_2| - u_1\bar{u}_2 ln|u_1|) =$$

$$i[(u_2\bar{u}_1 - u_1\bar{u}_2)ln|u_2| + (u_1\bar{u}_2 - u_2\bar{u}_1)ln|u_1|] = i(u_2\bar{u}_1 - u_1\bar{u}_2)(ln|u_2| - ln|u_1|) =$$

$$i(u_2(\bar{u}_1 - \bar{u}_2) + \bar{u}_2(u_2 - u_1))(ln|u_2| - ln|u_1|).$$

Therefore,

$$|I| \leq 2|u_2||u_1 - u_2||ln|u_2| - ln|u_1||. \tag{78}$$

Without loss of generality, we suppose that $|u_1| \geq |u_2| > 0$, i.e., $|I| \leq |u_1 - u_2|^2 \Rightarrow$

$$\frac{d}{dt}||w(t)||^2_{L^2} \leq 4|\lambda|||w(t)||^2_{L_2}, w(0) = 0, w \in C(\mathbf{R}; L^2(\mathbf{R}^n)).$$

The continuous function $0 \leq y(t) = ||w(t)||^2_{L^2}$, $y(0) = 0$ satisfies the inequality

$$y'(t) \leq Cy(t) \Rightarrow y(t) \equiv 0$$

(Gronwall Lemma)

Unicity is verified.

We know that $xlnx$ is not Lipschitz near 0. On the other hand, if $|w_1| > 0$, $|w_2| > 0$, then

$$|w_1 log|w_1| - w_2 log|w_2|| \leq |w_1 - w_2|(1 + max(|log|w_1|, |log|w_2|). \tag{79}$$

7. Discussion and Open Problems

In optics, higher-order PDE of Schrödinger type could appear, namely

$$iu_t + P_m(D_x)u = A|u|^2 R_n(D_x)u + B(|u|^2 u)_x, \tag{80}$$

P_m, R_n being polynomials of ξ, $1 \leq n < m - 2$.

A simple example is the Focas–Lenells equation (FLE)

$$iu_t + iau_{xxx} + bu_{xxxx} + |u|^2(cu + idu_x) = i[\alpha u_x + \lambda(|u|^2 u)_x + \mu(|u_x|^2)u], \tag{81}$$

where $a, b, c, d, \alpha, \lambda, \mu$ are real constants. We are looking for a solution of (80) having the form

$$u(x,t) = \Phi(x - vt)e^{i\psi(x,t)}, \xi = x - vt, \tag{82}$$

Φ-real, $\psi = -kx + \omega t$, all the coefficients being real-valued.

Taking (82) into (81) and splitting the real and imaginary parts, we obtain for $\Phi(\xi)$ a fourth-order ODE and third-order ODE with real coefficients. We take the coefficients of the third-order ODE to be zero and conclude that

$$\Phi^{iv} + C_3\Phi'' + C_4\Phi = C_5\Phi^3. \tag{83}$$

In general, $C_3C_4C_5 \neq 0$ and these constants are real.

One can easily guess the validity of the following two propositions.

Proposition 1. *The soliton-type function*

$$\Phi = \frac{A}{chB\xi}, \tag{84}$$

$AB \neq 1$, A, B-real, satisfies

$$\Phi'' + C_1\Phi = C_2\Phi^3, C_1 < 0, C_2 < 0 \tag{85}$$

if and only if $B = \pm\sqrt{-C_1}$, $A = \pm\sqrt{\frac{2C_1}{C_2}}$.

(84) does not verify (83).

Proposition 2. *Equation (83) has a soliton-type solution of the form*

$$\Phi = L + \frac{A}{ch^2 B\xi}, AB \neq 0 \tag{86}$$

in the following two cases:

(1) $L = 0$, $C_3 < 0$, $C_5 > 0$, $C_4 = \frac{4}{25}C_3^2$; $A = -\varepsilon_2\frac{C_3}{20}\sqrt{\frac{120}{C_5}}$, $B = \varepsilon_1\sqrt{\frac{-C_3}{20}}$, $\varepsilon_1^2 = \varepsilon_2^2 = 1$.

(2) $L = \varepsilon_3\sqrt{\frac{C_4}{C_5}}$, $C_4 > 0$, $C_5 > 0$, $\varepsilon_3^2 = 1$; $A = \varepsilon_2 B^2\sqrt{\frac{120}{C_5}}$, $\varepsilon_2^2 = 1$, $B = \varepsilon_4\sqrt{\frac{-(\varepsilon_2\varepsilon_3\sqrt{30C_4}+C_3)}{20}}$, $\varepsilon_2\varepsilon_3\sqrt{30C_4} + C_3 < 0$, *where* $C_3 = \frac{\sqrt{30C_4}}{12}(-7\varepsilon_2\varepsilon_3 - \sqrt{65}) < 0$, $\varepsilon_4^2 = 1$.

Equation (86) does not satisfy (85).

In a similar way, one can find a blowing up solution $\Phi = L + \frac{A}{sh^2 B\xi}$ of (83). We can reduce the order of (83) by two units, obtaining a non-autonomous second-order ODE for $p(\Phi) = |\Phi'_\xi|^{3/2}$.

The open problem is to find soliton (blowing up)-type solutions of (80). One can try to construct soliton solutions by using the method of a priori estimates.

8. Conclusions

In this paper, we construct exact solutions of several model examples of semilinear PDE arising in mathematical physics. At first we study the parabolic PDE with nonlinearity $\lambda u ln u^2$ and initial data $b_0 e^{-a_0 x^2}$, $a_0 < 0$. The corresponding Cauchy problem possesses a unique solution for $0 \leq t \leq T$ in the class of exponentially increasing functions u, $|u| \leq C(T)e^{ax^2}$, $a > 0$, $0 \leq t \leq T$. The solutions are either globally defined in t or the phase of $u = b(t)e^{-a(t)x^2}$ blows up for some finite $\bar{t}$. Much more complicated is the case of the Schrödinger equation with nonlinearity $\lambda u ln|u|^2$, $u \neq 0$, $\lambda = const \neq 0$. Here $u = b(t)e^{-a(t)\frac{x^2}{2}}$, $b(0) \neq 0$, $a(0) = \alpha_0 + i\beta_0$, $\alpha_0 < 0$, $\beta_0 \neq 0$. If $\lambda > 0$, there exists smooth periodic $a(t)$ such that $Rea(t) < 0$. It is interesting to note that $|b| > 0$ is periodic but we find here a necessary and sufficient condition for periodicity of $b(t)$ showing that $b(t)$ is very rarely periodic.

For the first time, we investigate the case $\lambda < 0$ and prove that the smooth curve $t \to a(t) \in \mathbf{C}^1$ is bounded and located in the half plane $Rea < 0$ being tangential to the imaginary axes at the origin, $a(\pm\infty) = 0$, $Ima > 0$ for $t \gg 1$, $Ima < 0$ for $t \ll -1$, while $b(t) : t \to \mathbf{C}^1$ describes a focus in $\mathbf{C}^1$ with center 0.

For the higher-order Schrödinger equations with power and power-logarithmic nonlinearities, we are looking for solutions of the type $u = \Phi(\xi)e^{i\psi(t,x)}$, Φ-real, $\xi = x - Vt$, $V \in \mathbf{R}^1$, ψ linear real-valued in x, t. We split the corresponding real and imaginary parts and reduce the solvability of that semilinear PDE to the solvability of appropriate second-order autonomous ODE. By means of the method of first integral (used in mechanical processes of one degree of freedom), we obtain directly the solutions of type solitons, kinks, blowing up, etc. We propose here solutions of the type *sech*, *cosech*, *sec*, *cosec*, Jacobi elliptic functions that are found directly. In the higher-order case, things are complicated. If the appropriate autonomous semilinear ODEs are of even order by using the method of first integral and the change $p(\Phi) = \Phi'_{\xi}$, we can reduce the order by (at least) two units, obtaining, for example, in the fourth-dimensional case, a second-order Emden–Fawler ODE.

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