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Gradient Ricci Solitons on Spacelike Hypersurfaces of Lorentzian Manifolds Admitting a Closed Conformal Timelike Vector Field

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Abstract: In this article, we investigate Ricci solitons occurring on spacelike hypersurfaces of Einstein Lorentzian manifolds. We give the necessary and sufficient conditions for a spacelike hypersurface of a Lorentzian manifold, equipped with a closed conformal timelike vector field $\bar{\xi}$, to be a gradient Ricci soliton having its potential function as the inner product of $\bar{\xi}$ and the timelike unit normal vector field to the hypersurface. Moreover, when the ambient manifold is Einstein and the hypersurface is compact, we establish that, under certain straightforward conditions, the hypersurface is an extrinsic sphere, that is, a totally umbilical hypersurface with a non-zero constant mean curvature. In particular, if the ambient Lorentzian manifold has a constant sectional curvature, we show that the compact spacelike hypersurface is essentially a round sphere.

Keywords: gradient Ricci soliton; Einstein manifold; conformal vector field; spacelike hypersurfaces with constant mean curvature

MSC: 53A10; 53C40; 53C42; 53C50; 53C65

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1. Introduction

It is a well-established fact that Ricci solitons are closely linked with Ricci flows, as outlined in [1]. Essentially, a pseudo-Riemannian metric g defined on M provides a Ricci soliton on a smooth manifold M if and only if there exists a positive function $\sigma(t)$ and a one-parameter family $\psi(t)$ of diffeomorphisms of M such that the one-parameter family of metrics $g(t) = \sigma(t)\psi(t)^* g$ satisfies the Ricci flow equation:

$$\frac{\partial}{\partial t} g(t) = -2Ric_g(t),$$

with the initial condition $g = g(0)$. Here, $\psi(t)^*$ denotes the pullback along the diffeomorphism $\psi(t)$, and $Ric_g(t)$ represents the Ricci curvature of $g(t)$.

A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if there exists a nonzero smooth vector field X and a constant λ satisfying

$$\frac{1}{2} L_X g + Ric = \lambda g, \tag{1}$$

where L_X is the Lie derivative with respect to X and Ric is the Ricci tensor with g. We denote a Ricci soliton by (M, g, X, λ). The concept of a Ricci soliton was first introduced by Hamilton [2,3].

Ricci solitons are a type of manifold in differential geometry that generalize the concept of Einstein metrics, that is, $Ric = cg$ for some constant c. The Ricci soliton is classified as shrinking, steady, or expanding based on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. The vector field X is referred to as the potential field of (M, g, X, λ). If the potential field X is the gradient of some smooth function f on M, that is $X = \nabla f$, then (M, g, f, λ) will denote the gradient Ricci soliton $(M, g, \nabla f, \lambda)$. In this case, Equation (1) takes the form

$$Ric + Hess(f) = \lambda g, \tag{2}$$

where $Hess(f)$ is the Hessian of the function f. The function f is called a potential function of the Ricci soliton (M, g, f, λ).

Additionally, if $L_X g = 0$, the Ricci soliton is considered trivial, and from Equation (1), M is an Einstein manifold.

One of the significant areas of focus in differential geometry and mathematical physics is the theory of submanifolds, which presents challenging topics related to submanifold geometry. In many research endeavors, the Gauss, Codazzi, and Ricci Equations for submanifolds play a crucial role as they can be formulated in a manageable manner. The exploration of Ricci solitons on hypersurfaces has gained traction, particularly in understanding the conditions under which hypersurfaces within Riemannian manifolds can exhibit Ricci soliton structures. While Ricci solitons on hypersurfaces in Riemannian manifolds have been extensively investigated, there is a relative scarcity of studies focusing on Ricci solitons in a Lorentzian manifold ambient space, despite their significance in terms of geometry and applications in theoretical physics. These circumstances have motivated our investigation into Ricci solitons on Riemannian hypersurfaces within Lorentzian manifolds.

The fascination with the geometry of Ricci solitons stems from its diverse applications in various disciplines, particularly in the context of hypersurfaces in Riemannian manifolds, as exemplified in [4–19]. This paper directs its attention to spacelike hypersurfaces in Lorentzian manifolds, which, to the best of our knowledge, represent an underexplored area in the existing literature. More specifically, our investigation centers on the analysis of gradient Ricci solitons on spacelike hypersurfaces of Lorentzian manifolds. These hypersurfaces are characterized by the presence of a closed conformal vector field of the ambient manifold, with the potential function denoted as θ, that is, the inner product between the closed conformal vector field and the timelike unit normal vector field to the hypersurface

This paper is organized as follows: the second section revisits essential concepts and formulas concerning spacelike hypersurfaces in Lorentzian manifolds. Section 3 presents the main results, focusing on characterizing conditions under which a spacelike hypersurface in a Lorentzian manifold, endowed with a closed conformal vector field, displays a gradient Ricci soliton structure with θ as the potential function.

The examination then focuses on compact gradient Ricci solitons, particularly when the ambient manifold is Einstein. We provide sufficient conditions to characterize spacelike hypersurfaces as extrinsic spheres, that is, totally umbilical hypersurfaces with a nonzero constant mean curvature. In the special case where the ambient manifold has a constant sectional curvature, it is deduced that the hypersurface is a round sphere. In the future, we look forward to generalizing this research in the case where the ambient manifold is a generalized Robertson Walker (GRW) spacetime.

2. Preliminaries

Let (M, g) be a hypersurface in an orientable Lorentzian manifold $(\overline{M}, \overline{g})$ of dimension (n + 1). Denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\overline{M}$, respectively. Two fundamental equations apply to all vector fields X and Y that are tangential to M.

$$\overline{\nabla}_X Y = \nabla_X Y - h(X, Y), \tag{3}$$

$$\overline{\nabla}_X N = -A(X). \tag{4}$$

Formula (3) is called Gauss' formula and Formula (4) is called Weingarten' formula, where h is the second fundamental form, and A is the shape operator of M derived from a normal vector field N to $\overline{M}$.

There is a relationship between the second fundamental form h and the shape operator A of M.

$$\overline{g}(A(X), Y) = \overline{g}(h(X, Y), N). \tag{5}$$

The Codazzi equation describes the normal part of the curvature $\overline{R}(X, Y)Z$ as follows:

$$(\overline{R}(X, Y)Z)^{\perp} = (\nabla_Y h)(X, Z) - (\nabla_X h)(Y, Z), \tag{6}$$

where X, Y, and Z are tangent to M, while N is normal to M, and $\overline{R}$ is the curvature tensor of $\overline{M}$, defined as follows:

$$\overline{R}(X, Y)Z = \overline{\nabla}_{[X,Y]}Z - [\overline{\nabla}_X, \overline{\nabla}_Y]Z.$$

The covariant derivative of h is denoted as ∇h, and it is defined as follows:

$$(\nabla_X h)(Y, Z) = \overline{\nabla}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

The Gauss–Codazzi equation is a mathematical formula that is widely known and used.

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \overline{g}(h(X, Z), h(Y, W)) - \overline{g}(h(X, W), h(Y, Z)), \tag{7}$$

for all X, Y, Z and W tangent to M, where R and $\overline{R}$ are the curvature tensors of M and $\overline{M}$, respectively.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of a pseudo-Riemannian manifold (M, g). Then, the Ricci curvature tensor on M is a symmetric tensor given by

$$Ric(X, Y) = \sum_{i=1}^{n} \epsilon_i R(X, e_i, Y, e_i),$$

where X and Y are tangent to M, and the scalar curvature S of M is defined by

$$S = \sum_{i=1}^{n} \epsilon_i Ric(e_i, e_i).$$

The divergence of the vector field X of M is defined by

$$div(X) = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i), \tag{8}$$

where $\epsilon_i = g(e_i, e_i)$. The trace of the curvature tensor is the Ricci curvature, and the trace of the Ricci is the scalar curvature.

The mean curvature H of a spacelike hypersurface M in a Lorentzian manifold $(\overline{M}, \overline{g})$ is define by

$$H = -\frac{1}{n} tr(A),$$

where $tr(A)$ is the trace of the shape operator A of M derived from a normal vector field N to $\overline{M}$.

Equation (7) results in a relationship between the Ricci curvatures Ric and $\overline{Ric}$ of M and $\overline{M}$, respectively. Furthermore, it can be expressed as follows:

$$Ric(X, Y) = \overline{Ric}(X, Y) + \overline{g}(\overline{R}(N, X)Y, N) + g(A(X), nHY + A(Y)). \tag{9}$$

The Hessian $Hess(f)$ of a smooth function f on a pseudo-Riemannian manifold (M, g) is a symmetric tensor defined by

$$Hess(f)(X, Y) = g(S_f(X), Y),$$

where S_f is the Hessian operator defined by $S_f = \nabla_X \nabla f$ and ∇f is the gradient of the function f.

A point p of a pseudo-Riemannian hypersurface M of $\overline{M}$ is called an umbilical point if the shape operator A at p, $A_p = \Phi I$, where Φ is scalar. M is called totally umbilical if every point of M is umbilical. In particular, M is called totally geodesic if $A = 0$.

A hypersurface M of a pseudo-Riemannian $(\overline{M}, \overline{g})$ is called an extrinsic sphere if it is a totally umbilical sphere with a non-zero constant mean curvature.

3. Ricci Solitons on Spacelike Hypersurfaces in Einstein Lorentzian Manifolds

Let (M, g) be an orientable spacelike hypersurface of a Lorentz manifold $(\overline{M}, \overline{g})$ of dimension $(n+1)$, and let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$ which means

$$\overline{\nabla}_X \bar{\xi} = \psi X,$$

for all $X \in \mathfrak{X}(\overline{M})$ ($\mathfrak{X}(\overline{M})$ is the set of all vector fields on $\overline{M}$) and ψ is called the conformal function, a smooth function on $\overline{M}$. The restriction of $\bar{\xi}$ to M is denoted by ξ. Let N be a unit timeline normal vector field on M, which can be chosen so that $\theta = \overline{g}(\xi, N) < 0$. Then, we can write

$$\xi = \xi^T - \theta N, \tag{10}$$

where ξ^T is the tangential component of ξ. By using Gauss and Weingarten formulas, it yields

$$\nabla_X \xi^T = \psi X - \theta A(X), \tag{11}$$

and

$$A(\xi^T) = -\nabla \theta. \tag{12}$$

From (8), it is straightforward to derive

$$div \xi^T = n(\psi + \theta H). \tag{13}$$

Let Q and $\overline{Q}$ be Ricci operators on M and $\overline{M}$, respectively, where Q and $\overline{Q}$ satisfy $Ric(X, Y) = g(QX, Y)$ and $\overline{Ric}(X, Y) = \overline{g}(\overline{Q}X, Y)$.

Some of the notation are reviewed, which are needed in our results. It is easy to see that $\overline{R}(N, X)N$ is tangent to M for all $X \in \mathfrak{X}(M)$, and, thus, we can define the normal Jacobi operator $R_N : TM \to TM$ by

$$R_N(X) = \overline{R}(N, X)N.$$

Define the operator $(\nabla A)\xi^T$ on M by

$$(\nabla A)\xi^T(X) = \nabla A(X, \xi^T) = (\nabla_X A)(\xi^T).$$

The following lemma is crucial for proving the main results.

Lemma 1.
$$tr((\nabla A)\xi^T) = -\overline{Ric}(\xi^T, N) - n\xi^T(H).$$

Proof. Let $\{e_1, \ldots, e_n\}$ denote a local orthonormal frame on M that can be taken as parallel. By using the Codazzi Equation (6), it yields

$$\begin{aligned} tr((\nabla A)\xi^T) &= \sum_{i=1}^{n} g((\nabla_{e_i} A)(\xi^T), e_i) \\ &= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\xi^T, e_i)e_i, N) - \sum_{i=1}^{n} g((\nabla_{\xi^T} A)(e_i), e_i) \\ &= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\xi^T, e_i)e_i, N) - \sum_{i=1}^{n} g(\nabla_{\xi^T}(A(e_i)), e_i) + \sum_{i=1}^{n} g(A(\nabla_{\xi^T} e_i), e_i) \\ &= -\sum_{i=1}^{n} \overline{g}(\overline{R}(\xi^T, e_i)e_i, N) + \overline{g}(\overline{R}(N,N)\xi^T, N) - \xi^T \sum_{i=1}^{n} g(A(e_i), e_i) \\ &= -\overline{Ric}(\xi^T, N) - n\xi^T(H). \end{aligned}$$

□

Our first result presents the conditions that a spacelike hypersurface must satisfy to be identified as a gradient Ricci soliton of the particular type (M, g, θ, λ).

Theorem 1. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Lorentzian manifold endowed with a timelike closed conformal vector field $\bar{\xi}$. Let (M, g) be a spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. (M, g, θ, λ) is a Ricci soliton if and only if the following equation is satisfied:*

$$\overline{Q} + R_N - (\nabla A)(\xi^T) + (nH - \psi)A + (1+\theta)A^2 = \lambda I. \tag{14}$$

Proof. By using Equations (3) and (4), it follows that

$$\begin{aligned} Hess(\theta)(X, Y) &= g(\nabla_X \nabla\theta, Y) \\ &= -g(\nabla_X(A(\xi^T)), Y) \\ &= -g((\nabla_X A)(\xi^T), Y) - g(A(\nabla_X \xi^T), Y) \\ &= -g((\nabla_X A)(\xi^T), Y) - g(\psi A(X) - \theta A^2(X), Y). \end{aligned}$$

From this last expression, we have

$$S_\theta = -(\nabla A)\xi^T - \psi A + \theta A^2.$$

By using Equation (2), it yields

$$Q = (\nabla A)\xi^T + \psi A - \theta A^2 + \lambda I. \tag{15}$$

By substituting (15) into (9), we obtain (14). □

The next result outlines a practical condition applicable to a spacelike hypersurface, establishing its characterization as a gradient Ricci soliton of the type (M, g, θ, λ).

Theorem 2. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Lorentzian manifold with a timelike closed conformal vector field $\bar{\xi}$ on $\overline{M}$. Let (M, g) be a spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, then*

$$\bar{S} + 2\overline{Ric}(N, N) + (\psi - nH)nH + (1+\theta)|A|^2 + \overline{Ric}(\xi^T, N) + n\xi^T(H) = n\lambda. \tag{16}$$

Proof. Formula (16) is obtained just by tracing Equation (14) and using Lemma 1. □

In the case of an Einstein ambient manifold, Theorem 2 yields the following implication.

Theorem 3. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Einstein Lorentzian manifold with $\overline{Ric} = n\bar{c}\overline{g}$, where $\bar{c}$ is a constant. Let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$. Let (M, g) be a spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, then*

$$(1+\theta)(|A|^2 - nH^2) + n(n-1)(\bar{c} - H^2) + ndiv(H\xi^T) = n\lambda. \tag{17}$$

Proof. Using (13) and (16), it follows that

$$\begin{aligned} \bar{S} + 2\overline{Ric}(N,N) + (1+\theta)(|A|^2 - nH^2) - n(n-1)H^2 + nHdiv(\xi^T) \\ + \overline{Ric}(\xi^T, N) + n\xi^T(H) = n\lambda. \end{aligned} \tag{18}$$

Since $\overline{M}$ is Einstein, then $\overline{Ric}(N,N) = -n\bar{c}$, $\overline{Ric}(\xi^T, N) = 0$ and $\bar{S} = n(n+1)\bar{c}$. It follows that Equation (18) becomes

$$(1+\theta)(|A|^2 - nH^2) + n(n-1)(\bar{c} - H^2) + nHdiv(\xi^T) + n\xi^T(H) = n\lambda. \tag{19}$$

Using $div(H\xi^T) = Hdiv(\xi^T) + \xi^T(H)$ yields (17). □

A simple consequence of the last theorem is the following result.

Theorem 4. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Einstein Lorentzian manifold with $\overline{Ric} = n\bar{c}\overline{g}$, where $\bar{c}$ is a constant, and let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$. Let (M, g) be a compact spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, then*

$$\int_M (1+\theta)(|A|^2 - nH^2)dV = n\int_M (\lambda - (n-1)(\bar{c} - H^2))dV. \tag{20}$$

There are interesting results after imposing certain assumptions on the function θ.

Theorem 5. *Consider the manifolds (M, g) and $(\overline{M}, \overline{g})$ as defined in Theorem 4, with the additional assumption that $\theta < -1$ (resp. $-1 < \theta < 0$) everywhere. If (M, g, θ, λ) is a non-trivial Ricci soliton, then $\lambda \le (n-1)(\bar{c} - H^2)$ (resp. $\lambda \ge (n-1)(\bar{c} - H^2)$), with equality holds if and only if M is an extrinsic sphere. In particular, if $(\overline{M}, \overline{g})$ has a constant sectional curvature, then M is necessarily a sphere with a constant sectional curvature $c = \bar{c} - H^2 > 0$. In this case, the Ricci soliton is shrinking.*

Proof. Applying Schwartz's inequality leads to the conclusion that $\lambda \le (n-1)(\bar{c} - H^2)$. It is a well-established fact that when equality is achieved, it indicates that M is totally umbilical. In [20], Lemma 35 on page 116 implies that M has a constant sectional curvature $c = \bar{c} - H^2$. As M is compact, it must be that M is a sphere with a constant positive curvature $c = \bar{c} - H^2$. This implies that $\lambda > 0$. Consequently, the Ricci soliton is shrinking. □

Remark 1. *In Theorem 5, assuming that $\overline{M}$ is a space form implies that it is isometric to the de Sitter Space $S_1^{n+1}(\bar{c})$, where $\bar{c} > 0$.*

The consequences derived from Equation (20) in Theorem 4 also lead to the following result.

Theorem 6. *Let $(\overline{M}, \overline{g})$ be an (n+1)-dimensional Einstein Lorentzian manifold with $\overline{Ric} = n\bar{c}\overline{g}$, where $\bar{c}$ is a constant, and let $\bar{\xi}$ be a timelike closed conformal vector field on $\overline{M}$. Let (M, g) be a compact spacelike hypersurface of $(\overline{M}, \overline{g})$, and let ξ, ξ^T, and θ be the same as above. If (M, g, θ, λ) is a Ricci soliton, such that either $\theta < -1$ and $\lambda \ge (n-1)(\bar{c} - H^2)$, or $-1 < \theta < 0$ and $\lambda \le (n-1)(\bar{c} - H^2)$, then M is totally umbilical, H is a constant, and M is an extrinsic sphere.*

Proof. Clearly, Equation (20) implies $|A|^2 - nH^2 = 0$ if $\theta < -1$ and $\lambda \ge (n-1)(\bar{c} - H^2)$ or $-1 < \theta < 0$ and $\lambda \le (n-1)(\bar{c} - H^2)$. It is concluded that M is totally umbilical with a constant mean curvature H, since $\lambda = (n-1)(\bar{c} - H^2)$. It must be that $H \neq 0$ because otherwise, M will be totally geodesic, a contradiction of the compactness of M. $\square$

4. Conclusions

Gradient Ricci solitons have been extensively studied in Riemannian manifolds, as discussed in the introduction. However, this concept has received limited attention in the Lorentzian context, with only a few papers published on the topic. In this paper, we investigate gradient Ricci solitons on spacelike hypersurfaces of Lorentzian manifolds, marking the first attempt to do so. We believe that our research offers several advantages and potential impacts compared to the existing literature, thereby advancing knowledge in Lorentzian geometry. By studying gradient Ricci solitons in Lorentzian manifolds, particularly on spacelike hypersurfaces, we aim to gain a deeper understanding of their properties and behavior. This understanding could have significant implications across various branches of physics, particularly in general relativity, where Lorentzian manifolds are fundamental. Our research also aims to contribute to the development of a more comprehensive theoretical framework applicable to diverse mathematical and physical fields. Additionally, we hope that our findings will inspire further research and potentially lead to practical applications in fields such as cosmology, gravitational physics, and geometric analysis. In summary, we think that our work represents a significant step forward in understanding gradient Ricci solitons on spacelike hypersurfaces of Lorentzian manifolds, with implications for both theoretical mathematics and applied physics.

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