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Existence of Kink and Antikink Wave Solutions of Singularly Perturbed Modified Gardner Equation

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Abstract: In this paper, the singularly perturbed modified Gardner equation is considered. Firstly, for the unperturbed equation, under certain parameter conditions, we obtain the exact expressions of kink wave solution and antikink wave solution by using the bifurcation method of dynamical systems. Then, the persistence of the kink and antikink wave solutions of the perturbed modified Gardner equation is studied by exploiting the geometric singular perturbation theory and the Melnikov function method. When the perturbation parameter is sufficiently small, we obtain the sufficient conditions to guarantee the existence of kink and antikink wave solutions.

Keywords: modified Gardner equation; kink and antikink waves; geometric singular perturbation theory; Melnikov function

MSC: 35Q35; 35L05; 74J30; 34D15

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1. Introduction

As we know, the investigation of nonlinear wave equations and their traveling wave solutions has attracted extensive attention in mathematical physics and the engineering field. A lot of effective methods have been developed to study the traveling wave solutions and their dynamical behaviors [1–3]. In 1968, the mathematician Gardner [4] derived the following Gardner equation:

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{1}$$

which can be used to describe the weakly nonlinear dispersive waves in situations where the higher-order nonlinearity effects play an important role. For a long time, the Gardner equation (1) has attracted much attention due to its significant nature in physical contexts. Various types of exact solutions of the Gardner equation (1) have been extensively studied [5–8]. For example, by applying the theory of dynamical systems and the bifurcation method, Chen and Liu [9] obtained the solitary wave solutions and kink wave solutions of (1). Recently, increasingly more interest has been paid to the traveling waves of singularly perturbed mathematical physics models [10–15]. For instance, Tang et al. [16] studied the Gardner equation with Kuramoto–Sivashinsky perturbation:

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0. \tag{2}$$

By using the geometric singular perturbation theory, the persistence of the solitary wave solution for Equation (2) is investigated. Wen [17] further showed that kink waves and antikink waves of Equation (2) persist. Zhang and Xia [18] studied the persistence of kink and antikink wave solutions for the perturbed double sine-Gordon equation.

Recently, Olivier et al. [19] first applied reductive perturbation analysis to a fluid model and derived the generalized KdV equation with supersolitons:

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u^3 u_x + \frac{1}{2} u_{xxx} = 0, \tag{3}$$

which is a higher-order form of the Gardner equation (1) named the modified Gardner equation. Tamang et al. [20] further derived the more general modified Gardner equation:

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u^3 u_x + \gamma u_{xxx} = 0. \tag{4}$$

By studying this equation, it becomes possible to unveil the nonlinear wave behavior and interactions in plasma. Jhangeer et al. [21] considered Equation (4) by using the Lie group analysis, power series technique, and bifurcation theory. All practicable types of phase portraits with regard to the parameters were plotted, and the traveling wave structures were studied.

To our knowledge, the modified Gardner equation (4) under perturbation has yet to be considered in the literature. Inspired by the above literature, we study the following modified Gardner equation with Kuramoto–Sivashinsky perturbation:

$$u_t + \alpha u u_x + \beta u^2 u_x + \delta u^3 u_x + \gamma u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{5}$$

where $\alpha, \beta, \gamma, \delta$ are parameters, and $\varepsilon > 0$ is a perturbation parameter. u_{xx} and u_{xxxx} represent the backward diffusion and dissipation terms, respectively.

Firstly, consider $\varepsilon = 0$: by using the dynamical system theory and bifurcation method, when the parameters of the modified Gardner equation (4) satisfy certain conditions, we provide several exact expressions of kink and antikink solutions of Equation (4). Note that when $\delta = 0$, Equation (4) becomes Equation (1). Compared with the methods and results in [9], the process of obtaining the exact solutions of Equation (4) with $\delta > 0$ is more complicated because the degree of the Hamiltonian function is five, and the system (8) admits a higher-order singular point.

The format of this article is as follows. In Section 2, by using the dynamical system theory and bifurcation method, when the parameters of the modified Gardner equation (4) satisfy certain conditions, we provide several exact expressions of kink and antikink solutions of Equation (4). In Section 3, when the perturbation parameter $\varepsilon > 0$ is sufficiently small, by exploiting the geometric singular perturbation theory and the Melnikov function method, the sufficient conditions are obtained to guarantee the existence of kink and antikink wave solutions of Equation (5). In Section 4, we perform numerical simulations to verify the theoretical results. Finally, the main conclusions of this paper are given.

2. Exact Solutions of the Unperturbed Modified Gardner Equation

In this section, we consider the unperturbed modified Gardner equation (4). Suppose the traveling wave solution of Equation (4) is $u(x,t) = \varphi(\xi)$, where $\xi = x - ct$, $c > 0$ is the wave speed. Then we obtain the following ordinary differential equation:

$$-c\varphi' + \alpha\varphi\varphi' + \beta\varphi^2\varphi' + \delta\varphi^3\varphi' + \gamma\varphi''' = 0. \tag{6}$$

The above equation can be integrated once to yield

$$-c\varphi + \frac{\alpha}{2}\varphi^2 + \frac{\beta}{3}\varphi^3 + \frac{\delta}{4}\varphi^4 + \gamma\varphi'' = 0, \tag{7}$$

where the integral constant is set to be zero.

Letting $y = \varphi'$, we obtain a planar system as follows:

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{1}{\gamma}\left(c\varphi - \frac{\alpha}{2}\varphi^2 - \frac{\beta}{3}\varphi^3 - \frac{\delta}{4}\varphi^4\right), \end{cases} \tag{8}$$

which is a Hamiltonian system with Hamiltonian function:

$$H(\varphi, y) = y^2 - \frac{1}{\gamma}\left(c\varphi^2 - \frac{\alpha}{3}\varphi^3 - \frac{\beta}{6}\varphi^4 - \frac{\delta}{10}\varphi^5\right). \tag{9}$$

Now, for the case of $\gamma > 0, \delta > 0$, we consider the phase portraits of system (8) (the other cases can be considered similarly). To state conveniently, assume that

$$\varphi_1 = \frac{2}{5}\sqrt[3]{\frac{10c}{\delta}}, \varphi_2 = \sqrt[3]{\frac{10c}{\delta}}.$$

When $\alpha = 9\sqrt[3]{\frac{\delta c^2}{10}}, \beta = -\frac{9}{5}\sqrt[3]{10\delta^2 c}$, it is easy to see that system (8) has three singular points $(0,0)$, $(\varphi_1, 0)$ and $(\varphi_2, 0)$. By calculating the characteristic values of the linearized system of system (8), we know that $(0,0)$ is a saddle point, $(\varphi_1, 0)$ is a center, and $(\varphi_2, 0)$ is a higher-order singular point.

According to the theory of dynamical systems, we easily know that there are two heteroclinic orbits L_1 and L_2 connecting the two singular points $(0,0)$ and $(\varphi_2, 0)$ of system (8), and we obtain the phase portrait of system (8) as shown in Figure 1 . We have the following result:

Theorem 1. *When $\alpha = 9\sqrt[3]{\frac{\delta c^2}{10}}, \beta = -\frac{9}{5}\sqrt[3]{10\delta^2 c}, \gamma > 0, \delta > 0$, Equation (4) has the kink wave solution $u_1(x,t)$, antikink wave solution $u_2(x,t)$, and blow-up wave solution $u_3(x,t)$, whose expressions are as follows:*

$$\frac{2\sqrt{\varphi_2}}{\sqrt{\varphi_2 - u_1(x,t)}} - \ln\frac{\sqrt{\varphi_2} + \sqrt{\varphi_2 - u_1(x,t)}}{\sqrt{\varphi_2} - \sqrt{\varphi_2 - u_1(x,t)}} = \sqrt{\frac{c}{\gamma}}(x - ct) + 2\sqrt{2} - 2\ln(1 + \sqrt{2}), \tag{10}$$

$$\frac{2\sqrt{\varphi_2}}{\sqrt{\varphi_2 - u_2(x,t)}} - \ln\frac{\sqrt{\varphi_2} + \sqrt{\varphi_2 - u_2(x,t)}}{\sqrt{\varphi_2} - \sqrt{\varphi_2 - u_2(x,t)}} = -\sqrt{\frac{c}{\gamma}}(x - ct) + 2\sqrt{2} - 2\ln(1 + \sqrt{2}), \tag{11}$$

$$\frac{2\sqrt{\varphi_2}}{\sqrt{\varphi_2 - u_3(x,t)}} - \ln\frac{\sqrt{\varphi_2} + \sqrt{\varphi_2 - u_3(x,t)}}{\sqrt{\varphi_2 - u_3(x,t)} - \sqrt{\varphi_2}} = -\sqrt{\frac{c}{\gamma}}|x - ct|. \tag{12}$$

Proof. From Figure 1, we can see that there are two heteroclinic orbits L_1, L_2 and one open orbit L_3 which have expressions as follows:

$$L_1, L_2 : y^2 = \frac{\delta}{10\gamma}\varphi^2(\varphi_2 - \varphi)^3, \quad 0 < \varphi < \varphi_2, \tag{13}$$

$$L_3 : y^2 = \frac{\delta}{10\gamma}\varphi^2(\varphi_2 - \varphi)^3, \quad -\infty < \varphi < 0. \tag{14}$$

Substituting (13)–(14) into $(d\varphi/d\xi) = y$ and integrating it along L_1, L_2, L_3, respectively, we have

$$\int_{\frac{\varphi_2}{2}}^{\varphi} \frac{ds}{s(\varphi_2 - s)\sqrt{\varphi_2 - s}} = \sqrt{\frac{\delta}{10\gamma}}\xi, \tag{15}$$

$$\int_{\frac{\varphi_2}{2}}^{\varphi} \frac{ds}{s(\varphi_2 - s)\sqrt{\varphi_2 - s}} = -\sqrt{\frac{\delta}{10\gamma}}\xi, \tag{16}$$

$$\int_{-\infty}^{\varphi} \frac{ds}{s(\varphi_2 - s)\sqrt{\varphi_2 - s}} = -\sqrt{\frac{\delta}{10\gamma}}|\xi|. \tag{17}$$

Solving (15)–(17) for φ and denoting them by $\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)$, respectively, we obtain $u_i(x,t) = \varphi_i(x - ct) = \varphi_i(\xi)(i = 1,2,3)$ as (10)–(12). $\square$

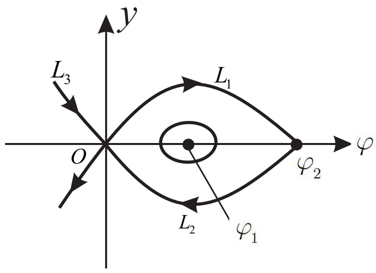


Figure 1. The phase portrait of system (8) when $\alpha = 9\sqrt[3]{\frac{\delta c^2}{10}}, \beta = -\frac{9}{5}\sqrt[3]{10\delta^2 c}, \gamma > 0, \delta > 0$.

3. Persistence of Kink and Antikink Wave Solutions

In this section, we consider the perturbed Equation (5). Substituting the traveling wave transformation $u(x,t) = \varphi(\xi), \xi = x - ct$ into Equation (5), we obtain

$$-c\varphi' + \alpha\varphi\varphi' + \beta\varphi^2\varphi' + \delta\varphi^3\varphi' + \gamma\varphi''' + \varepsilon(\varphi'' + \varphi'''') = 0, \tag{18}$$

which can be integrated once to obtain

$$-c\varphi + \frac{\alpha}{2}\varphi^2 + \frac{\beta}{3}\varphi^3 + \frac{\delta}{4}\varphi^4 + \gamma\varphi'' + \varepsilon(\varphi' + \varphi''') = 0, \tag{19}$$

where the integral constant is set to be zero. We can convert the above equation into the following system:

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = z, \\ \varepsilon\frac{dz}{d\xi} = c\varphi - \frac{\alpha}{2}\varphi^2 - \frac{\beta}{3}\varphi^3 - \frac{\delta}{4}\varphi^4 - \gamma z - \varepsilon y, \end{cases} \tag{20}$$

which is called the slow system. Therefore, studying the perturbed Equation (5) is equivalent to studying system (20). Formally, we can expect that there exists a two-dimensional invariant manifold near the surface

$$C_0 = \left\{ (\varphi, y, z) \in \mathbb{R}^3 : z = \frac{1}{\gamma}\left(c\varphi - \frac{\alpha}{2}\varphi^2 - \frac{\beta}{3}\varphi^3 - \frac{\delta}{4}\varphi^4 \right) \right\} \tag{21}$$

when $\varepsilon > 0$ is sufficiently small. Note that when $\alpha = 9\sqrt[3]{\frac{\delta c^2}{10}}, \beta = -\frac{9}{5}\sqrt[3]{10\delta^2 c}, \gamma > 0, \delta > 0$, system (20) has three equilibrium points $(0,0,0)$, $(\varphi_1, 0, 0)$, and $(\varphi_2, 0, 0)$, which are independent of $\varepsilon > 0$.

By making the transformation $\xi = \varepsilon\tau$, we obtain the following system:

$$\begin{cases} \frac{d\varphi}{d\tau} = \varepsilon y, \\ \frac{dy}{d\tau} = \varepsilon z, \\ \frac{dz}{d\tau} = c\varphi - \frac{\alpha}{2}\varphi^2 - \frac{\beta}{3}\varphi^3 - \frac{\delta}{4}\varphi^4 - \gamma z - \varepsilon y, \end{cases} \tag{22}$$

which is called the fast system, and it is equivalent to the slow system (20) for $\varepsilon > 0$. In order to obtain a two-dimensional invariant manifold of (20) for a sufficiently small

$\varepsilon > 0$, it suffices to verify the normal hyperbolicity of C_0. The linearized matrix of the fast system (22) restricted on C_0 is

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c - \alpha\varphi - \beta\varphi^2 - \delta\varphi^3 & 0 & -\gamma \end{bmatrix},$$

which has three eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -\gamma$, and, therefore, C_0 is normally hyperbolic. According to the geometric singular perturbation theory of Fenichel [22], there exists a two-dimensional invariant manifold C_ε of system (20) with $\varepsilon > 0$ sufficiently small, which can be written in the following form:

$$C_\varepsilon = \left\{ (\varphi, y, z) \in \mathbb{R}^3 : z = \frac{1}{\gamma}\left(c\varphi - \frac{\alpha}{2}\varphi^2 - \frac{\beta}{3}\varphi^3 - \frac{\delta}{4}\varphi^4 \right) + \zeta(\varphi, y, \varepsilon) \right\}, \tag{23}$$

where $\zeta(\varphi, y, \varepsilon)$ depends smoothly on φ, y, ε and satisfies $\zeta(\varphi, y, 0) = 0$. Therefore, the function $\zeta(\varphi, y, \varepsilon)$ can be expanded in ε as follows:

$$\zeta(\varphi, y, \varepsilon) = \varepsilon\zeta_1(\varphi, y) + O(\varepsilon^2). \tag{24}$$

Substituting it into the slow system (20) and comparing the coefficients of ε, we can obtain

$$\zeta_1(\varphi, y) = -\frac{1}{\gamma^2}\left(c - \alpha\varphi - \beta\varphi^2 - \delta\varphi^3 \right) y - \frac{1}{\gamma} y. \tag{25}$$

Thus, the dynamics of the slow manifold C_ε for system (20) are given as

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{1}{\gamma}\left(c\varphi - \frac{\alpha}{2}\varphi^2 - \frac{\beta}{3}\varphi^3 - \frac{\delta}{4}\varphi^4 \right) + \varepsilon\left(-\frac{1}{\gamma^2}(c - \alpha\varphi - \beta\varphi^2 - \delta\varphi^3) y - \frac{1}{\gamma} y \right) \\ \quad +O(\varepsilon^2). \end{cases} \tag{26}$$

Then, we are in a position to state our main theorem on the persistence of kink and antikink wave solutions for the perturbed modified Gardner Equation (5).

Theorem 2. *For Equation (5) with* $\alpha = 9\sqrt[3]{\frac{\delta c^2}{10}}, \beta = -\frac{9}{5}\sqrt[3]{10\delta^2 c}, \gamma > 0, \delta > 0$, *there exists* $c = \frac{39}{5}\gamma + O(\varepsilon)$ *such that Equation (5) has a kink wave solution and an antikink wave solution with wave speed c for* $0 < \varepsilon \ll 1$.

Proof. From Han [23] and Perko [24], the associated Melnikov function for system (26) is

$$M(c) = \int_{L_1} Q(\varphi, y(\varphi)) d\varphi, \tag{27}$$

where $Q(\varphi, y) = -\frac{1}{\gamma^2}(c - \alpha\varphi - \beta\varphi^2 - \delta\varphi^3) y - \frac{1}{\gamma} y$, and $y(\varphi)$ is the expression of the orbit L_1. Note that L_1 has the expression

$$y(\varphi) = \sqrt{\frac{\delta}{10\gamma}}\varphi(\varphi_2 - \varphi)\sqrt{\varphi_2 - \varphi}, \quad 0 < \varphi < \varphi_2. \tag{28}$$

Therefore, we have

$$
\begin{aligned}
&M(c)\\
&= \int_0^{\sqrt[3]{\frac{10c}{\delta}}} Q(\varphi, y(\varphi))d\varphi\\
&= -\frac{1}{\gamma}\sqrt{\frac{\delta}{10\gamma}}\int_0^{\sqrt[3]{\frac{10c}{\delta}}}\left(\varphi + \frac{c}{\gamma}\varphi - \frac{\alpha}{\gamma}\varphi^2 - \frac{\beta}{\gamma}\varphi^3 - \frac{\delta}{\gamma}\varphi^4\right)\left(\sqrt[3]{\frac{10c}{\delta}} - \varphi\right)^{\frac{3}{2}} d\varphi \qquad (29)\\
&= \frac{40}{273\sqrt[3]{10}}\frac{c^{\frac{7}{6}}}{\gamma^{\frac{5}{2}}\delta^{\frac{2}{3}}}\left(c - \frac{39}{5}\gamma\right).
\end{aligned}
$$

Thus, there exists a unique

$$\hat{c} = \frac{39}{5}\gamma$$

such that $M(\hat{c}) = 0$. Furthermore, by a simple calculation, we can obtain

$$\left.\frac{dM(c)}{dc}\right|_{c=\hat{c}} = \frac{4\sqrt[3]{100}}{35}\sqrt[6]{\frac{39}{5}}\frac{1}{\gamma^{\frac{4}{3}}\delta^{\frac{2}{3}}} \neq 0. \qquad (30)$$

Therefore, by the implicit function theorem, there exists $c = \frac{39}{5}\gamma + O(\varepsilon)$ such that system (26) has a pair of φ-axis symmetric heteroclinic orbits for $0 < \varepsilon \ll 1$. In other words, Equation (5) has a kink wave solution and an antikink wave solution with a wave speed c. □

4. Numerical Simulations

In this section, numerical simulations are performed to verify the theoretical results of the previous sections. Firstly, taking $\alpha = 90, \beta = -18, \gamma = 1, \delta = 1, c = 100$, we illustrate the profiles of the kink wave solution $u_1(x,t)$ and antikink wave solution $u_2(x,t)$ in Figure 2, which can verify the correctness of Theorem 1.

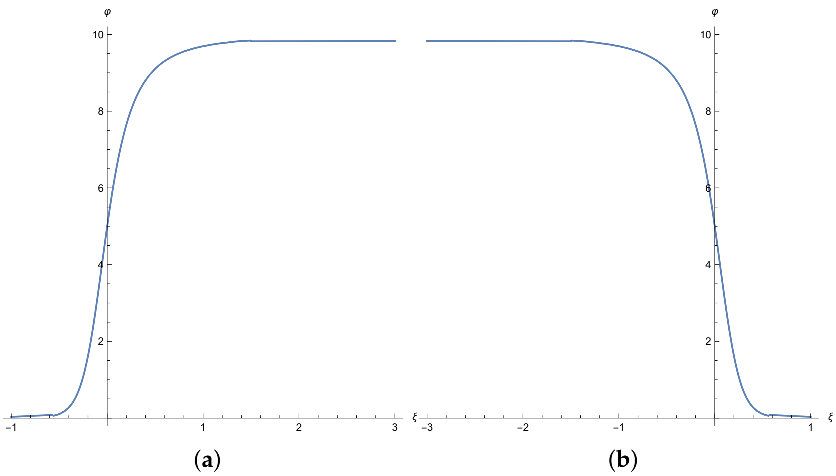
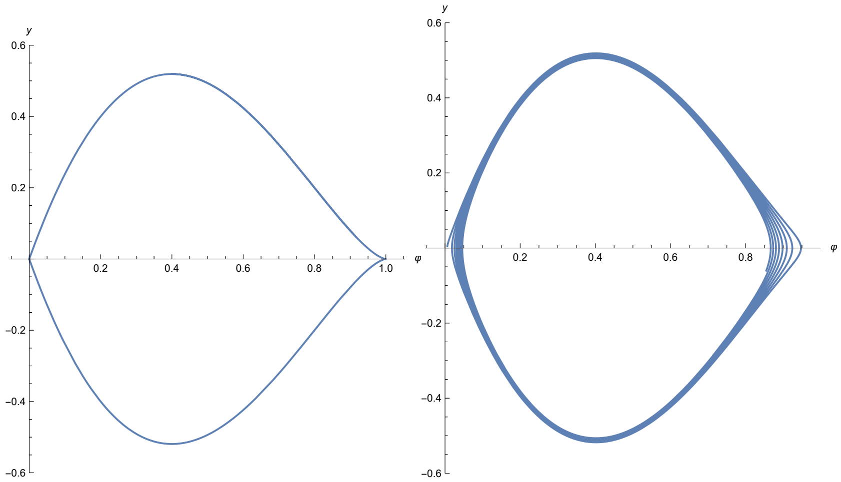


Figure 2. (**a**). The profile of kink wave solution (10). (**b**). The profile of antikink wave solution (11).

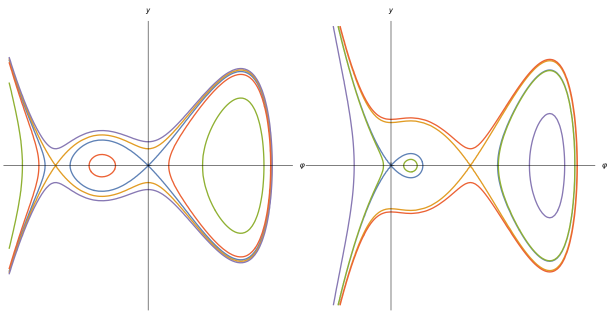
Now, we verify the persistence of the heteroclinic orbits of Equation (5) through system (20). Taking the parameters $\alpha = \frac{9}{2}$, $\beta = -9$, $\gamma = \frac{5}{78}$, $\delta = 5$, $c = \frac{1}{2}$, and the initial conditions $\varphi_0 = \frac{2}{5}, y_0 = \frac{18\sqrt{13}}{125}, z_0 = 0.001$. The persistence of the heteroclinic orbits for $\varepsilon = 0.001$ is illustrated in Figure 3a, and the break of the heteroclinic orbits for $\varepsilon = 0.1$ is illustrated in Figure 3b.



(**a**) (**b**)

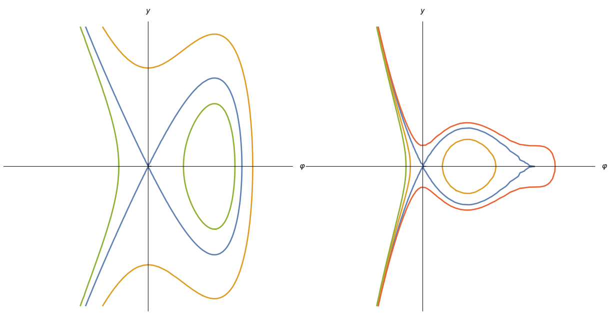
Figure 3. (**a**). The heteroclinic orbits persist for $\varepsilon = 0.001$. (**b**). The heteroclinic orbits break for $\varepsilon = 0.1$.

For $\varepsilon = 0$, taking four different classes of parameter values, we can give the phase portraits of system (8) in Figures 4 and 5. From this, we can see, under certain parameter conditions, that Equation (4) yields solitary wave solutions and periodic wave solutions. Thus, this study provides evidence for the existence of solitary waves, periodic waves, and kink waves in quantum electron–positron–ion magneto plasmas. These findings contribute to a better understanding of the nonlinear dynamics of ion-acoustic waves in plasmas and offer insights and guidance for both theoretical and practical applications in related fields.



(**a**) (**b**)

Figure 4. The phase portrait of system (8). (**a**). $\alpha = -2$, $\beta = \frac{3}{4}$, $\gamma = 1$, $\delta = 1, c = 1$. (**b**). $\alpha = \frac{11}{2}$, $\beta = -\frac{39}{8}$, $\gamma = 1$, $\delta = 1$, $c = 1$.



(**a**) (**b**)

Figure 5. The phase portrait of system (8). (**a**). $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, $c = 1$. (**b**). $\alpha = 45$, $\beta = -90$, $\gamma = 1$, $\delta = 50$, $c = 5$.

5. Conclusions

In [21], all practicable types of phase portraits of Equation (8) with regard to the parameters are plotted, but the study does not find the exact solutions of Equation (4) without solutions in power series form. In this paper, we provide the exact expressions of a pair of kink and antikink wave solutions of Equation (4) by applying the theory of dynamic systems, which are not in power series form. Furthermore, when $\varepsilon > 0$ is sufficiently small, we obtain the sufficient conditions that assure the persistence of kink and antikink wave solutions.

Furthermore, we intend to provide the exact expressions of solitary waves and periodic waves of Equation (4) by applying the theory of dynamic systems and reveal their relations. Note that if the integral constant is not set to be zero in Equation (7), then $(0,0)$ is not a singular point. Furthermore, the phase portraits of system (8) will be more complicated. Depending on the range of the integral constant, more expressions of exact solutions of Equation (4) can be found. However, we do not think these contents are consistent with the title of this article. So, we will study this topic in the future. Moreover, when $\varepsilon > 0$, we intend to investigate the persistence of these solitary waves and periodic waves by applying the geometric singular perturbation theory and Abelian integrals.

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