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Symmetry Analysis of the Two-Dimensional Stationary Gas Dynamics Equations in Lagrangian Coordinates

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Abstract: This article analyzes the symmetry of two-dimensional stationary gas dynamics equations in Lagrangian coordinates, including the search for equivalence transformations, the group classification of equations, the derivation of group foliations, and the construction of conservation laws. The consideration of equations in Lagrangian coordinates significantly simplifies the procedure for obtaining conservation laws, which are derived using the Noether theorem. The final part of the work is devoted to group foliations of the gas dynamics equations, including for the nonstationary isentropic case. The group foliations approach is usually employed for equations that admit infinite-dimensional groups of transformations (which is exactly the case for the gas dynamics equations in Lagrangian coordinates) and may make it possible to simplify their further analysis. The results obtained in this regard generalize previously known results for the two-dimensional shallow water equations in Lagrangian coordinates.

Keywords: Lie symmetry; conservation law; gas dynamics; stationary equations; Lagrangian coordinates; group foliation

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1. Introduction

Analyses of physical phenomena in continuum mechanics can be carried out using two different descriptions: Eulerian and Lagrangian. Usually, in hydrodynamics, the Eulerian description is employed, where the system describes the motion of the fluid at fixed points; the velocity, density, and other properties of the fluid particles are considered functions of time and fixed spatial coordinates. In contrast, in the Lagrangian description, particles are identified by the positions they occupy at some initial time. Lagrangian variables are rarely used to solve hydrodynamic problems. The reason for this is the more complicated appearance of continuum equations in Lagrangian form compared to their Eulerian counterparts. However, the application of Lagrangian variables is preferable for some types of problems [1–3].

The present paper is devoted to the analysis of the two-dimensional stationary (steady-state) gas dynamics equations. The model of stationary flows describes a wide class of real gas flows and therefore is systematically used to solve specific problems [4–8]. Typical examples of such problems are the outflow of a gas jet through a small hole from an infinite reservoir or a uniform translational motion of a body in an unbounded gas resting at infinity in a coordinate system moving with the body.

In addition to the stationarity assumption, it is often possible to simplify the model by considering potential or isentropic flows. For example, the Prandtl–Meyer waves [9], as well as their generalizations [10], are widely known for supersonic potential flows. Non-isentropic flows with separable state equations are invariant with respect to a family of the Munk–Prim transformations involving a smooth scalar function of the space variables that is constant along each individual streamline [11,12]. Although the inverse transformation might be problematic to obtain, the Munk–Prim transformations are known in the literature as a substitution principle [13]. The presence of the Munk–Prim transformations can be explained by the fact that the equations admit a specific infinite-dimensional group of transformations [4].

Transformations that simplify the forms of equations, reduce their order, and allow one to find conservation laws and exact solutions can be found by the group analysis method [14,15]. Symmetry analysis usually includes determining admitted Lie algebras of equations, their equivalence transformations, and group classifications; studying optimal systems of subalgebras and deriving their exact (invariant and partially invariant) solutions; finding conservation laws; and constructing automorphic and resolving systems (group foliations).

This article is devoted to the group analysis of stationary two-dimensional gas dynamics equations in Lagrangian coordinates. In contrast to the nonstationary case, stationary solutions in Lagrangian coordinates are determined by an overdetermined system of equations. In general, overdetermination results in a smaller number of determining equations, which can expand the set of symmetries compared to a determined system. In particular, for the stationary gas dynamics equations, we have obtained more arbitrary functions in symmetries than for the nonstationary case [16]. The use of Lagrangian coordinates and the found symmetries make it possible to apply the Noether theorem to derive conservation laws. This is due to the fact that equations in Lagrange coordinates, unlike equations in Euler coordinates, are of second order, and a Lagrangian can be found for them. Classifying Lie algebras containing arbitrary functions is a difficult task. One approach to such an analysis is group foliation [14], which is also presented in this article.

The paper is organized as follows. In the next section, the two-dimensional stationary gas dynamics equations in Eulerian and Lagrangian coordinates are presented. In Section 3, for the equations in Lagrangian coordinates, groups of equivalence transformations are found, and the results of group classification with respect to the entropy as an arbitrary element are presented. Conservation laws for stationary gas dynamics equations, derived using the Noether theorem, are given in Section 4. Section 5 is devoted to group foliations of the two-dimensional gas dynamics equations. Both the stationary nonisentropic case and the case of nonstationary isentropic flows are considered. The results are summarized in Section 6.

2. Stationary Equations of Gas Dynamics

2.1. Eulerian Coordinates

Consider the two-dimensional stationary gas dynamics equations of a polytropic gas

$$\begin{gathered} \rho(uu_x + vu_y) + p_x = 0, \\ \rho(uv_x + vv_y) + p_y = 0, \\ u\rho_x + v\rho_y + \rho(u_x + v_y) = 0, \\ uS_x + vS_y = 0, \end{gathered} \tag{1}$$

where x and y are Eulerian coordinates, u and v are the components of the two-dimensional velocity vector, ρ is the density, $p = S\rho^\gamma$ is the pressure, the polytropic exponent is

$$\gamma = 1 + \frac{R}{c_v} > 1,$$

where R is the gas constant, and c_v is the dimensionless specific heat capacity at constant volume. The function S is given by the formula [4]

$$S = Re^{(\widetilde{S} - \widetilde{S}_0)/c_v},$$

where $\widetilde{S}$ is the entropy and $\widetilde{S}_0$ is a constant.

In general, we assume the flows are nonisentropic, i.e., the entropy S can depend on the coordinates x and y. The standard two-dimensional shallow water equations [4,17] are a particular case of (1) for $\gamma = 2$ and $S = const$.

The admitted Lie algebra of (1) consists of the following generators.

$$X_1 = \frac{\partial}{\partial x}, \;\; X_2 = \frac{\partial}{\partial y}, \;\; X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \;\; X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},$$

$$X_g = g\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - 2\rho\frac{\partial}{\partial \rho} + 2\gamma S\frac{\partial}{\partial S}\right),$$

where g is an arbitrary function depending on the Bernoulli integral and the entropy

$$g = g\left(\frac{u^2+v^2}{2} + \frac{\gamma}{\gamma-1}S\rho^{\gamma-1}, S\right).$$

This result was obtained by a group of researchers under the leadership of L.V. Ovsiannikov as part of the work on the SUBMODELS program [18]. As is known [4], the presence of the generator X_g in the Lie algebra admitted by (1) provides a group interpretation for the existence of Munk–Prim transformations for these equations.

2.2. Lagrangian Coordinates

Relations between Eulerian coordinates (t, x, y) and Lagrangian coordinates $(\tilde{t}, \tilde{\xi}, \tilde{\eta})$ can be specified up to an equivalence transformation, which is determined from the conservation law of mass [19]

$$\rho = \frac{\rho_0}{\varphi_{1\tilde{\xi}}\varphi_{2\tilde{\eta}} - \varphi_{1\tilde{\eta}}\varphi_{2\tilde{\xi}}},$$

where $\rho_0 = \rho_0(\tilde{\xi}, \tilde{\eta}) > 0$ is the function of integration.

Here, we stay focused on the simplest case of *mass* Lagrangian coordinates when the equivalence transformation is chosen in such a way that $\rho_0 = 1$ and $\tilde{\xi}$ and $\tilde{\eta}$ are some specific functions of the new variables ξ and η (see details, for example, in [16]). Further, the mass Lagrangian coordinates are simply called the Lagrangian coordinates, and the symbol ˜ is omitted for brevity.

Thus, here, the Lagrangian coordinates are introduced as

$$\varphi_{1t}(t,\xi,\eta) = u(t,\varphi_1(t,\xi,\eta),\varphi_2(t,\xi,\eta)), \;\; \varphi_{2t}(t,\xi,\eta) = v(t,\varphi_1(t,\xi,\eta),\varphi_2(t,\xi,\eta)), \tag{2}$$

$$\rho(t,\varphi_1(t,\xi,\eta),\varphi_2(t,\xi,\eta)) = J^{-1}(t,\xi,\eta), \;\; S(t,\varphi_1(t,\xi,\eta),\varphi_2(t,\xi,\eta)) = S_0(\xi,\eta),$$

where $J = \varphi_{1\xi}\varphi_{2\eta} - \varphi_{1\eta}\varphi_{2\xi}$.

In Lagrangian coordinates, Equation (1) are brought to the form

$$\begin{aligned} &J^\gamma\varphi_{1tt} + S_{0\xi}\varphi_{2\eta} - S_{0\eta}\varphi_{2\xi} + \gamma J^{-1}S_0\big(\varphi_{2\eta}(\varphi_{1\eta}\varphi_{2\xi\xi} - \varphi_{2\eta}\varphi_{1\xi\xi}) \\ &\quad +\varphi_{2\xi}(\varphi_{1\xi}\varphi_{2\eta\eta} - \varphi_{1\eta\eta}\varphi_{2\xi}) + 2\varphi_{2\xi}\varphi_{2\eta}\varphi_{1\xi\eta} - (\varphi_{1\xi}\varphi_{2\eta} + \varphi_{1\eta}\varphi_{2\xi})\varphi_{2\xi\eta}\big) = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &J^\gamma\varphi_{2tt} - S_{0\xi}\varphi_{1\eta} + S_{0\eta}\varphi_{1\xi} + \gamma S_0 J^{-1}\big(\varphi_{1\eta}(\varphi_{2\eta}\varphi_{1\xi\xi} - \varphi_{1\eta}\varphi_{2\xi\xi}) \\ &\quad +\varphi_{1\xi}(\varphi_{2\xi}\varphi_{1\eta\eta} - \varphi_{1\xi}\varphi_{2\eta\eta}) + 2\varphi_{1\xi}\varphi_{1\eta}\varphi_{2\xi\eta} - (\varphi_{1\xi}\varphi_{2\eta} + \varphi_{1\eta}\varphi_{2\xi})\varphi_{1\xi\eta}\big) = 0. \end{aligned} \tag{4}$$

To derive conditions defining stationary solutions in Lagrangian coordinates, consider the functions $f^e(t,x,y)$ and $f^l(t,\xi,\eta)$, which are related as follows:

$$f^l(t,\xi,\eta) = f^e(t,\varphi_1(t,\xi,\eta),\varphi_2(t,\xi,\eta)).$$

Differentiating the latter with respect to t, ξ, and η, one obtains

$$\begin{gathered} f^l_t = f^e_t + f^e_x\varphi_{1t} + f^e_y\varphi_{2t}, \\ f^l_\xi = f^e_x\varphi_{1\xi} + f^e_y\varphi_{2\xi}, \qquad f^l_\eta = f^e_x\varphi_{1\eta} + f^e_y\varphi_{2\eta}. \end{gathered} \tag{5}$$

The derivatives of the function f with respect to the Eulerian coordinates x and y are expressed from the second and third equations of (5).

$$f^e_x = J^{-1}(f^l_\xi\varphi_{2\eta} - f^l_\eta\varphi_{2\xi}), \qquad f^e_y = J^{-1}(-f^l_\xi\varphi_{1\eta} + f^l_\eta\varphi_{1\xi}).$$

Substituting into the first Equation (5), one obtains the derivative with respect to time in Eulerian coordinates

$$f^e_t = f^l_t - \varphi_{1t}f^e_x - \varphi_{2t}f^e_y.$$

For stationary solutions, one should add to Equations (3) and (4) the constraints

$$u_t = 0, \qquad v_t = 0, \qquad \rho_t = 0, \qquad S_t = 0, \tag{6}$$

where the pairs (f^e, f^l) are the following:

$$(u, \varphi_{1t}), \qquad (v, \varphi_{2t}), \qquad (\rho, J^{-1}), \qquad (S, S_0).$$

Thus, the stationary solutions of the gas dynamics of Equation (1) in Eulerian coordinates correspond to the solutions of (3) and (4) along with the equations

$$\varphi_{1tt} + (\varphi_{2\xi}\varphi_{1t\eta} - \varphi_{2\eta}\varphi_{1t\xi})\varphi_{1t}J^{-1} + (\varphi_{1\eta}\varphi_{1t\xi} - \varphi_{1\xi}\varphi_{1t\eta})\varphi_{2t}J^{-1} = 0, \tag{7}$$

$$\varphi_{2tt} + (\varphi_{2\xi}\varphi_{2t\eta} - \varphi_{2\eta}\varphi_{2t\xi})\varphi_{1t}J^{-1} + (\varphi_{1\eta}\varphi_{2t\xi} - \varphi_{1\xi}\varphi_{2t\eta})\varphi_{2t}J^{-1} = 0, \tag{8}$$

$$\begin{aligned} &\Big((2\varphi_{2\xi}\varphi_{2\xi\eta} - \varphi_{2\eta}\varphi_{2\xi\xi})\varphi_{1\eta} + \varphi_{1\eta\eta}\varphi_{2\xi}^2 - (\varphi_{1\xi}\varphi_{2\eta\eta} + 2\varphi_{1\xi\eta}\varphi_{2\eta})\varphi_{2\xi} + \varphi_{1\xi\xi}\varphi_{2\eta}^2 + J\varphi_{2\xi\eta}\Big)\varphi_{1t} \\ &\quad + \varphi_{1\eta}^2\varphi_{2t}\varphi_{2\xi\xi} + ((2\varphi_{1\xi\eta}\varphi_{2\xi} - 2\varphi_{1\xi}\varphi_{2\xi\eta} - \varphi_{1\xi\xi}\varphi_{2\eta})\varphi_{2t} + J\varphi_{2t\xi})\varphi_{1\eta} \\ &\quad + \Big(\varphi_{1\xi}^2\varphi_{2\eta\eta} - \varphi_{1\xi}\varphi_{1\eta\eta}\varphi_{2\xi} + J\varphi_{1\xi\eta}\Big)\varphi_{2t} - J(\varphi_{1\xi}\varphi_{2t\eta} + \varphi_{1t\xi}\varphi_{2\eta} - \varphi_{1t\eta}\varphi_{2\xi}) = 0, \end{aligned} \tag{9}$$

$$S_{0\eta}(\varphi_{1t}\varphi_{2\xi} - \varphi_{1\xi}\varphi_{2t}) - S_{0\xi}(\varphi_{1t}\varphi_{2\eta} - \varphi_{1\eta}\varphi_{2t}) = 0, \tag{10}$$

which are constraints (6) written in Lagrangian coordinates.

In order to correctly find the admitted Lie algebra and the equivalence group of the system, the total derivatives of (10), D_t, D_ξ, and D_η with respect to the Lagrangian variables t, ξ, and η are considered as well. After differentiation, these equations are quite cumbersome and therefore are not presented here.

3. Group Analysis of the Stationary Gas Dynamics Equations

3.1. Equivalence Transformations

The initial stage of the symmetry analysis of Equations (3), (4), and (7)–(10) is to search for equivalence transformations that divide equations into classes [14]. The class of Equations (3), (4), and (7)–(10) is parameterized by the arbitrary element $S = S_0(\xi, \eta)$. Equivalence transformations preserve the structure of equations but allow arbitrary elements to be changed.

Finding a group of equivalence transformations for a system of equations is similar to finding an admitted Lie algebra with the different generator prolongations formulas [14,20].

Calculations show that the equivalence group for Equations (3), (4), and (7)–(10) corresponds to the generators

$$
\begin{gathered}
X_1^e = (\gamma-1)t\frac{\partial}{\partial t} - 2\xi\frac{\partial}{\partial \xi}, \quad X_2^e = F_1(S)\frac{\partial}{\partial t}, \quad X_3^e = (\gamma-1)S\frac{\partial}{\partial S} - \xi\frac{\partial}{\partial \xi}, \\
X_4^e = (\gamma-1)\left(\varphi_1\frac{\partial}{\partial \varphi_1} + \varphi_2\frac{\partial}{\partial \varphi_2}\right) + 2\gamma\xi\frac{\partial}{\partial \xi}, \\
X_5^e = \varphi_2\frac{\partial}{\partial \varphi_1} - \varphi_1\frac{\partial}{\partial \varphi_2}, \quad X_6^e = \frac{\partial}{\partial \varphi_1}, \quad X_7^e = \frac{\partial}{\partial \varphi_2}, \\
X_{\psi_0}^e = \psi_{0\xi}\frac{\partial}{\partial \eta} - \psi_{0\eta}\frac{\partial}{\partial \xi}.
\end{gathered}
\tag{11}
$$

Remark 1. *Equivalence transformations (11) can be obtained in two different ways. The first, the classical approach, which treats the entropy S as an arbitrary element of Equations (3), (4), and (7)–(10), derives equivalence transformations in the standard way: using the prolongation formulas [14,20], constructing determining equations, and solving them.*

Because the entropy is only a function of t, ξ, and η (the entropy does not depend on φ_1 and φ_2), another approach is possible. Consider $S(t,\xi,\eta)$, $\varphi_1(t,\xi,\eta)$, and $\varphi_2(t,\xi,\eta)$ as the set of the dependent variables of System (3), (4), and (7)–(10) and find the admitted Lie group of System (3), (4), and (7)–(10). This admitted Lie group allows for the entropy to be changed but leaves the structure of the equations unchangeable. The latter means that the transformations of the found Lie group belong to the equivalence group.

Calculations show that both approaches lead to the same result (11). It should be noted that the second way is much simpler from a practical point of view, as it allows the use of simpler generator prolongation formulas.

3.2. Admitted Lie Algebras

Constructing the determining equations for the system consisting of Equations (3) and (4) with the differential constraints (7)–(10) and solving them, one derives the admitted Lie algebras for nonisentropic and isentropic cases.

In the case of isentropic flows ($S_0 = \text{const}$), the admitted Lie algebra is

$$
\begin{gathered}
X_1 = (\gamma-1)t\frac{\partial}{\partial t} - 2\xi\frac{\partial}{\partial \xi}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = (\gamma-1)\left(\varphi_1\frac{\partial}{\partial \varphi_1} + \varphi_2\frac{\partial}{\partial \varphi_2}\right) + 2\gamma\xi\frac{\partial}{\partial \xi}, \\
X_4 = \varphi_2\frac{\partial}{\partial \varphi_1} - \varphi_1\frac{\partial}{\partial \varphi_2}, \quad X_5 = \frac{\partial}{\partial \varphi_1}, \quad X_6 = \frac{\partial}{\partial \varphi_2}, \\
X_{\psi_0} = \psi_{0\xi}\frac{\partial}{\partial \eta} - \psi_{0\eta}\frac{\partial}{\partial \xi},
\end{gathered}
\tag{12}
$$

where $\psi_0(\xi,\eta)$ is an arbitrary differentiable function of its arguments.

The generators X_2, X_5, and X_6 correspond to the shift transformations along the t, x, and y axes. The generator X_4 corresponds to the rotation transformation, X_1 and X_3 define the inhomogeneous scaling of the space, and the generator X_{ψ_0} corresponds to the relabeling transformation.

In the nonisentropic case ($S_0 \neq \text{const}$), one obtains

$$
\begin{gathered}
X_1 = \varphi_1\tfrac{\partial}{\partial \varphi_1} + \varphi_2\tfrac{\partial}{\partial \varphi_2} + \psi_1\left(S_{0\eta}\tfrac{\partial}{\partial \xi} - S_{0\xi}\tfrac{\partial}{\partial \eta}\right), \\
X_2 = S_{0\eta}^{-1}S_0\tfrac{\partial}{\partial \eta} + \psi_2\left(S_{0\eta}\tfrac{\partial}{\partial \xi} - S_{0\xi}\tfrac{\partial}{\partial \eta}\right), \\
X_3 = \varphi_2\tfrac{\partial}{\partial \varphi_1} - \varphi_1\tfrac{\partial}{\partial \varphi_2}, \quad X_4 = \tfrac{\partial}{\partial \varphi_1}, \quad X_5 = \tfrac{\partial}{\partial \varphi_2}, \quad X_{F_1} = F_1\tfrac{\partial}{\partial t}, \\
X_{F_2} = F_2\left(t\tfrac{\partial}{\partial t} - 2\gamma S_{0\eta}^{-1}S_0\tfrac{\partial}{\partial \eta} + \psi_3\left(S_{0\eta}\tfrac{\partial}{\partial \xi} - S_{0\xi}\tfrac{\partial}{\partial \eta}\right)\right), \\
X_{\psi_0} = \psi_0\left(S_{0\eta}\tfrac{\partial}{\partial \xi} - S_{0\xi}\tfrac{\partial}{\partial \eta}\right),
\end{gathered}
\tag{13}
$$

where $F_i(S_0)$ $(i = 1, 2)$ are arbitrary functions and the functions $\psi_i(\xi, \eta)$, $(i = 0, 1, 2, 3)$ satisfy the equations

$$S_{0\eta}\psi_{0\xi} - S_{0\xi}\psi_{0\eta} = 0, \;\; (\gamma - 1)(S_{0\eta}\psi_{1\xi} - S_{0\xi}\psi_{1\eta}) = 2\gamma,$$

$$(\gamma - 1)(S_{0\eta}\psi_{2\xi} - S_{0\xi}\psi_{2\eta}) = (\gamma - 1)\frac{S_0 S_{0\eta\eta}}{S_{0\eta}^2} - \gamma,$$

$$S_{0\eta}\psi_{3\xi} - S_{0\xi}\psi_{3\eta} = 2\left(\gamma + 1 - \gamma\frac{S_0 S_{0\eta\eta}}{S_{0\eta}^2}\right) + 2\gamma\frac{S_0 F_{2\xi}}{S_{0\xi}F_2}.$$

The generators X_{ψ_0} of (12) and (13) correspond to relabeling transformations. In the recent paper [21], a group foliation [14] of the two-dimensional shallow water equations in Lagrangian coordinates has been constructed with respect to a relabeling generator of the form X_{ψ_0}. As it turned out, the group foliation in this case has some specific properties. To derive more general results, in Section 5, group foliations of the gas dynamics equations for relabeling generators as well as for the generator X_1 of (13) are considered.

4. Conservation Laws of the Stationary Gas Dynamics Equations

As is known [3,16,22], Equations (3) and (4) are the Euler–Lagrange equations for the Lagrangian

$$\mathcal{L} = \frac{\varphi_{1t}^2 + \varphi_{2t}^2}{2} - \frac{1}{\gamma - 1}J^{1-\gamma}S,$$

for which one can find conservation laws by means of the Noether theorem [23,24].

Assume that the generator

$$X = \chi^t\frac{\partial}{\partial t} + \chi^\xi\frac{\partial}{\partial \xi} + \chi^\eta\frac{\partial}{\partial \eta} + \zeta^{\varphi_1}\frac{\partial}{\partial \varphi_1} + \zeta^{\varphi_2}\frac{\partial}{\partial \varphi_2}$$

satisfies the invariance condition

$$X\mathcal{L} + \mathcal{L}(D_t\chi^t + D_\xi\chi^\xi + D_\eta\chi^\eta) = D_t B^t + D_\xi B^\xi + D_\eta B^\eta$$

for some differentiable functions $B^t = B^1$, $B^\xi = B^2$, and $B^\eta = B^3$, where D_t, D_ξ, and D_η are operators of total differentiation with respect to the Lagrangian coordinates.

Then, according to the Noether theorem, the system of the corresponding Euler–Lagrange equations possess a local conservation law of the form

$$D_t T^t + D_\xi T^\xi + D_\eta T^\eta = 0,$$

where the conserved quantities $T^t = T^1$, $T^\xi = T^2$, and $T^\eta = T^3$ are given by the formulas

$$T^i = \chi^i\mathcal{L} + (\zeta^{\varphi_k} - \chi^j\varphi_j^k)\frac{\delta\mathcal{L}}{\delta\varphi_i^k} + \sum_{s=1} D_{i_1}\cdots D_{i_s}(\zeta^{\varphi_k} - \chi^j\varphi_j^k)\frac{\delta\mathcal{L}}{\delta\varphi_{ii_1\cdots i_s}^k} - B^i.$$

Here, $\frac{\delta}{\delta f}$ is the variational derivative with respect to f, and, for the sake of brevity, $\varphi_{i_1\cdots i_s}^k$ denotes the derivative of the function φ_k with respect to x_{i_1}, ..., x_{i_s}, where $x_1 = t$, $x_2 = \xi$, and $x_3 = \eta$.

To find conservation laws, it is necessary to consider a linear combination of all generators of the Lie algebra admitted by the system. To simplify the linear combination, it is convenient to introduce the functions ψ_{23} and ψ_{123}, satisfying the following equations:

$$\psi_3 = -(\gamma - 1)\psi_2 + \psi_{23}, \qquad \psi_1 = 2(2\gamma - 1)(\psi_{23}/(\gamma - 1) - \psi_2) + \psi_{123},$$

$$S_{0\eta}\psi_{23\xi} - \psi_{23\eta}S_{0\xi} = -(\gamma+1)\frac{S_0 S_{0\eta\eta}}{S_{0\eta}^2} + (\gamma+2),$$

$$S_{0\eta}\psi_{123\xi} - \psi_{123\eta}S_{0\xi} = 4\frac{\gamma(2\gamma-1)}{\gamma-1}\frac{S_0 S_{0\eta\eta}}{S_{0\eta}^2} - 2\frac{4\gamma^2+\gamma-2}{\gamma-1}.$$

Using the Noether theorem and taking into account the latter relations, one obtains the following set of conservation laws.

(1) The conservation law of energy:

$$T_6^\xi = -\frac{1}{2}\left(\varphi_{1t}^2 + \varphi_{2t}^2\right) - \frac{J^{1-\gamma}S_0}{\gamma-1}, \qquad T_6^\eta = -J^{-\gamma}S_0(\varphi_{1t}\varphi_{2\eta} - \varphi_{2t}\varphi_{1\eta}),$$

$$T_6^t = -J^{-\gamma}S_0(-\varphi_{1t}\varphi_{2\xi} + \varphi_{2t}\varphi_{1\xi});$$

(2) The angular momentum:

$$T_2^\xi = \varphi_{1t}\varphi_2 - \varphi_{2t}\varphi_1, \;\; T_2^\eta = J^{-\gamma}S_0(\varphi_1\varphi_{1\eta} + \varphi_2\varphi_{2\eta}), \;\; T_2^t = -J^{-\gamma}S_0(\varphi_1\varphi_{1\xi} + \varphi_2\varphi_{2\xi});$$

(3) The momentum along axis x:

$$T_3^\xi = \varphi_{1t}, \qquad T_3^\eta = J^{-\gamma}S_0\varphi_{2\eta}, \qquad T_3^t = -J^{-\gamma}S_0\varphi_{2\xi};$$

(4) The momentum along axis y:

$$T_4^\xi = \varphi_{2t}, \qquad T_4^\eta = -J^{-\gamma}S_0\varphi_{1\eta}, \qquad T_4^t = J^{-\gamma}S_0\varphi_{1\xi};$$

The physical interpretation of the remaining three conservation laws is not that clear, and they are listed below without specifying their names.

(5)

$$\begin{aligned}T_1^\xi = {} & (2\gamma-1)t\left(\varphi_{1t}^2 + \varphi_{2t}^2 + \frac{2}{\gamma-1}J^{1-\gamma}S_0\right) + (\gamma-1)(\varphi_1\varphi_{1t} + \varphi_2\varphi_{2t}) \\ & + (\gamma-1)\psi_{123}\big(\varphi_{1t}(S_{0\xi}\varphi_{1\eta} - S_{0\eta}\varphi_{1\xi}) + \varphi_{2t}(S_{0\xi}\varphi_{2\eta} - S_{0\eta}\varphi_{2\xi})\big) \\ & - 4\gamma(2\gamma-1)\frac{S_0}{S_{0\eta}}(\varphi_{1t}\varphi_{1\eta} + \varphi_{2t}\varphi_{2\eta}),\end{aligned}$$

$$\begin{aligned}T_1^\eta = {} & \frac{\gamma-1}{2}\psi_{123}S_{0\eta}\left(\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma}{\gamma-1}J^{1-\gamma}S_0\right) \\ & + J^{-\gamma}S_0\big(2(2\gamma-1)t(\varphi_{1t}\varphi_{2\eta} - \varphi_{2t}\varphi_{1\eta}) + (\gamma-1)(\varphi_1\varphi_{2\eta} - \varphi_{1\eta}\varphi_2)\big),\end{aligned}$$

$$\begin{aligned}T_1^t = {} & \left(\frac{1-\gamma}{2}\psi_{123}S_{0\xi} + (2\gamma-1)\frac{2\gamma S_0}{S_{0\eta}}\right)(\varphi_{1t}^2 + \varphi_{2t}^2) \\ & + \gamma J^{1-\gamma}\psi_{123}S_{0\xi}S_0 - 2(2\gamma-1)tJ^{-\gamma}S_0(\varphi_{1t}\varphi_{2\xi} - \varphi_{2t}\varphi_{1\xi}) \\ & + \frac{J^{-\gamma}S_0}{\gamma-1}\left((\gamma-1)^2(\varphi_{1\xi}\varphi_2 - \varphi_1\varphi_{2\xi}) - 4\frac{\gamma^2 J S_0}{S_{0\eta}}(2\gamma-1)\right);\end{aligned}$$

(6)

$$\begin{aligned}T_5^\xi = {} & -\frac{1}{2}t(\varphi_{1t}^2 + \varphi_{2t}^2) + \psi_{23}\big(\varphi_{1t}(S_{0\xi}\varphi_{1\eta} - S_{0\eta}\varphi_{1\xi}) + \varphi_{2t}(S_{0\xi}\varphi_{2\eta} - S_{0\eta}\varphi_{2\xi})\big) \\ & + (\gamma+1)\frac{S_0}{S_{0\eta}}(\varphi_{1t}\varphi_{1\eta} + \varphi_{2t}\varphi_{2\eta}) - \frac{J^{1-\gamma}S_0 t}{\gamma-1},\end{aligned}$$

$$T_5^{\eta} = \frac{1}{2}\psi_{23}S_{0\eta}\left(\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma J^{1-\gamma}S_0}{\gamma-1}\right) - J^{-\gamma}S_0 t(\varphi_{1t}\varphi_{2\eta} - \varphi_{2t}\varphi_{1\eta}),$$

$$T_5^t = -\frac{1}{2}\left(\psi_{23}S_{0\xi} + \frac{(\gamma+1)S_0}{S_{0\eta}}\right)\left(\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma J^{1-\gamma}S_0}{\gamma-1}\right) + J^{-\gamma}S_0 t(\varphi_{1t}\varphi_{2\xi} - \varphi_{2t}\varphi_{1\xi});$$

(7)

$$T_7^{\xi} = \psi_0\big(\varphi_{1t}(S_{0\xi}\varphi_{1\eta} - S_{0\eta}\varphi_{1\xi}) + \varphi_{2t}(S_{0\xi}\varphi_{2\eta} - S_{0\eta}\varphi_{2\xi})\big),$$

$$T_7^{\eta} = \psi_0\frac{1}{2}S_{0\eta}\left(\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma J^{1-\gamma}S_0}{\gamma-1}\right),\quad T_7^t = -\frac{1}{2}\psi_0 S_{0\xi}\left(\varphi_{1t}^2 + \varphi_{2t}^2 - \frac{2\gamma J^{1-\gamma}S_0}{\gamma-1}\right).$$

It is important to note that, although the same Lagrangian and equations are considered here as in [16], the set of conservation laws is different, as the admitted Lie algebra for stationary gas equations does not coincide with the Lie algebra studied in [16].

5. Group Foliations for the Two-Dimensional Gas Dynamics Equations in Lagrangian Coordinates

Here, group foliations for two-dimensional gas dynamics equations are considered. Group foliations [14] are carried out with respect to some generator X of an admitted Lie algebra and allow one to split the original system of equations into two: automorphic and resolving systems. An automorphic system is a set of dependencies between differential invariants of an admitted Lie algebra, and a resolving system is obtained as a set of conditions for the compatibility of the automorphic system and the original system of equations. The automorphic system always admits the generator X, and any of its solutions is translated into any of its other solutions by the action of the corresponding group transformation. Therefore, for X, it is advisable to choose a generator of the most general form possible (for example, including arbitrary functions). The set of Lie algebra generators admitted by the resolving system is isomorphic to the Lie algebra of the original system, with the exception of the generator X. This allows one to reduce the resolving system on subgroups and find invariant solutions to the resolving system, which can sometimes be easier than solving the original system of equations. If it is possible to find solutions to the resolving system, then with the help of these solutions, one can specify the form of the automorphic system. If one also manages to find at least a particular solution of such an automorphic system, then this is equivalent to finding the entire set of its solutions, since they are all connected by the action of the transformation group corresponding to the generator X. Moreover, the construction of invariant solutions using the group foliation approach can provide wider classes of invariant solutions than the standard approach, as has been shown by example in [25].

The construction of automorphic and resolving systems is often quite labor-intensive work (see, e.g., [26]). Although this is primarily a step towards obtaining invariant solutions of the original system, in some cases, the construction of group foliations is of independent theoretical interest. For example, in recent work [21], a group foliation has been constructed for the two-dimensional shallow water equations in Lagrangian coordinates. As a result, it turned out that the resolving system is isomorphic to the system of the shallow water equations in Eulerian coordinates.

Next, we study group foliations of gas dynamics equations in some special cases. To begin with, we focus on group foliations with respect to relabeling generators. As was already mentioned, a similar problem was considered recently for two-dimensional shallow water equations [21]. As one would expect, for nonstationary equations of isentropic flows, similar results are derived, generalizing those presented in [21]. On the contrary, for non-isentropic flows, the given entropy function S_0 appears among the differential invariants of the relabeling generator, and the group foliation structure is somewhat different.

Then, to illustrate the application of the approach, we dwell on the construction of a group foliation for one of the generators admitted by the stationary equations in the nonisentropic case. As an example, we consider the generator X_1 of Algebra (13). This generator defines uniform stretches in the x–y plane combined with relabeling. Here, the construction of the foliation allows one to reduce the consideration of a fairly wide class of corresponding solutions to the consideration of an automorphic system. Recall that, knowing any of its particular solutions, the entire set of its solutions can be derived by simple uniform scalings of the dependent variables in combination with relabeling transformations. It is also important that the 'finite' part of the generator X_1 does not include derivatives of the entropy function S_0, which simplifies the finding of its differential invariants and subsequent calculations. Even taking these simplifications into account, as one sees in the sequel, the corresponding resolving system is quite difficult to analyze.

In the process of constructing foliations, the main steps of the algorithm are briefly described. However, as a detailed discussion of the underlying theory is beyond the scope of this treatment, interested readers are referred to [14,25,27,28].

5.1. The Relabeling Symmetry X_{ψ_0}

We start with group foliations for Equations (3) and (4) with respect to the relabeling generator

$$X_{\psi_0} = \psi_0\left(S_{0\eta}\frac{\partial}{\partial\xi} - S_{0\xi}\frac{\partial}{\partial\eta}\right),$$

where the function $\psi(\xi,\eta)$ satisfies the relation

$$S_{0\eta}\psi_{0\xi} - S_{0\xi}\psi_{0\eta} = 0.$$

5.1.1. Isentropic Flows, Nonstationary Case

First, assume $S_0 =$ const. To find a universal invariant, one consequently applies the generator and its first prolongation to some function F of 11 variables

$$(t,\xi,\eta,\varphi_1,\varphi_2,\varphi_{1t},\varphi_{1\xi},\varphi_{1\eta},\varphi_{2t},\varphi_{2\xi},\varphi_{2\eta})$$

and splits the result with respect to ψ_0 and its derivatives. The resulting system of equations is

$$\begin{gathered}
F_\xi = 0, \qquad F_\eta = 0, \\
\varphi_{1\eta}F_{\varphi_{1\xi}} + \varphi_{2\eta}F_{\varphi_{2\xi}} = 0, \qquad \varphi_{1\xi}F_{\varphi_{1\eta}} + \varphi_{2\xi}F_{\varphi_{2\eta}} = 0, \\
\varphi_{1\xi}F_{\varphi_{1\xi}} + \varphi_{2\xi}F_{\varphi_{2\xi}} - \varphi_{1\eta}F_{\varphi_{1\eta}} - \varphi_{2\eta}F_{\varphi_{2\eta}} = 0.
\end{gathered}$$

From the latter system, one can derive the universal invariant

$$F(t,x,y,x_t,y_t,J),$$

which depends on six invariant arguments.

To construct an automorphic system, one must define the relationships between three (according to the number of independent variables) zero-order invariants and the remaining first-order differential invariants, as follows.

$$J^{-1} = W(t,x,y), \qquad x_t = U(t,x,y), \qquad y_t = V(t,x,y).$$

The resolving system is derived as compatibility conditions of the automorphic system and Equations (3) and (4) and has the following form:

$$U_t + UU_x + VU_y + \gamma S_0 W^{\gamma-2}W_x = 0,$$

$$V_t + UV_x + VV_y + \gamma S_0 W^{\gamma-2}W_y = 0,$$

$$W_t + (UW)_x + (VW)_y = 0.$$

This generalizes the result obtained in [21].

5.1.2. Nonisentropic Flows ($S_0 \neq$ const), Stationary Case

Here, we consider the nonisentropic case along with the constraints (7)–(10).
In the case that $S_0 \neq$ const, the universal invariant is found from the system

$$\begin{gathered}
S_{0\xi}F_\eta - S_{0\eta}F_\xi = 0,\\
(S_{0\eta}\varphi_{1\xi} - S_{0\xi}\varphi_{1\eta})S_{0\xi}F_{\varphi_{1\xi}} + (S_{0\eta}\varphi_{1\xi} - S_{0\xi}\varphi_{1\eta})S_{0\eta}F_{\varphi_{1\eta}}\\
+(F_{\varphi_{2\xi}}S_{0\xi} + F_{\varphi_{2\eta}}S_{0\eta})(S_{0\eta}\varphi_{2\xi} - S_{0\xi}\varphi_{2\eta}) = 0,\\
(S_{0\eta\eta}\varphi_{2\xi} - S_{0\xi\eta}\varphi_{2\eta})F_{\varphi_{2\eta}} + (S_{0\eta\eta}\varphi_{1\xi} - S_{0\xi\eta}\varphi_{1\eta})F_{\varphi_{1\eta}} + (S_{0\xi\eta}\varphi_{2\xi} - S_{0\xi\xi}\varphi_{2\eta})F_{\varphi_{2\xi}}\\
+(S_{0\xi\eta}\varphi_{1\xi} - S_{0\xi\xi}\varphi_{1\eta})F_{\varphi_{1\xi}} = 0.
\end{gathered}$$

There are nine independent differential invariants, and the universal invariant depending on them is of the form

$$F(t, S_0, x, y, x_t, y_t, J, J_1, J_2),$$

where

$$J_1 = J^{-1}(S_{0\xi}\varphi_{2\eta} - S_{0\eta}\varphi_{2\xi}) = S_{0x}, \qquad J_2 = J^{-1}(S_{0\eta}\varphi_{1\xi} - S_{0\xi}\varphi_{1\eta}) = S_{0y}.$$

Thus, the automorphic system is

$$\begin{gathered}
J^{-1} = W(t,x,y), \qquad x_t = U(t,x,y), \qquad y_t = V(t,x,y),\\
S_0 = Z(t,x,y), \qquad J_1 = A(t,x,y), \qquad J_2 = B(t,x,y).
\end{gathered}$$

The resolving system is derived in the similar way as in the previous case:

$$\begin{gathered}
U_t = 0, \qquad V_t = 0, \qquad Z_x - A = 0, \qquad A_y = B_x,\\
Z_t + UA + BV = 0, \qquad W_t + (UW)_x + (VW)_y = 0,\\
U(A_t + UA_x + VA_y) + V(B_t + UB_x + VB_y) + A(UU_x + VU_y) + B(UV_x + VV_y) = 0.
\end{gathered}$$

5.2. Symmetry X_1, Nonisentropic Case

Consider a group foliation of Equations (3), (4), and (7)–(10) with respect to the generator

$$X_1 = \varphi_1\frac{\partial}{\partial\varphi_1} + \varphi_2\frac{\partial}{\partial\varphi_2} + \psi_1\left(S_{0\eta}\frac{\partial}{\partial\xi} - S_{0\xi}\frac{\partial}{\partial\eta}\right),$$

where

$$(\gamma - 1)(S_{0\eta}\psi_{1\xi} - S_{0\xi}\psi_{1\eta}) = 2\gamma.$$

One can apply the generator and its first prolongation to a function of 11 arguments and split the resulting equations with respect to ψ_1 and its derivatives. This leads to the system for the universal invariant F.

$$\varphi_1F_{\varphi_1} + \varphi_2F_{\varphi_2} = 0, \qquad S_{0\eta}F_\xi - S_{0\xi}F_\eta = 0,$$

$$\begin{aligned}
(2S_{0\xi}\gamma\varphi_{1\eta} - \varphi_{1\xi}(\gamma+1)S_{0\eta})F_{\varphi_{1\xi}} + (2\gamma\varphi_{2\eta}S_{0\xi} - \varphi_{2\xi}(\gamma+1)S_{0\eta})F_{\varphi_{2\xi}} +\\
+ S_{0\eta}(\varphi_{1t}F_{\varphi_{1t}} + \varphi_{2t}F_{\varphi_{2t}} + \varphi_{1\eta}F_{\varphi_{1\eta}} + \varphi_{2\eta}F_{\varphi_{2\eta}})(\gamma - 1) = 0,
\end{aligned}$$

$$
\begin{aligned}
S_{0\xi}(S_{0\xi}\varphi_{1\eta} - S_{0\eta}\varphi_{1\xi})F_{\varphi_{1\xi}} + S_{0\eta}(S_{0\xi}\varphi_{1\eta} - S_{0\eta}\varphi_{1\xi})F_{\varphi_{1\eta}}& \\
- (S_{0\xi}F_{\varphi_{2\xi}} + S_{0\eta}F_{\varphi_{2\eta}})(S_{0\eta}\varphi_{2\xi} - S_{0\xi}\varphi_{2\eta}) &= 0,
\end{aligned}
$$

$$
\begin{aligned}
(S_{0\xi\xi}\varphi_{1\eta} - S_{0\xi\eta}\varphi_{1\xi})F_{\varphi_{1\xi}} + (S_{0\xi\eta}\varphi_{1\eta} - S_{0\eta\eta}\varphi_{1\xi})F_{\varphi_{1\eta}} + (S_{0\xi\xi}\varphi_{2\eta} - S_{0\xi\eta}\varphi_{2\xi})F_{\varphi_{2\xi}} +& \\
+ (S_{0\xi\eta}\varphi_{2\eta} - S_{0\eta\eta}\varphi_{2\xi})F_{\varphi_{2\eta}} &= 0.
\end{aligned}
$$

One thus obtains the solution

$$
F = F\left(t, S_0, \frac{\varphi_2}{\varphi_1}, \frac{\varphi_{1t}}{\varphi_1}, \frac{\varphi_{2t}}{\varphi_1}, J\varphi_1^{\frac{2}{\gamma-1}}, J_1, J_2\right),
$$

where

$$
J_1 = J^{-1}(S_{0\xi}\varphi_{2\eta} - S_{0\eta}\varphi_{2\xi})\varphi_1 = \varphi_1 S_{0x}, \qquad J_2 = J^{-1}(S_{0\eta}\varphi_{1\xi} - S_{0\xi}\varphi_{1\eta})\varphi_2 = \varphi_2 S_{0y}.
$$

Thus, the automorphic system is

$$
\begin{gathered}
J\varphi_1^{\frac{2}{\gamma-1}} - W(t, S_0, q) = 0, \\
\varphi_{1t} - \varphi_1 U(t, S_0, q) = 0, \qquad \varphi_{2t} - \varphi_1 V(t, S_0, q) = 0, \\
J_1 = A(t, S_0, q), \qquad J_2 = B(t, S_0, q),
\end{gathered}
$$

where $q = \varphi_2/\varphi_1$.

In contrast to the previous cases, the resolving system turns out to be quite cumbersome and consists of 10 equations:

$$
\begin{gathered}
qUA + BV = 0, \\
A_t = 0, \qquad B_t = 0, \qquad V_t = 0, \qquad W_t = 0, \\
WB - \gamma S_0(BW_{S_0} + qW_q) - q((qU - V)V_q - UV)W^\gamma = 0, \\
(qU - V)WA_q + (A + B)WV_q + A((qU - V)W_q + 2UW) + \frac{2AUW}{\gamma - 1} = 0, \\
(U_t + (V - qU)U_q + U^2)W^\gamma + \gamma S_0(qW_q - AW_{S_0}) + (A + \kappa_1 S_0)W = 0, \\
A\Big(q(qU - V)(WU_q - UW_q) - W\Big(qU_t + UV + \kappa_1 qU^2\Big)\Big) + (qU - V)VWB_q = 0,
\end{gathered}
$$

$$
\begin{gathered}
q\{(qU - V)(VB_q + q(AU)_q + AU(1 - B_{S_0})) + U(A + B)((V_q + A_{S_0})qU + qAU_{S_0} + BV_{S_0})\}AW_q \\
- \Big\{q((qU - V)VB_q^2 - (A + B)BV_q^2) - qU\big((A + B)(qAU_{S_0} + BV_{S_0}) + \kappa_2 qAU\big)A_q \\
+ \Big(q((2qU - V)A + qBU)V_q + q^2(qU - V)UA_q \\
- \Big((qU - V)(qUA_{S_0} + VB_{S_0} - q^2U_q) - qU(qU + \kappa_3 V)\Big)\Big)AB_q \\
- \Big((A + B)(q^2(AU)_q + qABU_{S_0} + B^2V_{S_0}) + (A - \kappa_3 B)qUA\Big)V_q \\
+ A\big[q\big(Uq(A + B)A_{S_0} - A((qU - V)B_{S_0} + \kappa_2 qU)\big)U_q \\
+ qU(A - \kappa_1 B)UA_{S_0} + (\kappa_1 qU + V)AUB_{S_0} - \kappa_2(qBU_{S_0}A + B^2V_{S_0} + qAU)U\big]\Big\}W = 0,
\end{gathered}
$$

where, for brevity, we have denoted

$$
\kappa_1 = \frac{\gamma + 1}{\gamma - 1}, \qquad \kappa_2 = \kappa_1 + 1 = \frac{2\gamma}{\gamma - 1}, \qquad \kappa_3 = -\kappa_1 - 2 = \frac{1 - 3\gamma}{\gamma - 1}.
$$

Here, one assumes that $\varphi_1\varphi_2 W \not\equiv 0$.

Our analysis shows that these 10 equations are functionally independent. Notice also that the first equation of the resolving system (namely, $qUA + BV = 0$) corresponds to the condition $D_t^e(S) = 0$, where D_t^e is the total derivative with respect to time in Eulerian coordinates.

6. Conclusions

Symmetries, conservation laws, and group foliations of the two-dimensional stationary gas dynamics equations in mass Lagrangian coordinates were considered in this paper. The system of equations in Lagrangian coordinates was supplemented with conditions (in Eulerian coordinates) for the independence of density, velocity, and entropy from time. As a result of the group classification of the equations, two cases were distinguished: the isentropic and nonisentropic cases. The classification showed that the admitted Lie algebras for the stationary case differ from the Lie algebras obtained recently in [16] for nonstationary equations. Due to this, using the Noether theorem, one can obtain new conservation laws specific to the stationary equations.

The last section was devoted to the group foliations of the gas dynamics equations with respect to relabeling the generators X_{ψ_0} and the generator X_1, which correspond to a combination of relabeling transformations and uniform stretching in the x–y plane. The resulting foliations generalize and complement previously known results for the two-dimensional shallow water equations in Lagrangian coordinates [21]. Recall that group foliation allows one to move from the original system of equations to two systems: automorphic and resolving systems. An automorphic system relates differential invariants, whereas a resolving system is obtained as a set of compatibility conditions for the original and automorphic systems. Often, the automorphic and resolving systems have a simpler form than the original system, which is suitable further analysis.

Calculations showed, similarly to [21], that for isentropic flows, the resolving system for nonstationary gas dynamics equations in mass Lagrangian coordinates with respect to the relabeling generator are isomorphic to the system of gas dynamics equations in Eulerian coordinates. For nonisentropic flows, such isomorphism no longer takes place, and the resolving system has a more complex form. The resolving system for the generator X_1 turned out to be of an even more complex structure. A compatibility analysis of the derived resolving systems and the search for invariant solutions based on the certain group foliations are beyond the scope of the present study and may become a subject for future research.

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