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Dirac Geometric Approach for the Unimodular Holst Action

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Abstract: We perform a Hamiltonian analysis of unimodular gravity in its first-order formulation, specifically a modification of the Holst action. In order to simplify the analysis, prior studies on this theory have introduced (for several reasons) additional elements, such as parametrization, complex fields, or considering the Barbero–Immirzi parameter as imaginary. We show that, by using a geometric implementation of the Dirac algorithm, a comprehensive analysis of the theory can be conducted without relying on these additional ingredients. The resulting theory reproduces the behavior of metric unimodular gravity.

Keywords: unimodular gravity; Holst action; geometric Dirac algorithm; Hamiltonian formulation

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1. Introduction

The Hamiltonian formulation of the Lagrangian theory is a valuable description in various scenarios, particularly at different stages of the quantization process. However, when dealing with singular systems—those for which the fiber derivative of the Lagrangian is not a local diffeomorphism—a more intricate analysis becomes necessary, warranting a cautious approach. Several methods exist for obtaining the Hamiltonian description of a system, the most well-known among them being the Dirac program [1,2], where, after finding the primary constraints, one checks their consistency via Poisson brackets, obtaining secondary constraints, classifying them into first or second class, and acting accordingly. Among the many interesting features of this algorithm, one may emphasize the following: since it is performed entirely in an ambient symplectic manifold, one has a symplectic structure during the entire process, which enables the calculation of the Poisson brackets. These brackets can be used to draw an analogy between the classical theory and a hypothetical quantum theory, which is why this method has been preferred among physicists. However, this breaks down when there are second-class constraints, for then one is forced to introduce Dirac brackets, rendering this advantage null in that case. An equivalent approach of a geometric nature is provided by the Gotay–Nester–Hinds (GNH) algorithm [3–9]. The central idea is to search for a "stable submanifold" supporting a Hamiltonian vector field whose integral curves, suitably projected, provide the solutions to the equations of motion. This algorithm bypasses some of the problems found in the Dirac program, such as the difficulties that may arise when defining the Poisson brackets in certain field theories, especially those with boundaries. The price to pay is that the final manifold obtained by this procedure is not necessarily a symplectic space, in the sense that there may not be a well-defined symplectic form that one could relate to the commutators between quantum operators or to the volume form used in geometric quantization. As long as we are only interested in the classical theory, this is not necessarily a drawback, but it could become one when trying to construct a quantum theory. In addition to this, one also loses the "Dirac multipliers" found in the Dirac algorithm, which, as we will see, can provide non-trivial information on the system.

Luckily, it is not necessary to give up the advantages of either of these two methods; it is possible to export the geometric character of the GNH approach to the context of the Dirac method. Following [10] (see also [11,12] for applications), we will refer to this adaptation as the geometric Dirac algorithm. By employing this approach, there is no necessity to explicitly employ Poisson brackets. Moreover, when dealing with boundaries, this method helps circumvent issues related to the use of formal expressions involving functional derivatives for the symplectic form. There is a further advantage from a purely computational point of view: the recurring calculations via the Poisson brackets are replaced by the study of tangency requirements, which is faster and makes the user less prone to mistakes.

In this article, we will employ the geometric Dirac algorithm to study unimodular gravity (UG) in its first-order formulation. The reasons for this are manifold. To begin with, this theory has been widely discussed in the literature [13–21] and has been regaining traction recently [22–28], which allows us to compare the results obtained via the geometric Dirac algorithm with those obtained by other methods in a physically relevant system, as opposed to a simplified model. A more pressing issue is obtaining a full, consistent analysis of the constraints of unimodular gravity. Prior studies have introduced additional elements, such as the condition of the Barbero–Immirzi parameter being $\pm i$ [29], complex fields, parametrizations [30], and others, to simplify the process. Furthermore, it is very common to introduce a gauge fixing condition, such as the time gauge, midway through the calculation. Although the results obtained by doing this may be correct, there is something unsettling about this choice for two reasons: first, how can one, from a purely mathematical point of view, know which gauges can be fixed until the full study of the constraints has finished? Even if this leads to a correct result, it would not be something obtained by the Dirac algorithm. Secondly, even if the gauge-fixing is correct, introducing it midway through the analysis raises the question of which results are dependent on it and which are not.

The purpose of this work is to apply the geometric Dirac algorithm to unimodular gravity with a Holst term, demonstrating that the analysis can be performed without introducing the additional elements used in prior literature. Subsequently, we will compare the results with those obtained in different ways, including the cases where $\gamma = \pm i$, the parametrized version, gauge-fixings, and the GNH algorithm. The structure of this paper is as follows. After this introduction, in Section 2, we discuss the unimodular version of the Holst action, the field equations, and its Hamiltonian formulation using the geometric Dirac algorithm. Finally, we give our conclusions in Section 3.

2. Unimodular Holst Action

Let us consider an orientable four-dimensional manifold $\mathcal{M}$ diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ is a closed-orientable three-dimensional manifold. A modification of the Holst action that describes unimodular general relativity is

$$S(\boldsymbol{e}, \boldsymbol{\omega}, \Lambda) = \int_{\mathcal{M}} \left(P_{IJKL} \boldsymbol{e}^I \wedge \boldsymbol{e}^J \wedge \boldsymbol{F}^{KL} + \Lambda \left(\mathbf{vol} - \frac{1}{12} \epsilon_{IJKL} \boldsymbol{e}^I \wedge \boldsymbol{e}^J \wedge \boldsymbol{e}^K \wedge \boldsymbol{e}^L \right) \right), \qquad (1)$$

where the one-forms $\boldsymbol{e}^I$ (the cotetrads) are required to be non-degenerate, i.e.,

$$\epsilon_{IJKL} \boldsymbol{e}^I \wedge \boldsymbol{e}^J \wedge \boldsymbol{e}^K \wedge \boldsymbol{e}^L$$

is a volume form in $\mathcal{M}$, with the Levi–Civita symbol ϵ_{IJKL}, is totally antisymmetric and is chosen to satisfy $\epsilon_{0123} = +1$. The field $\boldsymbol{\omega}^I{}_J$ is a $\mathfrak{so}(1,3)$-valued connection one-form with curvature

$$\boldsymbol{F}^I{}_J := \mathrm{d}\boldsymbol{\omega}^I{}_J + \boldsymbol{\omega}^I{}_K \wedge \boldsymbol{\omega}^K{}_J.$$

The internal indices $I, J, \ldots$ take the values $0, 1, 2, 3$, and are raised (lowered) with the (internal) metric $\eta_{IJ} = \mathrm{diag}(\varepsilon, +1, +1, +1)$ with $\varepsilon = -1$ (we have included ε to keep track of the spacetime signature and facilitate the extension of our results to the Euclidean case). Finally, Λ is a scalar field, **vol** is a non-dynamical volume form on $\mathcal{M}$, and the invariant $SO(1,3)$ tensor is

$$P_{IJKL} := \frac{1}{2}\left(\epsilon_{IJKL} + \frac{\varepsilon}{\gamma}\eta_{IK}\eta_{JL} - \frac{\varepsilon}{\gamma}\eta_{JK}\eta_{IL}\right) = P_{KLIJ}\,. \tag{2}$$

This tensor has an inverse if $\gamma^2 \neq \varepsilon$, which is the case in which we are working. The real parameter γ is the called Barbero–Immirzi parameter.

The field equations derived from action (1) are obtained by varying with respect to the dynamical variables $\boldsymbol{\omega}^I{}_J$, $\boldsymbol{e}^I$, and Λ, respectively:

$$P_{IJKL}\boldsymbol{e}^I \wedge \boldsymbol{D}\boldsymbol{e}^J = 0\,, \tag{3a}$$

$$2P_{IJKL}\boldsymbol{e}^J \wedge \boldsymbol{F}^{KL} - \frac{\Lambda}{3}\epsilon_{IJKL}\boldsymbol{e}^J \wedge \boldsymbol{e}^K \wedge \boldsymbol{e}^L = 0\,, \tag{3b}$$

$$\mathbf{vol} - \frac{1}{12}\epsilon_{IJKL}\boldsymbol{e}^I \wedge \boldsymbol{e}^J \wedge \boldsymbol{e}^K \wedge \boldsymbol{e}^L = 0\,, \tag{3c}$$

where

$$\boldsymbol{D}\boldsymbol{e}^I := \mathrm{d}\boldsymbol{e}^I + \boldsymbol{\omega}^I{}_J \wedge \boldsymbol{e}^J.$$

Using the inverse of P_{IJKL} and the fact that the cotretrads are non-degenerated, Equation (3a) implies

$$\boldsymbol{D}\boldsymbol{e}^I = 0\,.$$

Computing the covariant differential of (3b) using the previous result ($\boldsymbol{D}\boldsymbol{e}^I = 0$), the Bianchi identity $\boldsymbol{D}\boldsymbol{F}_{IJ} = 0$, and the non-degeneracy of the cotetrads, we obtain

$$\mathrm{d}\Lambda = 0.$$

This last equation implies that Λ is constant. Furthermore, taking the covariant differential on both sides of the equation $\boldsymbol{D}\boldsymbol{e}^I = 0$, we obtain

$$\boldsymbol{e}_I \wedge \boldsymbol{F}^{IJ} = 0\,.$$

Plugging this into (3b) makes it straightforward to see that the γ-dependent terms disappear. Then, Equation (3b) becomes the usual Einstein–Palatini equation, and Λ is identified with the cosmological constant. Finally, Equation (3c) is the so-called unimodular condition, which forces the volume defined by the tetrads to coincide with the non-dynamic volume form **vol** that appears in the Lagrangian. A consequence of this is that the invariance under diffeomorphism is reduced to the **vol**-preserving diffeomorphisms.

The Hamiltonian analysis of a particular case of (1) was carried out in reference [29], where the authors chose $\gamma = i$ (the imaginary unit), simplifying the Hamiltonian constraint compared to the case with real values of γ. Additionally, they implemented the time gauge from the beginning, working with a self-dual connection (and thus, dealing with complex fields). However, as mentioned in the introduction, while this approach may yield correct results, there is no theoretical justification for fixing the gauge at the outset. In fact, knowledge of gauge symmetries is one of the outcomes of performing the Hamiltonian analysis. In contrast, we do not rely on any of the aforementioned simplifications.

On the other hand, in reference [30], a parametrized version of action (1) was investigated. The idea of parametrizing a theory was mainly developed by Karel V. Kuchař [31], aiming at exploring whether the proposal of Unruh and Wald [32], who suggested that the time problem could be addressed in this way, holds. Unfortunately, this does not work. This formulation introduced additional field variables, namely dynamical diffeomorphism. The authors of [30] showed that, by considering the variation in the action with respect to diffeomorphisms, the unimodular condition is directly obtained. The same conclusion was also arrived at in the Hamiltonian formulation, providing a direct pathway to the unimodular condition. It is important to note that the authors employed the GNH method and mentioned that it should be possible to achieve the same results using the Dirac algorithm. Although parametrization simplifies the analysis in this specific case, it may not be necessary in order to study unimodular gravity. It should be feasible to perform the Hamiltonian analysis without involving diffeomorphisms as field variables. In the next section, we demonstrate that this is indeed possible.

2.1. Dirac Geometric Analysis

Dirac's geometrical analysis, as described in [10], proceeds as follows: First, a 3 + 1 decomposition of the field is performed. After that, conjugate momenta are identified for each variable in the configuration space. However, the variables in the phase space could not be independent, being related to each other through relations called primary constraints, which define a submanifold $\mathcal{M}_0$ of the phase space. Second, using the canonical symplectic structure, we find the Hamiltonian vector field associated with the usual Hamiltonian function supplemented with the mentioned constraints through Dirac multipliers. Third, for the consistency of the dynamics, this Hamiltonian vector must be tangent to the submanifold $\mathcal{M}_0$ (a condition often expressed as the preservation of constraints under time evolution). This condition could fix (some of) the Dirac multipliers and/or give rise to secondary constraints. In the latter case, we must iterate the procedure by requiring appropriate tangency conditions until it stops, i.e., no more constraints arise. Now, we apply these steps to the action (1).

2.2. The Hamiltonian Setup

We begin by performing the 3 + 1 decomposition. Taking into account that our manifold $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \Sigma$, we introduce a foliation of $\mathcal{M}$ defined by the level surfaces Σ_t of a scalar function t (and inclusion $\jmath_t : \Sigma_t \hookrightarrow \mathcal{M}$), together a vector field ∂_t, such that $\mathrm{d}t(\partial_t) = 1$. Then, we expand a differential form $\boldsymbol{a}$ as $\boldsymbol{a} = \mathrm{d}t \wedge \boldsymbol{a}_0 + \underline{\boldsymbol{a}}$, where $\boldsymbol{a}_0 := \iota_{\partial_t}\boldsymbol{a}$ (ι denotes the interior product) and $\underline{\boldsymbol{a}} := \boldsymbol{a} - \mathrm{d}t \wedge \boldsymbol{a}_0$. Also, we define $a_0 := \jmath_t^* \boldsymbol{a}_0$ and $a := \jmath_t^* \underline{\boldsymbol{a}}$. Applying these definitions to our field variables, we have that the configuration space $\mathcal{Q}$ of our theory consists of the scalar fields $e_0^I, \omega_0{}^I{}_J, \Lambda \in C^\infty(\Sigma)$, and the one-forms $e^I, \omega^I{}_J \in \Omega^1(\Sigma)$. Also, the points in the tangent bundle of the configuration space of our system are denoted as v_q (where $q = (e_0^I, e^I, \omega_0^{IJ}, \omega^{IJ}, \Lambda)$ denotes a point in $\mathcal{Q}$) with components $(v_{e0}^I, v_e^I, v_{\omega 0}^{IJ}, v_\omega^{IJ}, v_\Lambda)$ that can be interpreted as velocities, $\mathbf{v}_q \in T_q\mathcal{Q}$. We have $v_{e0}^I, v_{\omega 0}^{IJ}, v_\Lambda \in C^\infty(\Sigma)$, and $v_e^I, v_\omega^{IJ} \in \Omega^1(\Sigma)$. In terms of these objects, the Lagrangian can be written as

$$L(\mathrm{v}_q) = \int_\Sigma \Big(P^{IJ}{}_{KL} \big(2\, e_0^K e^L \wedge F_{IJ} + e^K \wedge e^L \wedge (v_{\omega IJ} - D\omega_{0IJ}) \big) + \Lambda \big(\mathsf{vol} - \frac{1}{3} \epsilon_{IJKL} e_0^I e^J \wedge e^K \wedge e^L \big) \Big) , \tag{4}$$

where the curvature is $F_{IJ} := \mathrm{d}\omega_{IJ} + \omega_I{}^K \wedge \omega_{KJ}$, D is the covariant exterior derivative with respect to the connection ω^{IJ} on Σ, and $\mathsf{vol} := \mathsf{vol}_0$.

Next, we take v and w in the same fiber of $T\mathcal{Q}$ and compute the fiber derivative $FL : T\mathcal{Q} \to T^*\mathcal{Q}$ to obtain the momenta $\mathbf{p}$, with components $(\mathbf{p}_{e0I}, \mathbf{p}_{eI}, \mathbf{p}_{\omega 0IJ}, \mathbf{p}_{\omega IJ}, \mathbf{p}_\Lambda)$, we obtain

$$\mathbf{p}(\mathrm{w}) = \langle FL(\mathrm{v}) | \mathrm{w} \rangle = \frac{d}{dt} L(q, \mathrm{v} + t\mathrm{w}) \Big|_{t=0} = \int_\Sigma P_{IJKL} w_\omega^{IJ} \wedge e^K \wedge e^L . \tag{5}$$

Then, we obtain the following five primary constraints

$$\mathbf{c}_{0I}\left(w_{e0}^{I}\right) := \mathbf{p}_{e0I}(w_{e0}^{I}) = 0\,,\quad \mathbf{c}_{I}\left(w_{e}^{I}\right) := \mathbf{p}_{eI}(w_{e}^{I}) = 0\,,\quad \mathbf{C}_{0IJ}\left(w_{\omega 0}^{IJ}\right) := \mathbf{p}_{\omega 0IJ}(w_{\omega 0}^{IJ}) = 0\,,$$
$$\mathbf{C}_{IJ}\left(w_{\omega}^{IJ}\right) := \mathbf{p}_{\omega IJ}(w_{\omega}^{IJ}) - \int_{\Sigma} P_{IJKL} w_{\omega}^{IJ} \wedge e^{K} \wedge e^{L} = 0\,,\qquad \mathbf{c}(w_{\Lambda}) := \mathbf{p}_{\Lambda}[w_{\Lambda}] = 0\,, \tag{6}$$

acting on $w_{e0}^{I}, w_{e}^{I}, w_{\omega 0}^{IJ}, w_{\omega}^{IJ}, w_{\Lambda}$, respectively.

The Hamiltonian, which is defined on the primary constraint submanifold, takes the form

$$H = \int_{\Sigma} P_{IJKL}\left(e^{I} \wedge e^{J} \wedge D\omega_{0}^{KL} - 2e_{0}^{I} e^{J} \wedge F^{KL}\right) - \Lambda\left(\mathsf{vol} - \frac{1}{3}\epsilon_{IJKL} e_{0} e^{I} \wedge e^{J} \wedge e^{K} \wedge e^{L}\right). \tag{7}$$

To extend it to the full phase space, we add the primary constraints multiplied by some undetermined objects (at this stage) that we refer to as "Dirac multipliers". This provides us with a family of extensions of the Hamiltonian beyond the primary constraint submanifold, with all of them equal on it to (7).

Now, we look for the Hamiltonian vector field associated with the mentioned (total) Hamiltonian, i.e., we look for the vector field $\mathbb{X} \in \mathfrak{X}(T^*\mathcal{Q})$ satisfying

$$\Omega(\mathbb{X}, \mathbb{Y}) = \mathbb{d}H(\mathbb{Y}) + \langle u | \mathbb{d}\Phi \rangle(\mathbb{Y})\,, \tag{8}$$

where Ω is the canonical symplectic form in $T^*\mathcal{Q}$, Φ denotes the sum of the primary constraints (6), acting on their corresponding Dirac multipliers (denoted by u's), and $\mathbb{Y}$ is an auxiliary vector field defined on the phase space.

In this case, the vector fields take the form

$$\mathbb{Z} = ((Z_{e0}^{I}, Z_{e}^{I}, Z_{\omega 0}^{IJ}, Z_{\omega}^{IJ}, Z_{\Lambda}), (\mathbf{Z}_{e0}^{I}, \mathbf{Z}_{e}^{I}, \mathbf{Z}_{\omega 0}^{IJ}, \mathbf{Z}_{\omega}^{IJ}, \mathbf{Z}_{\Lambda}))\,.$$

From (8), we obtain

$$X_{e0}^{I} = u_{0}^{I}\,,\quad X_{e}^{I} = u^{I}\,,\quad X_{\omega 0}^{IJ} = u_{0}^{IJ}\,,\quad X_{\omega}^{IJ} = u^{IJ}\,,\quad X_{\Lambda} = u_{\Lambda}\,,$$
$$\mathbf{X}_{e0I}[Y_{e0}^{I}] = -\int_{\Sigma} Y_{e0}^{I}\left(2P_{IJKL} e^{J} \wedge F^{KL} - \frac{\Lambda}{3}\epsilon_{IJKL} e^{J} \wedge e^{K} \wedge e^{L}\right),$$
$$\mathbf{X}_{eI}[Y_{e}^{I}] = \int_{\Sigma} Y_{e}^{I} \wedge \left(2P_{IJKL} e^{J} \wedge \left(D\omega_{0}^{KL} - u^{KL}\right) + e_{0}^{J}\left(2P_{IJKL} F^{KL} - \Lambda\epsilon_{IJKL} e^{K} \wedge e^{L}\right)\right),$$
$$\mathbf{X}_{\omega 0IJ}[Y_{\omega 0}^{IJ}] = -\int_{\Sigma} Y_{\omega 0}^{IJ} D(P_{IJKL} e^{K} \wedge e^{L})\,,$$
$$\mathbf{X}_{\omega IJ}[Y_{\omega}^{IJ}] = \int_{\Sigma} Y_{\omega}^{IJ} \wedge \left(-2D(P_{IJKL} e_{0}^{K} e^{L}) + 2P_{IKLM} {\omega_{0J}}^{K} e^{L} \wedge e^{M}\right),$$
$$\mathbf{X}_{\Lambda}[Y_{\Lambda}] = -\int_{\Sigma} Y_{\Lambda}\left(\mathsf{vol} - \frac{1}{3}\epsilon_{IJKL} e_{0}^{I} e^{J} \wedge e^{K} \wedge e^{L}\right).$$

2.3. Tangency Conditions to the Primary Constraints

Now, we need to verify that the vector field $\mathbb{X}$ is tangent to the surface defined by the primary constraints (6). This requires ensuring the satisfaction of the consistency equation, i.e.,

$$\imath_{\mathbb{X}} \mathbb{d}\mathbf{C} = 0,$$

where $\mathbf{C}$ represents any of the constraints (6). By computing the tangency condition for the constraints $\mathbf{c}_I$ and $\mathbf{C}_{IJ}$, we obtain

$$
\begin{aligned}
&0 = \imath_{\mathbb{X}} \mathrm{d}\mathbf{c}_I = \mathbf{X}_{eI} \\
&\quad \Rightarrow e^J \wedge P_{IJKL}\left(u^{KL} - D\omega_0^{KL}\right) - e_0^J\left(P_{IJKL}F^{KL} - \Lambda * (e_I \wedge e_J)\right) = 0\,, \qquad \text{(9a)}\\
&0 = \imath_{\mathbb{X}} \mathrm{d}\mathbf{C}_{IJ}(\cdot) = \mathbf{X}_{\omega IJ}(\cdot) - \int_\Sigma (\cdot)^{IJ} \wedge 2P_{IJKL}X_e^K \wedge e^L \\
&\quad \Rightarrow D\left(P_{IJKL}e_0^K e^L\right) + \omega^K_{0[I}P_{|K|J]MN}e^M \wedge e^N - P_{IJKL}u^K \wedge e^L = 0\,. \qquad \text{(9b)}
\end{aligned}
$$

The implications are there because the tangency condition must be valid for all test functions (denoted by w in Equation (6)). Equations (9a) and (9b) are equations for the Dirac multipliers u^I and u^{IJ}.

The tangency condition over the constraints $\mathbf{c}_{0I}$, $\mathbf{C}_{0IJ}$, and $\mathbf{c}$ gives

$$0 = \imath_{\mathbb{X}} \mathrm{d}\mathbf{C}_{0IJ} = \mathbf{X}_{\omega 0IJ} \qquad \Rightarrow \quad C_{IJ} := D(e_I \wedge e_J) = 0\,, \qquad \text{(10a)}$$

$$0 = \imath_{\mathbb{X}} \mathrm{d}\mathbf{c}_0 = \mathbf{X}_\Lambda \qquad \Rightarrow \quad c := \mathsf{vol} - \frac{1}{3}\epsilon_{IJKL}e_0^I e^J \wedge e^K \wedge e^L = 0\,, \qquad \text{(10b)}$$

$$0 = \imath_{\mathbb{X}} \mathrm{d}\mathbf{c}_{0I} = \mathbf{X}_{e0I} \qquad \Rightarrow \quad C_I := e^J \wedge \left(P_{IJKL}F^{KL} - \frac{\Lambda}{6}e_I \wedge e_J\right) = 0\,. \qquad \text{(10c)}$$

Equations (10a)–(10c) are secondary constraints.

The procedure to follow now is the following: first, we must solve (9a) and (9b) for the Dirac multipliers, and then study whether the tangency of the vector field to the constraint submanifolds defined by the constraints C_I, C_{IJ}, and c yields secondary constraints or further conditions on the multipliers (this process may iteratively continue). The aim is to conclude with either a complete set of constraints or the identification of a contradiction.

At this point, it is useful to compare our findings with those derived in the parametrized version, as analyzed in [30]. We must mention that, in that study, the dynamical diffeomorphisms only coupled to the non-dynamical volume form. This fact has many consequences, as we explained before, at the level of the action. At the Hamiltonian level, we observe the following distinctions:

(d1) There are no momenta associated with the g-spacelike embeddings, $X : \Sigma \hookrightarrow \mathcal{M}$, because they are not present in our Lagrangian. In the case of [30], they look like

$$\mathbf{p}_X(w_X) = \int_\Sigma \varepsilon n_X(w_X)\Lambda \mathsf{vol}_{\gamma_X}$$

where n_X denotes the future directed normal vector field over X associated with the embedding.

(d2) Our Hamiltonian incorporates the term $\Lambda\mathsf{vol}$, which does not have a counterpart in the parametrized version.

(d3) For the action used in [30], while solving the analog equation of (8) for the Hamiltonian vector field in the GNH formalism, the authors obtain the unimodular condition $\mathrm{d}\Lambda = 0$ as a consequence of the non-trivial evolution of the embedding variables X. In our case, as we demonstrate, several subsequent steps are necessary to arrive at this.

(d4) We have the constraint (10b), which is not present in [30]. Of course, they obtain an analogous equation, but instead of being a constraint, it is an equation of motion involving the component $Z_{\mathsf{X}}^\perp$ of the Hamiltonian field associated with the diffeomorphism variable:

$$Z_{\mathsf{X}}^\perp\, \mathsf{vol}_{\gamma_X} = \frac{1}{3}\epsilon_{IJKL}e_{\mathsf{t}}^I(e^J \wedge e^K \wedge e^L)\,.$$

This equation must then be solved. In our case, we need to verify the tangency condition of (10b).

Despite these differences, several similarities exist:

(s1) The equations for the Dirac multipliers (9a) and (9b) appear there as equations for some components of the Hamiltonian vector field. Explicity,

$$Z_e^{[I} \wedge e^{J]} = D(e_{\mathrm{t}}^{[I} e^{J]}) - [P^{-1}]^{IJ}{}_{MN} P^{MK}{}_{LP}\, \omega_{\mathrm{t}}{}^{N}{}_{K}\, e^L \wedge e^P \,, \tag{11a}$$

$$2P_{IJKL} e^J \wedge Z_\omega^{KL} = 2P_{IJKL}\big(e^J \wedge D\omega_{\mathrm{t}}^{KL} + e_{\mathrm{t}}^J F^{KL}\big) - \Lambda\, \epsilon_{IJKL} e_{\mathrm{t}}^J e^K \wedge e^L \,. \tag{11b}$$

Then, the resolution of these equations follows the same steps.

(s2) The constraints (10a) and (10c), also appear there as secondary constraints. This has important consequences. In particular, the tangency condition on (10a) mirrors the (long) procedure outlined in [30]. In the following we provide a summary of the results.

Although Dirac multipliers u^{IJ} can be determined, the complete expressions are long and not particularly illuminating, as detailed in [30]. On the other hand, Equation (9b) for u^I can only be solved if the following conditions hold

$$\mathsf{D}^{ij} := \frac{De^i \wedge e^j + De^j \wedge e^i}{\mathsf{w}} = 0\,, \tag{12}$$

where we have introduced the volume form $\mathsf{w} := \dfrac{1}{3!}\epsilon_{ijk} e^i \wedge e^j \wedge e^k$. The interpretation of the above quotient is as follows: given any top form α, it is always possible to find a unique smooth function f, such that $\alpha = f\mathsf{w}$ (notice that f depends both on α and w). We denote this function as $\dfrac{\alpha}{\mathsf{w}}$. The tangency condition imposed on this constraint does not provide any further conditions. Additionally, it is possible to prove (see [30]) that the constraints (10a) and (12) are equivalent to the condition

$$De^I = 0\,. \tag{13}$$

Using this form of the constraints, the solution for the Dirac multiplier in (9b) is given by

$$u^I = De_0^I - \omega_{0J}^I e^J \,.$$

Furthermore, the constraint (10c) simplifies to

$$\epsilon_{IJKL} e^J \wedge \left(F^{KL} - \frac{1}{3}\Lambda e^K \wedge e^L\right) = 0\,, \tag{14}$$

which can also be written as

$$e^I \wedge F^{JK} + e^J \wedge F^{KI} + e^K \wedge F^{IJ} = \Lambda e^I \wedge e^J \wedge e^K \,. \tag{15}$$

To continue with the algorithm, we need to impose the tangency condition on constraints (10b) and (10c). For the first, we obtain the following

$$0 = \imath_{\mathbb{X}} \mathrm{d}\mathrm{c} = -\epsilon_{IJKL}\left(\frac{1}{3} u_0^I e^J \wedge e^K \wedge e^L + 3 e_0^I u^J \wedge e^K \wedge e^L\right). \tag{16}$$

Using the solution for the Dirac multiplier u^I, we observe that

$$\epsilon_{IJKL} e_0^I u^J \wedge e^K \wedge e^L = 0\,.$$

Therefore, Equation (16) indicates that the parameters u_0^I are not independent. In fact, by breaking down the Lorentz group, we can rewrite the above equation as follows

$$u_0^0 = -\frac{\epsilon_{ijk} u_0^i e^0 \wedge e^j \wedge e^k}{2\mathsf{w}}\,. \tag{17}$$

Therefore, the Dirac multipliers u_0^i are arbitrary, while u_0^0 must satisfy Equation (17). It is well-known (see for instance [30,33]) that the u_0^I parameters are related to the diffeomorphism symmetry. This can also be seen directly from the Hamiltonian vector field. The significance of (17) is that admissible diffeomorphisms are those that preserve the fixed volume form (for the analog equation in metric variables, see [13]).

For the constraint (10c), we obtain

$$u^J \wedge \left(P_{IJKL}F^{KL} - \frac{\Lambda}{2}\epsilon_{IJKL}e^K \wedge e^L \right) + e^J \wedge P_{IJKL}Du^{KL} - \frac{u_\Lambda}{6}\epsilon_{IJKL}e^J \wedge e^K \wedge e^L = 0\,. \tag{18}$$

The term involving u^{KL} can be derived by applying the covariant derivative D to (9a). Using the identity $D(D\omega_0^{KL}) = 2F^K{}_P\omega_0^{PL}$, that holds on the constraint surface, we obtain

$$\begin{aligned} e^J \wedge P_{IJKL}Du^{KL} =& 2e^J \wedge P_{IJKL}F^K{}_P\omega_0^{PL} - De_0^J \wedge (P_{IJKL}F^{KL} - \frac{\Lambda}{2}\epsilon_{IJKL}e^K \wedge e^L) \\ &+ \frac{1}{2}\epsilon_{IJKL}e_0^J \mathrm{d}\Lambda \wedge e^K \wedge e^L\,. \end{aligned} \tag{19}$$

Using this result and that $u^I = De_0^I - \omega_0^I{}_J e^J$ in (18), we obtain an equation that depends only on the Lagrange multiplier u_Λ, given by

$$\begin{aligned} 0 =& \omega_0^J{}_P e^P \wedge (P_{IJKL}F^{KL} - \frac{\Lambda}{2}\epsilon_{IJKL}e^K \wedge e^L) + 2e^J \wedge P_{IJKL}F^K{}_P\omega_0^{PL} + \frac{1}{2}\epsilon_{IJKL}e_0^J d\Lambda \wedge e^K \wedge e^L \\ &- \frac{u_\Lambda}{6}\epsilon_{IJKL}e^J \wedge e^K \wedge e^L\,. \end{aligned} \tag{20}$$

This can be significantly simplified by employing the constraint C_I itself, not as defined in Equation (10c) but written in the form in (15). If we do so, it reduces to

$$0 = \epsilon_{IJKL}\left(e_0^J d\Lambda \wedge e^K \wedge e^L - \frac{u_\Lambda}{3}e^J \wedge e^K \wedge e^L\right). \tag{21}$$

Let us analyze the consequences of this equation, which is not present in [30]. Multiplying the equation by e_0^I allows us to annihilate the first term, resulting in

$$0 = \frac{u_\Lambda}{3}\epsilon_{IJKL}e_0^I e^J \wedge e^K \wedge e^L = u_\Lambda \mathsf{vol}\,, \tag{22}$$

where, in the last equality, we used the constraint c defined in (10b). As vol is a volume form, we are left with $u_\Lambda = 0$, implying $X_\Lambda = 0$. As a consequence, Λ does not evolve. Then, Equation (21) reduces to

$$0 = \epsilon_{IJKL}e_0^J \mathrm{d}\Lambda \wedge e^K \wedge e^L\,. \tag{23}$$

Given that e^i spans a basis for the one-forms in Σ, we can express e^0 as $e^0 = \mu_i e^i$. By defining $E^{ij} := \dfrac{e^i \wedge e^j \wedge e^0}{\mathsf{w}}$, we obtain $\mu_i = \dfrac{\epsilon_{ijk}E^{jk}}{2}$. Using the previous expressions, writing $\mathrm{d}\Lambda = a_i e^i$ and breaking the Lorentz group in (23), we obtain

$$I = 0\,, \qquad a_i e_\perp^i = 0\,, \tag{24a}$$

$$I = i\,, \qquad e_\perp^0 a_i + \epsilon_{ijk}e_\perp^j E^{lk}a_l = 0\,. \tag{24b}$$

Using $E^{ij} = \epsilon^{ijk}\mu_k$ and (24a) in (24b) we obtain

$$0 = a_i\left(e_\perp^0 - e_0^j\mu_j\right). \tag{25}$$

In [30], it was demonstrated that $\epsilon_{IJKL}e_0^I e^J \wedge e^K \wedge e^L = 6\left(e_\perp^0 - e_0^i\mu_i\right)\mathsf{w}$. Consequently, the term in the brackets in (25) is non-zero due to the non-degeneracy of the tetrads. Therefore, Equation (25) implies $a_i = 0$, and subsequently, $\mathrm{d}\Lambda = 0$. This last constraint, together with $X_\Lambda = 0$, asserts that Λ is a constant on $\mathcal{M}$. Given its role in the equations of motion, it is identified with the cosmological constant.

The tangency condition over the constraint $\mathrm{d}\Lambda = 0$ gives no further conditions.

3. Conclusions

This paper explores the application of the geometric Dirac algorithm for unimodular gravity with a Holst term and addresses several issues that have not been treated in detail in the literature. Through a comprehensive analysis, we demonstrate the algorithm's applicability without the need for the additional ingredients found in the existing literature. Wherever possible, we compare our results with previous findings, managing to reproduce their final results despite the expected additional steps due to the absence of the simplifications provided by those additional ingredients. Notably, we uncover an extra generation of constraints, specifically in (21), as we have explained.

As a byproduct of our analysis, we identify the Dirac multipliers that are responsible for the reduction in the group of diffeomorphisms to the group of diffeomorphisms that preserve a fixed volume.

It is interesting to note that, as described in detail in [33], by using time-gauge fixing, the real Ashtekar formulation for general relativity can also be readily derived from the results presented here, incorporating, in this case, the unimodularity condition.

The techniques shown in this paper for the case of the unimodular Holst action can be implemented in an analogous way to other first-order theories and, in particular, to the many competing alternative gravitational theories [34], whose experimental status can be seen in [35–37]. In particular, it can be interesting to discuss the application of the geometric Dirac program to the study of theories with boundaries, where, as in the case of generalized general relativity in 2 + 1 dimensions [10,38–40], they have shown improvement over other methods.

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References

1. Dirac, P.A.M. *Lectures on Quantum Mechanics*; Dover Books on Physics; Dover Publications: Mineola, NY, USA, 2013.
2. Dirac, P.A.M. Generalized hamiltonian dynamics. *Can. J. Math.* **1950**, *2*, 129–148. [CrossRef]
3. Gotay, M.J.; Nester, J.M.; Hinds, G. Presymplectic manifolds and the Dirac–Bergmann theory of constraints. *J. Math. Phys.* **1978**, *19*, 2388. [CrossRef]
4. Gotay, M.J. Presymplectic Manifolds, Geometric Constraint Theory and the Dirac-Bergmann Theory of Constraints. Ph.D. Thesis, Center for Theoretical Physics, University of Maryland, College Park, MD, USA, 1979.
5. Gotay, M.J.; Nester, J.M. Generalized constraint algorithm and special presymplectic manifolds. In *Proceedings of the Geometric Methods in Mathematical Physics. Lecture Notes in Mathematics vol 775*; Kaiser, G., Marsden, J.E., Eds.; Springer: Berlin/Heidelberg, Germany, 1980; p. 78.
6. Gotay, M.J.; Nester, J.M. Presymplectic lagrangian systems. I: The constraint algorithm and the equivalence theorem. *Annales de l'institut Henri Poincaré. Section A Physique Théorique* **1979**, *30*, 129–142.

7. Gotay, M.J.; Nester, J.M. Presymplectic lagrangian systems. II: The second-order equation problem. *Annales de l'institut Henri Poincaré. Section A Physique Théorique* **1980**, *32*, 1–13.
8. Barbero González, J.F.; Prieto, J.; Villaseñor, E.J.S. Hamiltonian treatment of linear field theories in the presence of boundaries: A geometric approach. *Class. Quantum Gravity* **2014**, *31*, 045021. [CrossRef]
9. Margalef-Bentabol, J. Towards General Relativity through Parametrized Theories. Ph.D. Thesis, Universidad Carlos III de Madrid, Getafe, Spain, 2018.
10. Barbero González, J.F.; Díaz, B.; Margalef-Bentabol, J.; Villaseñor, E.J.S. Dirac's algorithm in the presence of boundaries: A practical guide to a geometric approach. *Class. Quantum Gravity* **2019**, *36*, 205014. [CrossRef]
11. Barbero González, J.F.; Díaz, B.; Margalef-Bentabol, J.; Villaseñor, E.J.S. Generalizations of the Pontryagin and Husain-Kuchař actions to manifolds with boundary. *J. High Energy Phys.* **2019**, *2019*, 121. [CrossRef]
12. Barbero González, J.F.; Díaz, B.; Margalef-Bentabol, J.; Villaseñor, E.J.S. Edge observables of the Maxwell-Chern-Simons theory. *Phys. Rev. D* **2022**, *106*, 025011. [CrossRef]
13. Henneaux, M.; Teitelboim, C. The cosmological constant and general covariance. *Phys. Lett. B* **1989**, *222*, 195. [CrossRef]
14. Van Der Bij, J.; Van Dam, H.; Ng, Y.J. The exchange of massless spin-two particles. *Phys. A Stat. Mech. Its Appl.* **1982**, *116*, 307–320. [CrossRef]
15. Unruh, W.G. Unimodular theory of canonical quantum gravity. *Phys. Rev. D* **1989**, *40*, 1048–1052.
16. Ng, Y.J.; van Dam, H. Unimodular theory of gravity and the cosmological constant. *J. Math. Phys.* **1991**, *32*, 1337–1340. [CrossRef]
17. Finkelstein, D.R.; Galiautdinov, A.A.; Baugh, J.E. Unimodular relativity and cosmological constant. *J. Math. Phys.* **2001**, *42*, 340–346. [CrossRef]
18. Nobbenhuis, S. Categorizing Different Approaches to the Cosmological Constant Problem. *Found. Phys.* **2006**, *36*, 613–680.
19. Smolin, L. Quantization of unimodular gravity and the cosmological constant problems. *Phys. Rev. D* **2009**, *80*, 084003. [CrossRef]
20. Smolin, L. Unimodular loop quantum gravity and the problems of time. *Phys. Rev. D* **2011**, *84*, 044047. [CrossRef]
21. Klusoň, J. Canonical analysis of unimodular gravity. *Phys. Rev. D* **2015**, *91*, 064058. [CrossRef]
22. Bufalo, R.; Oksanen, M.; Tureanu, A. How unimodular gravity theories differ from general relativity at quantum level. *Eur. Phys. J. C* **2015**, *75*, 477. [CrossRef]
23. García-Aspeitia, M.A.; Martínez-Robles, C.; Hernández-Almada, A.; Magaña, J.; Motta, V. Cosmic acceleration in unimodular gravity. *Phys. Rev. D* **2019**, *99*, 123525. [CrossRef]
24. Barvinsky, A.O.; Kolganov, N.; Kurov, A.; Nesterov, D. Dynamics of the generalized unimodular gravity theory. *Phys. Rev. D* **2019**, *100*, 023542. [CrossRef]
25. Carballo-Rubio, R.; Garay, L.J.; García-Moreno, G. Unimodular gravity vs general relativity: A status report. *Class. Quantum Gravity* **2022**, *39*, 243001.
26. Barvinsky, A.O.; Nesterov, D.V. Restricted gauge theory formalism and unimodular gravity. *Phys. Rev. D* **2023**, *108*, 065004. [CrossRef]
27. Bengochea, G.R.; León, G.; Perez, A.; Sudarsky, D. A clarification on prevailing misconceptions in unimodular gravity. *J. Cosmol. Astropart. Phys.* **2023**, *2023*, 011. [CrossRef]
28. Anero, J.; Martin, C.P. Unimodular gravity and the gauge/gravity duality. *Phys. Rev. D* **2023**, *107*, 046001. [CrossRef]
29. Yamashita, S. Hamiltonian analysis of unimodular gravity and its quantization in the connection representation. *Phys. Rev. D* **2020**, *101*, 086007. [CrossRef]
30. Barbero González, J.F.; Díaz, B.; Margalef-Bentabol, J.; Villaseñor, E.J.S. Hamiltonian Gotay-Nester-Hinds analysis of the parametrized unimodular extension of the Holst action. *Phys. Rev. D* **2021**, *103*, 064062. [CrossRef]
31. Kuchař, K.V. Does an unspecified cosmological constant solve the problem of time in quantum gravity? *Phys. Rev. D* **1991**, *43*, 3332–3344. [CrossRef]
32. Unruh, W.G.; Wald, R.M. Time and the interpretation of canonical quantum gravity. *Phys. Rev. D* **1989**, *40*, 2598–2614. [CrossRef] [PubMed]
33. Barbero González, J.F.; Díaz, B.; Margalef-Bentabol, J.; Villaseñor, E.J.S. Concise symplectic formulation for tetrad gravity. *Phys. Rev. D* **2021**, *103*, 024051. [CrossRef]
34. Olmo, G.J.; Rubiera-Garcia, D.; Wojnar, A. Stellar structure models in modified theories of gravity: Lessons and challenges. *Phys. Rep.* **2020**, *876*, 1–75.
35. Yunes, N.; Siemens, X. Gravitational-wave tests of general relativity with ground-based detectors and pulsar-timing arrays. *Living Rev. Relat.* **2013**, *16*, 1–124.
36. Will, C.M. The confrontation between general relativity and experiment. *Living Rev. Relat.* **2014**, *17*, 4.
37. Will, C.M. *Theory and Experiment in Gravitational Physics*; Cambridge University Press: Cambridge, UK, 2018.
38. Romano, J.D. Geometrodynamics vs. connection dynamics. *Gen. Relativ. Gravit.* **1993**, *25*, 759–854. [CrossRef]
39. Ashtekar, A.; Romano, J.D. Chern-Simons and Palatini actions and (2 + 1)-gravity. *Phys. Lett. B* **1989**, *229*, 56–60. [CrossRef]
40. Witten, E. 2 + 1 dimensional gravity as an exactly soluble system. *Nucl. Phys. B* **1988**, *311*, 46–78. [CrossRef]