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Chains with Connections of Diffusion and Advective Types

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Abstract: The local dynamics of a system of oscillators with a large number of elements and with diffusive- and advective-type couplings containing a large delay are studied. Critical cases in the problem of the stability of the zero equilibrium state are singled out, and it is shown that all of them have infinite dimensions. Applying special methods of infinite normalization, we construct quasinormal forms, namely, nonlinear boundary value problems of the parabolic type, whose nonlocal dynamics determine the behavior of the solutions of the initial system in a small neighborhood of the equilibrium state. These quasinormal forms contain either two or three spatial variables, which emphasizes the complexity of the dynamical properties of the original problem.

Keywords: boundary value problem; delay; stability; normal form; dynamics; asymptotics of solutions; bifurcations; singular perturbations; oscillators

MSC: 34K25

1. Introduction

We consider the dynamics of chains with diffusive and advective couplings containing a large delay. The second-order equation with cubic nonlinearity,

$$\ddot{u} + a\dot{u} + u + f(u, \dot{u}) = 0, \tag{1}$$

$$f(u, \dot{u}) = b_1 u^3 + b_2 u^2 \dot{u} + b_3 u \dot{u}^2 + b_4 \dot{u}^3, \tag{2}$$

serves as a basic example.

A chain of N equations of the form in (1) has the form

$$\ddot{u}_j + a\dot{u}_j + u_j + f(u_j, \dot{u}_j) = d \sum_{k=1}^{N} a_{j-k} u_k(t - T), \tag{3}$$

where $T > 0$ is the delay time, a_k denotes the coefficients of the couplings, and $u_k(t)$ denotes N-periodic functions of the index k:

$$u_{k \pm N} \equiv u_k.$$

The dynamics of chains of this kind have been studied by many authors, such as [1–3], where chains without a delay were considered, and [4–12], where chains with a delay were studied. The main assumption is that the number N of oscillators is sufficiently large; i.e., the value $\varepsilon = 2\pi N^{-1}$ is sufficiently small:

$$0 < \varepsilon \ll 1. \tag{4}$$

Functions $u_k(t)$ are conveniently associated with the values of a function of two variables, $u_k(t) = u(t, x_k)$, where x_k denotes points with angular coordinates uniformly distributed on some circle: $x_k = 2\pi k N^{-1}$. Condition (4) gives reason to transition from the system in (3)

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to the problem of studying functions of two variables, $u(t,x)$, with a continuous spatial variable $x \in (-\infty,\infty)$ and with the periodicity condition

$$u(t,x+2\pi) \equiv u(t,x), \tag{5}$$

for which

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + u + f\left(u, \frac{\partial u}{\partial t}\right) = d\int\limits_{-\infty}^{\infty} \Phi(s,\varepsilon)u(t-T,x+s)ds. \tag{6}$$

The values of the function $\Phi(s,\varepsilon)$ are determined by coupling coefficients a_k. Let us describe the classes of functions $\Phi(s,\varepsilon)$ that will be studied in this paper. We arbitrarily set $\sigma > 0$ and introduce a Gaussian function:

$$F_\varepsilon(s) = \frac{1}{\sigma\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(s-\varepsilon)^2}{2\varepsilon^2\sigma^2}\right).$$

Let $\Phi_0(s,\varepsilon)$ denote the function

$$\Phi_0(s,\varepsilon) = F_\varepsilon(s) - 2F_0(s) + F_{-\varepsilon}(s). \tag{7}$$

Due to the fact that, for every continuous function $u(x)$,

$$\lim_{\sigma\to 0}\int\limits_{-\infty}^{\infty} \Phi_0(s,\varepsilon)u(x+s)ds = u(x+\varepsilon) - 2u(x) + u(x-\varepsilon), \tag{8}$$

it is natural to call (7) a diffusion-type coupling, since the right part of this equality resembles the expression for the standard difference approximation of the diffusion operator $\partial^2 u/\partial x^2$. Such couplings were used, for example, in [8,12–14]. Let us also note the work in [15], where chains of systems of laser equations were considered.

Let us introduce two more functions:

$$\Phi_1(s,\varepsilon) = F_\varepsilon(s) - F_{-\varepsilon}(s) \tag{9}$$

and

$$\Phi_2(s,\varepsilon) = F_\varepsilon(s) - F_0(s). \tag{10}$$

For each fixed continuous function $u(x)$ bounded on the interval $(-\infty,\infty)$, we have the following equations:

$$\lim_{\sigma\to 0}\int\limits_{-\infty}^{\infty} \Phi_1(s,\varepsilon)u(x+s)ds = u(x+\varepsilon) - u(x-\varepsilon), \tag{11}$$

$$\lim_{\sigma\to 0}\int\limits_{-\infty}^{\infty} \Phi_2(s,\varepsilon)u(x+s)ds = u(x+\varepsilon) - u(x). \tag{12}$$

The right-hand sides of (11) and (12) usually arise, for example, when applying the standard difference approximation of the advection (transfer) operator $\partial u/\partial x$. Therefore, it is natural to call the right-hand side in (6) an advection-type coupling.

Another assumption that paves the way for the application of asymptotic methods is that the value of T is sufficiently large: for some $c > 0$, we have

$$T = c\varepsilon^{-1}. \tag{13}$$

In Equation (6), we perform time normalization $t \to Tt$. As a result, we arrive at the singularly perturbed equation

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \varepsilon a \frac{\partial u}{\partial t} + u + f\left(u, \varepsilon \frac{\partial u}{\partial t}\right) = d \int_{-\infty}^{\infty} \Phi(s, \varepsilon) u(t - c, x + s) ds. \tag{14}$$

Note that the degenerate at $\varepsilon = 0$ in Equation (14) does not give information about the behavior of solutions. We will use classical asymptotic methods based on the application of methods of many scales: methods characteristic of the theory of averaging (see, for example, [16]) and methods of singular perturbations [17–19]. In order to study the dynamical properties of solutions under conditions (4) and (13), we will use the special asymptotic methods of local analysis developed in [20,21].

Let us study the behavior of all solutions of the boundary value problem (14) as $t \to \infty$ with initial functions sufficiently small in the norm $C^1_{[-c,0]} \times C_{[0,2\pi]}$ and 2π-periodic in the spatial variable x.

In the study of the local—in the neighborhood of the zero equilibrium state—behavior of solutions, the linearized boundary value problem

$$\varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \varepsilon a \frac{\partial u}{\partial t} + u = d \int_{-\infty}^{\infty} \Phi(s, \varepsilon) u(t - c, x + s) ds, \tag{15}$$

$$u(t, x + 2\pi) \equiv u(t, x). \tag{16}$$

plays an important role. Its characteristic equation, which we obtain by substituting the Euler solutions $u = \exp(ikx + \lambda t)$ into (15), has the form

$$\varepsilon^2 \lambda^2 + \varepsilon a \lambda + 1 = d\gamma(z) \exp(-c\lambda), \tag{17}$$

where, in the case of diffusion coupling,

$$\gamma(z) = -4 \sin^2 \frac{z}{2} \cdot \exp\left(-\frac{1}{2}\sigma^2 z^2\right), \quad z = \varepsilon k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

In advective coupling (9),

$$\gamma(z) = 2i \sin z, \tag{18}$$

and at the connection of the form in (10),

$$\gamma(z) = \exp(iz) - 1. \tag{19}$$

In the case where all roots of Equation (17), for all $k = 0, \pm 1, \pm 2, \ldots$, have negative real parts that move away from zero as $\varepsilon \to 0$, the solutions of the boundary value problem (15), (16) are asymptotically stable, and the solutions of (14), (16) with sufficiently small and ε-independent (by the norm $C^1_{[-c,0]} \times C_{[0,2\pi]}$) initial conditions tend to zero as $t \to \infty$. If Equation (17) has a root with a positive real part that moves away from zero as $\varepsilon \to 0$, then the solutions of (15), (16) are unstable, and the dynamics problem (14), (16) becomes nonlocal.

Here, we will consider the critical case where there are no roots with a positive real part that moves away from zero in (17), but there are roots that tend to the imaginary axis as $\varepsilon \to 0$. Note that, in the case of the finite dimensionality of the critical case, the methodology for the study of local dynamics is well known. It relies on the method of integral manifolds and the method of normal forms (see, e.g., [22,23]). A characteristic feature of all of the problems considered below is the fact that they realize infinite-dimensional critical cases when infinitely many roots of the characteristic equation tend to the imaginary axis as $\varepsilon \to 0$. Therefore, the methods of integral manifolds and normal forms are not directly

applicable. The approach developed in [20,21], which is related to the construction of infinite-dimensional quasinormal forms, is essentially used here.

Let us briefly look at the research design used below. First, a linearized boundary value problem is considered, and its characteristic equation is studied. We determine those parameters at which a critical case occurs in the problem of the stability of solutions. Then, we obtain the asymptotics of those roots of the characteristic equation that tend to the imaginary axis as the small parameter tends to zero. Since there are infinitely many such roots, there are also infinitely many solutions corresponding to the linearized boundary value problem. The set of such solutions can be written in a special form using another spatial variable. Therefore, it is possible to determine the structure of the main approximation of solutions to a nonlinear boundary value problem. Let us denote it conditionally by εU_1.

The solutions of the nonlinear boundary value problem are then found in the form of a formal series in powers of ε, the coefficients of which are periodic in t. Since, for simplicity, there is no quadratic nonlinearity in the equation, then, as a consequence, there are no terms of order ε^2 in the formal asymptotic series. Substituting the formal series into the original equation, we obtain a special linear inhomogeneous boundary value problem for the elements of this series. Using the solvability conditions for the resulting equation, we arrive at an equation for the unknown slowly varying amplitudes included in U_1. These equations are called quasinormal forms. They describe the local behavior of the original boundary value problem.

Note that the form of the notation in (7) is convenient from a purely technical point of view. Below, we will use the equality

$$\int_{-\infty}^{\infty} F_{\pm}(s,\varepsilon)\exp(iks)ds = \exp(\pm ik\varepsilon)\exp\left(-\frac{\sigma^2\varepsilon^2k^2}{2}\right).$$

The σ parameter defines the set of chain elements that significantly affect each specific element. In addition, it also sets the strength of the corresponding influence: the farther the elements are from each other, the weaker this influence is.

At $\sigma = 0$, an additional critical case arises; therefore, this work examines the dynamics of the system under the condition $\sigma \ll 1$. As it turns out, in these cases, the quasinormal form acquires an additional spatial variable. It follows that, for $\sigma \to 0$, there is a tendency for the dynamic properties of solutions to become more complex.

The corresponding results are given in Sections 2.3 and 3.5.

Chains of this type without a delay were studied in [14]. The presence of a delay, on the one hand, allows one to obtain explicitly formal expressions for critical cases. On the other hand, the dimensionality of critical cases increases, and the corresponding quasinormal forms become even more complicated.

This paper consists of two parts. The first part studies diffusion-type couplings, whereas the second part deals with advection-type couplings.

2. Diffusion-Type Coupling

Linear analysis has a central role in the study of the boundary value problem (14), (16).

2.1. Linear Analysis

Let us consider the roots of the characteristic Equation (17). Recall that critical cases in the stability problem (15), (16) are realized when Equation (17) has a root with a zero or sufficiently close to zero real part for some k. In this connection, for some real value of ω, let us set $\lambda = i\omega\varepsilon^{-1}$ in (17). As a result, we obtain the following:

$$1 - \omega^2 + ia\omega = d\gamma(z)\exp\left(-i\omega\varepsilon^{-1}c\right), \quad z = \varepsilon k, \quad k = 0, \pm1, \pm2, \ldots. \tag{20}$$

Let $p(\omega)$ denote the modulus of the left part of (20):

$$p(\omega) = \left[(1-\omega^2)^2 + a^2\omega^2\right]^{1/2},$$

and let

$$p_0 = \min_{-\infty<\omega<\infty} p(\omega) = p(\omega_0).$$

Here,

$$\omega_0 = \begin{cases} 0, & \text{if } a^2 \geq 2, \\ \left(1 - \dfrac{a^2}{2}\right)^{1/2}, & \text{if } a^2 < 2, \end{cases} \quad p_0 = \begin{cases} 1, & \text{if } a^2 \geq 2, \\ \dfrac{a^2}{2}(4-a^2)^{1/2}, & \text{if } a^2 < 2. \end{cases}$$

Note that $p_0 = 0$ for $a = 0$.

In this section, we will focus on the first case, where

$$\sigma > 0. \tag{21}$$

The case

$$\sigma = \varepsilon\sigma_1. \tag{22}$$

will be discussed in Section 2.3.

Let condition (21) be satisfied. For each fixed z and under the condition

$$d|\gamma(z)| < p_0$$

Equation (20) has no real roots. Below, we assume that

$$\gamma_0 = \max_{-\infty<z<\infty} \gamma(z) = \gamma(z_0) \quad (z_0 \geq 0). \tag{23}$$

The value of z_0 is defined in a unique way and is found simply. From the condition $\gamma'(z_0) = 0$, we find that z_0 is the first positive root of the equation

$$\frac{z}{2} = 2(\sigma^2 z)^{-1}.$$

Given $d|\gamma_0| < p_0$ and sufficiently small ε, all roots of Equation (17) have negative real parts that move away from zero as $\varepsilon \to 0$. Given $d|\gamma_0| > p_0$, we find z_0 such that Equation (17) has a root with a positive real part that moves away from zero as $\varepsilon \to 0$.

Let us restrict ourselves to the case where the parameter d is positive. The value of the parameter d_0, which distinguishes the critical case in the stability problem (15), (16), is determined by the equality

$$d_0 = p_0|\gamma_0|^{-1}.$$

In this connection, we assume below that, for an arbitrary fixed value d_1 for the parameter d, we have

$$d = d_0 + \varepsilon^2 d_1. \tag{24}$$

Under this condition, let us consider the asymptotics of all those roots of the characteristic Equation (17) whose real parts tend to zero as $\varepsilon \to 0$. We note at once that there are infinitely many such roots, so the critical case has infinite dimensionality.

Let us introduce some more notations. Let $\Omega_0 = \Omega_0(\omega_0)$ be a real value for which

$$1 - \omega_0^2 + i\omega = p_0 \exp(i\Omega_0).$$

We let $\theta_\omega = \theta_\omega(\varepsilon) \in [0, 2\pi)$ denote an expression that complements the value $\omega_0(c\varepsilon)^{-1}$ to an integer multiple of 2π. When $\omega_0 = 0$, then $\theta_\omega = 0$. We will similarly let $\theta_z = \theta_z(\varepsilon) \in [0, 1)$ denote an expression that complements the value $z_0\varepsilon^{-1}$ to an integer. Given $z_0 = 0$, we consider that $\theta_z = 0$.

Let us formulate two simple statements about the asymptotics of the roots of (17).

Lemma 1. *Let*

$$a^2 > 2. \tag{25}$$

Then, $d_0 = p_0 = 1, \omega_0 = 0$, *and for the roots* $\lambda_{kn}(\varepsilon)$ $(k, n = 0, \pm 1, \pm 2, \ldots)$ *of* (17), *the real parts of which tend to zero as* $\varepsilon \to 0$, *the asymptotic equations*

$$\lambda_{kn}(\varepsilon) = \pi i c^{-1}(2n+1) + \varepsilon\lambda_{1kn} + \varepsilon^2\lambda_{2kn} + \ldots, \tag{26}$$

are satisfied, where

$$\lambda_{1kn} = -c^{-2}ia\pi(2n+1),$$

$$\lambda_{2kn} = c^{-3}\left(1 - \frac{1}{2}a^2\right)(\pi(2n+1))^2 - ic^{-3}a^2\pi(2n+1) + c^{-1}d_1\gamma_0 p_0^{-1} + \\ + \frac{1}{2}c^{-1}\gamma_0''(z_0)(\theta_z + k)^2(p_0\gamma_0)^{-1}.$$

Lemma 2. *Let*

$$0 < a^2 < 2. \tag{27}$$

Then, $\omega_0 > 0$, *and for the roots* $\lambda_{kn}(\varepsilon)$ $(k, n = 0, \pm 1, \pm 2, \ldots)$ *of* (17) *whose real parts tend to zero as* $\varepsilon \to 0$, *the asymptotic equations*

$$\lambda_{kn}(\varepsilon) = i\frac{\omega_0}{\varepsilon} + \lambda_{0n} + \varepsilon\lambda_{1kn} + \varepsilon^2\lambda_{2kn} + \ldots \tag{28}$$

hold, where

$$\lambda_{0n} = ic^{-1}\left[\pi(2n+1) + \theta_\omega - \Omega_0\right], \quad \varkappa = p_0\exp(i\Omega_0),$$
$$\lambda_{1kn} = ic^{-1}\varkappa^{-1}(2\omega_0 - ia)\lambda_{0n}, \tag{29}$$

$$\lambda_{2kn} = c^{-1}\left[\left(\varkappa^{-1} - \frac{1}{2}(-2\omega_0 + ia)^2\varkappa^{-2}\right)\lambda_{0n}^2 + d_1 p_0^{-1} - \right. \\ \left. + \frac{1}{2}\gamma''(z_0)(\theta_z + k)^2(p_0\gamma_0)^{-1} - (c\varkappa)^{-1}2i\omega_0\varkappa^{-1}(2\omega_0 - ia)\lambda_{0n} - i\varkappa^{-1}a\lambda_{0n}\right].$$

Note that the following conditions hold:

$$\Re\left(\varkappa^{-1} - \frac{1}{2}(-2\omega_0 + ia)^2\varkappa^{-2}\right) < 0, \quad \Re\lambda_{1kn} = 0. \tag{30}$$

The first condition in (30) is obvious. Regarding the second equality in (30), it suffices to prove that the expression

$$(2\omega_0 - ia)\varkappa^{-1}$$

is purely imaginary. In this case, $P(\omega) = p(\omega)\exp(i\Omega(\omega))$ and $P'(\omega) = \left(p'(\omega) + i\Omega'(\omega)p(\omega)\right)$ $\exp\left(i\Omega(\omega)\right)$; hence,

$$P'(\omega) = i\Omega'(\omega_0)p_0\exp(i\Omega_0) = -2\omega_0 + ia.$$

Therefore, we conclude that $(-2\omega_0 + ia)\varkappa^{-1} = i\Omega'(\omega_0) = 2ia^{-1}$.

The roots $\lambda_{kn}(\varepsilon)$ of the characteristic Equation (17) allow us to determine solutions to the linear boundary value problem (15), (16):

$$u_{kn}(t, x, \varepsilon) = \exp\left(i(z_0\varepsilon^{-1} + \theta_z + k)x + \lambda_{kn}(\varepsilon)t\right),$$

and hence, the formal set of solutions is

$$u(t,x,\varepsilon) = \sum_{k,n=-\infty}^{\infty} \Big(\xi_{kn} u_{kn}(t,x,\varepsilon) + \bar{\xi}_{kn} \bar{u}_{kn}(t,x,\varepsilon) \Big), \tag{31}$$

where ξ_{kn} denotes arbitrary complex constants.

Remark 1. *Together with the roots $\lambda_{kn}(\varepsilon)$ of Equation (17), there are roots $\overline{\lambda}_{kn}(\varepsilon)$, which correspond to the solutions of the boundary value problem (15), (16):*

$$\overline{u}_{kn}(t,x,\varepsilon) = \exp\big(-i(z_0\varepsilon^{-1} + \theta_z + k)x + \overline{\lambda}_{kn}(\varepsilon)t \big).$$

Note that for the parameters z and $-z$, the roots in (17) are the same, since the dependence of the right-hand side of (17) on z is even. This means that for the modes of $-(z_0\varepsilon^{-1} + \theta_z + k)$, the roots are the same: $\lambda_{kn}(\varepsilon)$. Therefore, the problem (15), (16) has the solutions

$$\tilde{u}_{kn}(t,x,\varepsilon) = \exp\big(-i(z_0\varepsilon^{-1} + \theta_z + k)x + \overline{\lambda}_{kn}(\varepsilon)t \big).$$

Under the conditions of Lemma 1, we have the following:

$$\tilde{u}_{kn}(t,x,\varepsilon) = \overline{u}_{kn}(t,x,\varepsilon),$$

which is not the case under the conditions of Lemma 2.

2.2. Nonlinear Analysis

We separately consider the cases where $a^2 > 2$ and where $0 < a^2 < 2$.

2.2.1. Case $a^2 > 2$

In this case, we have the equality $\omega_0 = 0, \Omega_0 = 0, p_0 = 1$. The critical case in the stability problem is defined by the equality

$$d_0|\gamma_0| = 1. \tag{32}$$

We will base the following on the representation in (31). Let us write it in a more convenient form:

$$\begin{aligned} u(t,x,\varepsilon) &= E(x) \sum_{k,n=-\infty}^{\infty} \xi_{kn} \exp\big(ikx + ic^{-1}\pi(2n+1)(1-\varepsilon c^{-1}a)t + (\lambda_{2kn} + O(\varepsilon))\tau \big) = \\ &= E(x) \sum_{k,n=-\infty}^{\infty} \xi_{kn}(\tau) \exp\big(ikx + i\pi(2n+1)x_1 \big) = E(x)\xi(\tau,x,x_1), \end{aligned} \tag{33}$$

where $\tau = \varepsilon^2 t$ is the "slow" time, $E(x) = \exp\big(i(z_0\varepsilon^{-1} + \theta_z)x\big)$, and $\xi_{kn}(\tau) = \xi_{kn} \exp\big((\lambda_{2kn} + O(\varepsilon))\tau\big)$ denotes the coefficients of the expansion $\xi(\tau,x,x_1)$ into a Fourier series by the 2π-periodic argument x and 1-antiperiodic argument $x_1 = c^{-1}(1-\varepsilon c^{-1}a)t$.

The solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u = \varepsilon\big(E(x)\xi(\tau,x,x_1) + \overline{cc}\big) + \varepsilon^3 u_3(\tau,x,x_1) + \dots. \tag{34}$$

Here and below, $\overline{cc}$ denotes the term that is complex conjugate to the previous one. The unknown complex function $\xi(\tau,x,x_1)$ is to be defined. Let us substitute (34) into (14) and

equate the coefficients of the various powers of ε. Then, at the first degree of ε, we obtain the identity. Equating the coefficients of ε^3, we arrive at the equation

$$\begin{aligned} c\frac{\partial \xi}{\partial \tau} =&(2c^2)^{-1}(a^2-2)\frac{\partial^2 \xi}{\partial x_1^2} + (2\gamma_0)^{-1}\gamma''(z_0)\frac{\partial^2 \xi}{\partial x^2} - i\gamma_0^{-1}\gamma''(z_0)\theta_z\frac{\partial \xi}{\partial x} - \\ &- ac^{-2}\frac{\partial \xi}{\partial x_1} + \big((2\gamma_0)^{-1}\gamma''(z_0)\theta_z^2 - \gamma_0^{-1}d_1\big)\xi + 3b_1\xi|\xi|^2 \end{aligned} \tag{35}$$

with the boundary conditions

$$-\xi(\tau, x, x_1+1) \equiv \xi(\tau, x, x_1) \equiv \xi(\tau, x+2\pi, x_1). \tag{36}$$

Here, we take into account the relations

$$\begin{aligned} \frac{d\xi}{dt} =&\varepsilon^2\frac{\partial \xi}{\partial \tau} + \frac{\partial \xi}{\partial x_1}(1-\varepsilon a c^{-1}), \\ \xi_{t-c} =&\xi(\tau - \varepsilon^2 c, x, x_1 - c(1-\varepsilon a c^{-1})) = \\ =&\xi(\tau, x, x_1) - \varepsilon^2 c\frac{\partial \xi}{\partial \tau} + \varepsilon a\frac{\partial \xi}{\partial x_1} + \frac{1}{2}\varepsilon^2 a^2\frac{\partial^2 \xi}{\partial x_1^2} + o(\varepsilon^2). \end{aligned}$$

Let us introduce the following notation. We arbitrarily fix the value $\theta_{0z} \in [0,1)$ and let $\varepsilon_n = \varepsilon_n(\theta_{0z})$ denote a sequence such that $\varepsilon_n \to 0$, for $n \to \infty$, and $\theta_z(\varepsilon_n, \theta_{0z}) = \theta_{0z}$. The above constructions justify the following result.

Theorem 1. *Let $a^2 > 2$ and conditions (24) and (32) be satisfied. Let $\theta_{0z} \in [0,1)$ be arbitrarily fixed, and let the boundary value problem (35), (36) for $\theta_z = \theta_{0z}$ have a bounded solution $\xi(\tau, x, x_1)$ for $\tau \to \infty$, $x \in [0, 2\pi]$, $x_1 \in [0,1]$. Then, the function $u(t,x,\varepsilon) = \varepsilon\big(E(x)\xi(\tau, x, x_1) + \overline{c}\overline{c}\big) + \varepsilon^3 u_3(\tau, x, x_1)$ satisfies the boundary value problem (14), (16) up to $o(\varepsilon^3)$.*

Thus, the parabolic boundary value problem (35), (36) is a quasinormal form for the boundary value problem (14), (16).

2.2.2. Case $0 < a^2 < 2$

The dynamical properties in this case are significantly more complicated. The principal parts of the roots $\lambda_{kn}(\varepsilon)$ of the characteristic equation are close to $i\omega_0\varepsilon^{-1}$: i.e., they are asymptotically large. Therefore, it is natural to expect that the oscillations in the boundary value problem (14), (16) will be rapid.

Note that, in this case,

$$\omega_0 = \left(1 - \frac{a^2}{2}\right)^{1/2}, \quad p_0 = \frac{a^2}{2}(4-a^2)^{1/2}, \quad d_0 = p_0|\gamma_0|^{-1}. \tag{37}$$

The roots of $\lambda_{kn}(\varepsilon)$ correspond to the Euler solutions of the linear boundary value problem (15), (16):

$$u_{kn}^{\pm}(t,x,\varepsilon) = \exp\big(\pm i(z_0\varepsilon^{-1} + \theta_z + k)x + \lambda_{kn}(\varepsilon)t\big).$$

It is more convenient to write these functions in the form

$$u_{kn}^{\pm}(t,x,\varepsilon) = E^{\pm}(t,x)\exp\big(ikx + i\pi(2n+1)x_1 + (\lambda_{2kn} + O(\varepsilon))\tau\big),$$

where

$$\begin{aligned} E^{\pm}(t,x) =&\exp\big(\pm i(c^{-1}\omega_0\varepsilon^{-1} + c^{-1}(\theta_\omega - \Omega_0) + \varepsilon c^{-1}\varkappa^{-1}(2\omega_0 - ia)\cdot \\ &\cdot ic^{-1}(\theta_\omega - \Omega_0))t + i(z_0\varepsilon^{-1} + \theta_z)x\big), \end{aligned}$$

$$R = \varkappa^{-1}(2\omega_0 - ia) \cdot ic^{-1}(\theta_\omega - \Omega_0) = 2(ca)^{-1}, \quad \Im R = 0,$$

$\tau = \varepsilon^2 t$, $x_1 = c^{-1}(1 - \varepsilon c^{-1} R)t$. Hence, we find that

$$\sum_{k,n=-\infty}^{\infty} \xi_{kn}^{\pm} u_{kn}^{\pm}(t,x,\varepsilon) = E^{\pm}(t,x) \sum_{k,n=-\infty}^{\infty} \xi_{kn}^{\pm}(\tau) \exp\left(ikx + i\pi(2n+1)x_1\right) = \\ = E^{\pm}(t,x)\xi^{\pm}(\tau,x,x_1).$$

Here, $\xi_{kn}^{\pm}$ denotes arbitrary complex constants, and $\xi_{kn}^{\pm}(\tau) = \xi_{kn}^{\pm} \exp\left((\lambda_{2kn} + O(\varepsilon))\tau\right)$. The functions $\xi_{kn}^{\pm}(\tau)$ are the Fourier coefficients of the function $\xi^{\pm}(\tau, x, x_1)$, which is 2π-periodic with respect to x and 1-antiperiodic with respect to x_1.

The solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u(t,x) = u^{+}(t,x) + u^{-}(t,x), \tag{38}$$

$$u^{\pm}(t,x) = \varepsilon\left(\xi^{\pm}(\tau,x,x_1)E^{\pm}(t,x) + \overline{cc}\right) + \varepsilon^3 u_3(t,\tau,x,x_1) + \ldots,$$

where the dependence on t, x and x_1 is periodic.

Let us substitute (38) into (14) and equate the coefficients of the same powers ε. In the first step, we obtain the identity, and for ε^3, we obtain an equation for u_3. From the condition in the specified class of functions, we arrive at the relation. Let us substitute (38) into (14) and equate the coefficients of the same powers ε. In the first step, we obtain the identity, while, by equating the coefficients of ε^3, we obtain an equation for u_3. From its solvability condition in the specified class of functions, we arrive at the relation

$$\frac{\partial \xi^{\pm}}{\partial \tau} = A_1 \frac{\partial^2 \xi^{\pm}}{\partial x_1^2} + A_2 \frac{\partial \xi^{\pm}}{\partial x_1} + A_3 \xi^{\pm} + B_1 \frac{\partial^2 \xi^{\pm}}{\partial x^2} + B_2 \frac{\partial \xi^{\pm}}{\partial x} + c^{-1}\beta\xi^{\pm}(|\xi^{\pm}|^2 + 2|\xi^{\mp}|^2), \tag{39}$$

in which

$$\begin{aligned}
A_1 =& -c^{-3}\left[\varkappa^{-1} - \frac{1}{2}(ia - 2\omega_0)^2\varkappa^{-2}\right],\\
A_2 =& c^{-3}\Big[-2(\varkappa^{-1} - \frac{1}{2}(ia - 2\omega_0)^2\varkappa^{-2}(\theta_\omega - \Omega_0)) + c\varkappa^{-2}2\omega_0(2\omega_0 - ia) +\\
&+ c^2\varkappa^{-1}a(\theta_\omega - \Omega_0)\Big],\\
A_3 =& c^{-3}\left[\frac{1}{2}(ia - 2\omega_0)\varkappa^2(\theta_\omega - \Omega_0)^2 - \varkappa^{-1}\right] + d_1 c^{-1} p_0^{-1} +\\
&+ \frac{1}{2}\gamma''(z_0)c^{-1}\theta_z^2(p_0\gamma_0)^{-1} - i(c\varkappa)^{-2}2i\omega_0(2\omega_0 - ia)\cdot\\
&\cdot(\theta_\omega - \Omega_0) - i(c\varkappa)^{-1}a(\theta_\omega - \Omega_0),\\
B_1 =& \frac{1}{2}c^{-1}\gamma''(z_0)(p_0\gamma_0)^{-1},\\
B_2 =& c^{-1}\gamma''(z_0)\theta_z(p_0\gamma_0)^{-1},\\
\beta =& b_1 + i\omega_0 b_2 - \omega_0^2 b_3 - i\omega_0^3 b_4.
\end{aligned}$$

Recall that the function $\xi(\tau, x, x_1)$ satisfies the boundary conditions

$$-\xi(\tau, x, x_1 + 1) \equiv \xi(\tau, x, x_1) \equiv \xi(\tau, x + 2\pi, x_1). \tag{40}$$

In order to formulate the final result, we introduce some notations. We arbitrarily fix $\theta_{0\omega} \in [0, 2\pi)$ and let the sequence $\varepsilon_s = \varepsilon_s(\theta_{0\omega})$ be defined by the condition $\theta_\omega(\varepsilon_s(\theta_{0\omega})) = \theta_{0\omega}$ $(s = 1, 2, \ldots)$. Let $\Gamma(\theta_{0\omega})$ denote all limit points of the sequence $\theta_z(\varepsilon_s(\theta_{0\omega}))$ from the

interval $[0,1]$. Let θ_{0z} denote the limit element of $\Gamma(\theta_{0\omega})$ and let the subsequence ε_{s_Γ} of the sequence ε_s be such that

$$\lim_{\Gamma\to\infty} \theta_z(\varepsilon_{s_\Gamma}) = \theta_{0z}.$$

Note that it is possible that the set $\Gamma(\theta_{0\omega})$ coincides with the segment $[0,1]$, and it is possible that this set consists of a single element.

Theorem 2. *Let $0 < a^2 < 2$ and $d_0 = p_0|\gamma_0|^{-1}$. We arbitrarily fix $\theta_{0\omega} \in [0,2\pi)$ and let $\theta_{0z} \in \Gamma(\theta_{0\omega})$. Let $\xi^{\pm}(\tau,x,x_1)$ be the solution of the boundary value problem (39), (40) that is bounded for $\tau \to \infty$, $x \in [0,2\pi]$, $x_1 \in [0,1]$. Then, the function*

$$u(t,x,\varepsilon) = \varepsilon\big(\xi^{+}(\tau,x,x_1)E^{+}(t,x) + \overline{cc} + \xi^{-}(\tau,x,x_1)E^{-}(t,x) + \overline{cc}\big) + \varepsilon^3 u_3(t,\tau,x,x_1)$$

satisfies the boundary value problem (14), (16) up to $o(\varepsilon_{s_\Gamma}^3)$ for $\tau = \varepsilon^2 t$, $x_1 = (1-\varepsilon c^{-1}R)t$, for the sequence $\varepsilon = \varepsilon_{s_\Gamma}$.

Thus, the boundary value problem (39), (40) is a quasinormal form for the original boundary value problem (14), (16) in this critical case.

2.3. Small Values of Parameter σ

Below, we will consider important questions about the dynamical properties of the boundary value problem (14), (16) for small values of σ. We will assume that for some fixed value of σ_1, equality (22) is satisfied.

The interest in this case is due to the fact that, first, as is shown above for small σ, the corresponding integral expressions in the boundary value problem (14), (16) are close to being written in the form of a finite difference on the spatial variable.

Second, it follows from (17) that the value of $\exp\big(-\sigma^2 z^2/2\big)$ on the right-hand side of (17) is small, and hence, the critical cases are determined by the periodic function $\gamma(z)$. Thus, the critical values of z_0 in (23) are obviously not unique. There are obviously infinitely many such values. This suggests that the quasinormal form becomes significantly more complex, and the dynamical properties more interesting and diverse.

Under condition (22) for the function $\gamma(z)$, we have the equality

$$\gamma(z) = -4\sin^2\left(\frac{z}{2}\right)\cdot\exp\left(-\frac{1}{2}\varepsilon^2\sigma_1^2 z^2\right).$$

Let

$$\gamma_0(z) = -4\sin^2\frac{z}{2}.$$

Then,

$$\gamma(z) = \gamma_0(z)\left(1 - \frac{1}{2}\varepsilon^2\sigma_1^2 z^2 + O(\varepsilon^4)\right).$$

The largest value $|\gamma_0(z)| = 4$, and for all values z_m at which this value is reached, we have the equations

$$z_m = \pi(2m+1), \quad m = 0, \pm1, \pm2, \ldots.$$

Recall that $\varepsilon = 2\pi N^{-1}$. Below, we will assume that the value N is even, so all values of $\pi(2m+1)\varepsilon^{-1}$ are integers for all integers m.

Consider the set of integers $\pi(2m+1)\varepsilon^{-1} + k$, $k, m = 0, \pm1, \pm2, \ldots$. Let $u_{kmn}(t,x)$ denote the Euler solutions of the linear problem (15), (16):

$$u_{kmn}(t,x) = \exp\big[i\big(\pi(2m+1)\varepsilon^{-1} + k\big)x + \lambda_{kmn}(\varepsilon)t\big].$$

Here, $\lambda_{kmn}(\varepsilon)$ represents the roots of the characteristic equation (17) whose real parts tend to zero as $\varepsilon \to 0$.

2.3.1. Building A Quasinormal Form For $A^2 > 2$

Recall that, given $a^2 > 2$, we have $\omega_0 = \Omega_0 = 0$, $p_0 = 1, |\gamma_0| = 4$, and $d_0 = 1/4$. Let us first consider the asymptotics of $\lambda_{kmn}(\varepsilon)$.

Lemma 3. *Let conditions* (22), (24) *and* (25) *be satisfied. Then, there are asymptotic equalities:*

$$\lambda_{kmn}(\varepsilon) = c^{-1} i\pi(2n+1) + \varepsilon\lambda_{1kmn} + \varepsilon^2\lambda_{2kmn} + \dots,$$

where

$$\begin{aligned}\lambda_{1kmn} =& iac^{-2}\pi(2n+1),\\ \lambda_{2kmn} =& \frac{1}{2}(2-a^2)c^{-3}\big(\pi(2n+1)\big)^2 - ia^2c^{-3}\pi(2n+1) + 4d_1 - \frac{1}{2}\sigma_1^2\big(\pi(2m+1)\big)^2 - \\ &- \frac{1}{4}(\theta_z + k)^2.\end{aligned}$$

The set of Euler solutions of the linear boundary value problem (15), (16)

$$u(t,x,\varepsilon) = \sum_{k,m,n=-\infty}^{\infty} \xi_{kmn} \exp\big(i(\pi(2m+1)\varepsilon^{-1} + k)x + \lambda_{kmn}(\varepsilon)t\big)$$

can be written in the form

$$\begin{aligned}u(t,x,\varepsilon) = \sum_{k,m,n=-\infty}^{\infty} \xi_{kmn}(\tau) \exp\big(ikx + i\pi(2m+1)y + i\pi(2n+1)x_1\big) =\\ = \xi(\tau,x,y,x_1).\end{aligned} \tag{41}$$

Here,

$$\xi_{kmn}(\tau) = \xi_{kmn} \exp\big(\lambda_{2kmn} + O(\varepsilon)\tau\big), \quad y = x\varepsilon^{-1}, \quad x_1 = (1 + \varepsilon ac^{-1})t.$$

Based on the representation in (41), we will look for solutions of the nonlinear boundary value problem (14), (16) of the form

$$u(t,x) = \varepsilon\xi(\tau,x,y,x_1) + \varepsilon^3 u_3(\tau,x,y,x_1) + \dots. \tag{42}$$

After substituting (42) into (14) and following the standard steps, we arrive at the boundary value problem for determining the unknown function $\xi(\tau,x,y,x_1)$:

$$\begin{aligned}\frac{\partial\xi}{\partial\tau} =& \frac{a^2-2}{2c}\cdot\frac{\partial^2\xi}{\partial x_1^2} - \frac{a^2}{c^2}\cdot\frac{\partial\xi}{\partial x_1} + \frac{\sigma_1}{2c}\cdot\frac{\partial^2\xi}{\partial y^2} + \frac{1}{4c}\cdot\frac{\partial^2\xi}{\partial x^2} - \frac{i\theta_z}{2c}\cdot\frac{\partial\xi}{\partial x} +\\ &+ \left(\frac{4d_1}{c} - \frac{1}{4c}\theta_z^2\right)\xi + \frac{b_1}{c}\xi^3,\end{aligned} \tag{43}$$

$$-\xi(\tau,x,y+1,x_1) \equiv \xi(\tau,x,y,x_1) \equiv \xi(\tau,x+2\pi,y,x_1), \tag{44}$$

$$-\xi(\tau,x,y,x_1+1) \equiv \xi(\tau,x,y,x_1). \tag{45}$$

As a result of the above constructions, we come to the justification of the following result.

Theorem 3. *Let conditions* (22), (24) *and* (25) *be satisfied. Let* $\theta_{z0} \in [0,1)$ *be arbitrarily fixed, and let* $\xi(\tau,x,y,x_1)$ *be a solution of the boundary value problem* (43)–(45) *bounded for* $\tau \to \infty$, $x \in [0,2\pi]$, $y \in [0,1]$, $x_1 \in [0,1]$. *Then, for the sequence* $\varepsilon_s(\theta_z(\varepsilon_s(\theta_{z0}) = \theta_{z0})$, *the function*

$$u(t,x) = \varepsilon\xi(\tau,x,y,x_1) + \varepsilon^3 u_3(\tau,x,y,x_1)$$

satisfies the boundary value problem of (14), (16) *up to* $o(\varepsilon_s^3)$ *for* $\theta_z = \theta_{z0}$.

2.3.2. Building Quasinormal Forms for $0 < A^2 < 2$

Recall that, in this case, Equation (37) holds.

Let us consider the asymptotics of such roots of the characteristic Equation (17) whose real parts tend to zero as $\varepsilon \to 0$.

In the following, θ_ω denotes such a quantity that complements the expression $\omega_0\varepsilon^{-1}$ to a value that is an odd multiple of πc^{-1}.

Lemma 4. *Let $0 < a^2 < 2$ and let conditions (22) and (24) be satisfied. Then, for $\lambda_{kmn}(\varepsilon)$, $k, m, n = 0, \pm1, \pm2, \ldots$, the asymptotic equalities take place:*

$$\begin{aligned}
&\lambda_{kmn}(\varepsilon) = i\big(\omega\varepsilon^{-1} + \theta_\omega - c^{-1}\Omega_0 + c^{-1}\pi(2n+1)\big) + \varepsilon\lambda_{1kmn} + \varepsilon^2\lambda_{2kmn} + \ldots,\\
&\lambda_{1kmn} = -2i(ac)^{-1}K, \quad K = \theta_\omega - c^{-1}\Omega_0 + c^{-1}\pi(2n+1),\\
&\lambda_{2kmn} = -D_1K^2 + D_2K - \frac{1}{2}\sigma_1^2\big(\pi(2m+1)\big)^2 + d_1d_0^{-1} - K^2,\\
&D_1 = 2a^{-2}c^{-3} - (1 + ia\omega_0 - \omega_0^2)^{-1}c^{-3},\\
&D_2 = 2i(2i\omega_0 + a)\big(p_0\exp(i\Omega_0)ca\big)^{-1}.
\end{aligned}$$

Note that $\Re D_1 > 0$.

The set of Euler solutions of the linear boundary value problem (15), (16) can then be represented as

$$\begin{aligned}
u(t,x) &= \sum_{k,m,n=-\infty}^{\infty} \xi_{kmn}\exp\big(i(\pi(2m+1)\varepsilon^{-1} + k)x + \lambda_{kmn}(\varepsilon)t\big) =\\
&= E(t)\sum_{k,m,n=-\infty}^{\infty} \xi_{kmn}(\tau)\exp\big(i\pi(2m+1)y + ikx + i\pi(2n+1)x_1\big) =\\
&= E(t)\xi(\tau,x,y,x_1).
\end{aligned} \tag{46}$$

Here, $\tau = \varepsilon^2 t$, $E(t) = \exp\big[i(\omega_0\varepsilon^{-1} + (\theta_\omega - c^{-1}\Omega_0)(1 - 2\varepsilon(ca)^{-1}))t\big]$, $\xi_{kmn}(\tau) = \xi_{kmn}\cdot\exp\big((\lambda_{2kmn} + O(\varepsilon))\tau\big)$, $y = x\varepsilon^{-1}$, $x_1 = c^{-1}(1 - 2\varepsilon(ca)^{-1})t$. Based on the representation in (46), we will look for solutions of the nonlinear boundary value problem (14), (16) of the form

$$u(t,x) = \varepsilon\big(\xi(\tau,x,y,x_1)E(t) + \overline{c\xi}\big) + \varepsilon^3u_3(t,\tau,x,y,x_1) + \ldots, \tag{47}$$

where the dependence on x, y, x_1 and t is periodic.

By substituting (47) into (14) and performing some straightforward calculations, we arrive at an equation for u_3. From its solvability condition in the specified class of functions, we obtain

$$\begin{aligned}
\frac{\partial\xi}{\partial\tau} =& D_1\frac{\partial^2\xi}{\partial x_1^2} + i(2D_1 + D_2)\frac{\partial\xi}{\partial x_1}(\theta_\omega - c^{-1}\Omega_0) + \frac{1}{2c}\sigma_1^2\frac{\partial^2\xi}{\partial y^2} +\\
&+ \frac{1}{c}\frac{\partial^2\xi}{\partial x^2} + \big(-D_1(\theta_\omega - \Omega_0)^2 + D_2(\theta_\omega - \Omega_0)\big)\xi + c^{-1}\beta\xi|\xi|^2.
\end{aligned} \tag{48}$$

For this equation, the boundary conditions are satisfied:

$$-\xi(\tau,x,y,x_1+1) \equiv \xi(\tau,x,y,x_1) \equiv \xi(\tau,x+2\pi,y,x_1), \tag{49}$$

$$-\xi(\tau,x,y+1,x_1) \equiv \xi(\tau,x,y,x_1). \tag{50}$$

Let us summarize.

Theorem 4. *Let conditions* (22), (24) *and* (27) *be satisfied. We arbitrarily fix* $\theta_{0\omega} \in [0, c^{-1}\pi)$ *and let* $\xi(\tau, x, y, x_1)$ *be a solution of the boundary value problem* (48)–(50) *for* $\theta_\omega = \theta_{0\omega}$ *bounded for* $\tau \to \infty$, $x \in [0, 2\pi]$, $y \in [0, 1]$, $x_1 \in [0, 1]$. *Then, the function*

$$u(t, x) = \varepsilon\big(\xi(\tau, x, y, x_1)E(t) + \overline{cc}\big) + \varepsilon^3 u_3(t, \tau, x, y, x_1)$$

satisfies the boundary value problem (14), (16) *up to* $o(\varepsilon^3)$ *for* $\tau = \varepsilon^2 t$, $x_1 = c^{-1}(1 - 2\varepsilon(ca)^{-1})t$ *and* $\varepsilon = \varepsilon_s(\theta_{0\omega})$.

Thus, in this section, we construct quasinormal forms, namely, boundary value problems of the parabolic type, (43)–(45) and (48)–(50), with three spatial variables. They play the role of the normal forms of the original boundary value problem (14), (16) in the above critical cases.

3. Advective-Type Coupling

3.1. The Results of Linear Analysis in the Case $\Phi(s) = \Phi_1(s)$

At each fixed z and under the condition

$$d|\gamma(z)| < p_0$$

Equation (20) has no real roots. Let us assume that

$$\gamma_0 = \max_{-\infty<z<\infty} |\gamma(z)| = |\gamma(z_0)| \quad (z_0 \geq 0). \tag{51}$$

The value of z_0 is defined in a unique way and is found simply. From the condition $|\gamma(z_0)|' = 0$, we find that z_0 is the first positive root of equation

$$z = 2(\sigma^2 z)^{-1}. \tag{52}$$

Given $d|\gamma_0| < p_0$ and sufficiently small ε, all roots of Equation (17) have negative real parts that move away from zero as $\varepsilon \to 0$. Given $d|\gamma_0| > p_0$, we find a z_0 such that Equation (17) has a root with a positive real part that moves away from zero as $\varepsilon \to 0$.

Let us restrict ourselves to the case where the parameter d is positive. The value of the parameter d_0, which distinguishes the critical case in the stability problem (15), (16), is determined by the equality

$$d_0 = p_0|\gamma_0|^{-1}.$$

In this connection, we assume below that, for an arbitrary fixed value d_1 for the parameter d, we have

$$d = d_0 + \varepsilon^2 d_1. \tag{53}$$

Under this condition, let us consider the asymptotics of all those roots of the characteristic Equation (17) whose real parts tend to zero as $\varepsilon \to 0$. There are infinitely many such roots, so the critical case has infinite dimensionality.

Let us introduce some more notations. Let $\Omega_0 = \Omega_0(\omega_0)$ be a real value for which

$$1 - \omega_0^2 + i\omega = p_0 \exp(i\Omega_0).$$

As above, we let $\theta_\omega = \theta_\omega(\varepsilon) \in [0, 2\pi)$ denote an expression that complements the value of $c\omega_0\varepsilon^{-1}$ to an integer multiple of 2π. Given $\omega_0 = 0$, we consider $\theta_\omega = 0$. We similarly let $\theta_z = \theta_z(\varepsilon) \in [0, 1)$ denote an expression that complements the value of $z_0\varepsilon^{-1}$ to an integer. Given $z_0 = 0$, we consider that $\theta_z = 0$.

We shall now formulate a statement about the asymptotics of the roots of (17) in the case of (18).

Lemma 5. *Let* $\gamma(z) = \gamma_1(z)$ *and*

$$a^2 > 2. \tag{54}$$

Then, $d_0\gamma_0 = p_0 = 1, \omega_0 = 0$, and for the roots $\lambda_{kn}(\varepsilon)$ $(k, n = 0, \pm1, \pm2, \ldots)$ of Equation (17), the real parts of which tend to zero as $\varepsilon \to 0$, the asymptotic equations are satisfied:

$$\lambda_{kn}(\varepsilon) = \pi i c^{-1}\left(2n + \frac{1}{2}\right) + \varepsilon\lambda_{1kn} + \varepsilon^2\lambda_{2kn} + \ldots, \tag{55}$$

where

$$\lambda_{1kn} = -c^{-2}ia\pi\left(2n + \frac{1}{2}\right),$$

$$\lambda_{2kn} = c^{-3}\left(1 - \frac{1}{2}a^2\right)\left(\pi\left(2n + \frac{1}{2}\right)\right)^2 - ic^{-3}a^2\pi\left(2n + \frac{1}{2}\right) + c^{-1}d_1\gamma_0 p_0^{-1} + \\ + \frac{1}{2}c^{-1}\gamma_0''(z_0)(\theta_z + k)^2(p_0\gamma_0)^{-1}.$$

Lemma 6. *Let $\gamma(z) = \gamma_1(z)$ and*

$$0 < a^2 < 2. \tag{56}$$

Then, $\omega_0 > 0$, and for the roots $\lambda_{kn}(\varepsilon)$ $(k, n = 0, \pm1, \pm2, \ldots)$ of Equation (17) whose real parts tend to zero as $\varepsilon \to 0$, the asymptotic equations are satisfied:

$$\lambda_{kn}(\varepsilon) = i\omega_0\varepsilon^{-1} + \lambda_{0n} + \varepsilon\lambda_{1kn} + \varepsilon^2\lambda_{2kn} + \ldots, \tag{57}$$

where

$$\lambda_{0n} = ic^{-1}\left[\pi\left(2n + \frac{1}{2}\right) + \theta_\omega - \Omega_0\right], \quad \varkappa = p_0\exp(i\Omega_0),$$

$$\lambda_{1kn} = ic^{-1}\varkappa^{-1}(2\omega_0 - ia)\lambda_{0n}, \tag{58}$$

$$\lambda_{2kn} = c^{-1}\left[\left(\varkappa^{-1} - \frac{1}{2}(-2\omega_0 + ia)^2\varkappa^{-2}\right)\lambda_{0n}^2 + d_1p_0^{-1} - \right. \\ \left. + \frac{1}{2}\gamma''(z_0)(\theta_z + k)^2(p_0\gamma_0)^{-1} - (c\varkappa)^{-1}2i\omega_0\varkappa^{-1}(2\omega_0 - ia)\lambda_{0n} - i\varkappa^{-1}a\lambda_{0n}\right].$$

Note that

$$\Re\left(\varkappa^{-1} - \frac{1}{2}(-2\omega_0 + ia)^2\varkappa^{-2}\right) < 0, \quad \Re\lambda_{1kn} = 0. \tag{59}$$

The roots $\lambda_{kn}(\varepsilon)$ of the characteristic Equation (17) allow us to determine solutions to the linear boundary value problem (15), (16):

$$u_{kn}(t, x, \varepsilon) = \exp\left(i(z_0\varepsilon^{-1} + \theta_z + k)x + \lambda_{kn}(\varepsilon)t\right),$$

and hence, the formal set of solutions is

$$u(t, x, \varepsilon) = \sum_{k,n=-\infty}^{\infty}\left(\xi_{kn}u_{kn}(t, x, \varepsilon) + \bar{\xi}_{kn}\bar{u}_{kn}(t, x, \varepsilon)\right), \tag{60}$$

where ξ_{kn} denotes arbitrary complex constants.

Remark 2. *Together with the roots $\lambda_{kn}(\varepsilon)$ of Equation (17), there are the roots $\overline{\lambda}_{kn}(\varepsilon)$, which correspond to the solutions of the boundary value problem (15), (16):*

$$\overline{u}_{kn}(t, x, \varepsilon) = \exp\left(-i(z_0\varepsilon^{-1} + \theta_z + k)x + \overline{\lambda}_{kn}(\varepsilon)t\right).$$

Note that for the parameters z and $-z$, the roots in (17) are the same, since the dependence of the right-hand side of (17) on z is even. This means that for the modes of $-(z_0\varepsilon^{-1}+\theta_z+k)$, the roots are the same, i.e., $\lambda_{kn}(\varepsilon)$. Therefore, the problem (15), (16) has the following solutions:

$$\tilde{u}_{kn}(t,x,\varepsilon)=\exp\big(-i(z_0\varepsilon^{-1}+\theta_z+k)x+\overline{\lambda}_{kn}(\varepsilon)t\big).$$

Under the conditions of Lemma 5, we have

$$\tilde{u}_{kn}(t,x,\varepsilon)=\overline{u}_{kn}(t,x,\varepsilon),$$

and under the conditions of Lemma 6, this is no longer the case.

3.2. The Results of Linear Analysis in the Case $\Phi(s)=\Phi_2(s)$

In the case of (19), the value of $z_0>0$ is defined as the first positive root from the equation

$$\frac{z}{2}=(2\sigma^2 z)^{-1}. \tag{61}$$

Lemma 7. *Let condition (19) be satisfied and $a^2>2$. Then, $d_0\gamma_0=p_0=1,\omega_0=0$, and for the roots $\lambda_{kn}(\varepsilon)$ $(k,n=0,\pm1,\pm2,\ldots)$ of Equation (17) whose real parts tend to zero as $\varepsilon\to0$, the asymptotic equations are satisfied:*

$$\lambda_{kn}(\varepsilon)=\left[i\pi\left(\frac{1}{2}+2n\right)+\frac{i}{2}(z_0+\varepsilon(\theta_z+k))\right]c^{-1}+\varepsilon\lambda_{1kn}+\varepsilon^2\lambda_{2kn}+\ldots, \tag{62}$$

where

$$\lambda_{1kn}=-ic^{-2}a\left(\pi\left(\frac{1}{2}+2n\right)+\frac{z_0}{2}\right)-\frac{1}{2}c^{-1}(\theta_z+k),$$

$$\lambda_{2kn}=\left(\frac{2\pi n}{c}\right)^2\left[\frac{2-a^2}{c}\right]+\frac{2\pi n}{c}\left[\frac{\pi}{2c^2}(2-a^2)+\frac{z_0}{c^2}(2-a^2)+i\frac{a^2}{c}\right]+d_0c^{-1}\gamma''(z_0)k^2+$$
$$+2d_0c^{-1}\gamma''(z_0)\theta_z k+B_1,$$

$$B_1=c^{-1}d_1\gamma_0+c^{-1}d_0\gamma''(z_0)\theta_z^2+\frac{\pi^2}{4c^3}(2-a^2)+\frac{\pi^2}{2c^3}z_0(2-a^2)+\frac{z^2}{4c^3}(2-a^2)-\frac{ia^2}{2c^2}(\pi+z_0).$$

Lemma 8. *Let condition (19) be satisfied and*

$$0<a^2<2. \tag{63}$$

Then, $\omega_0>0$, and for the roots $\lambda_{kn}(\varepsilon)$ $(k,n=0,\pm1,\pm2,\ldots)$ of Equation (17), the real parts of which tend to zero as $\varepsilon\to0$, the asymptotic equations are fulfilled:

$$\lambda_{kn}(\varepsilon)=i\big[\omega_0\varepsilon^{-1}+c^{-1}\lambda_{0n}\big]+\varepsilon\left(\frac{i}{2}c^{-1}(\theta_z+k)+\lambda_{1kn}\right)+\varepsilon^2\lambda_{2kn}+\ldots, \tag{64}$$

where

$$\lambda_{0n}=\pi\left(2n+\frac{1}{2}\right)+\theta_\omega-\Omega_0+\frac{z_0}{2},\quad \varkappa=p_0\exp(i\Omega_0),$$
$$\lambda_{1kn}=-\frac{2i}{ac^2}\lambda_{0n}, \tag{65}$$

$$\lambda_{2kn}=-2c^{-3}a^{-2}\lambda_{0n}^2+d_1(cd_0)^{-1}+d_0(c\gamma_0)^{-1}\gamma''(z_0)(\theta_z+k)^2+$$
$$+(cp_0\exp(i\Omega_0))^{-1}\left[c^{-2}\lambda_{0n}^2-(2i\omega_0+a)\left(-\frac{2i}{ac}\lambda_{0n}+\frac{1}{2}(\theta_z+k)\right)\right].$$

Remark 3. *The roots $\lambda_{kn}(\varepsilon)$ of the characteristic Equation (17) allow us to determine solutions to the linear boundary value problem (15), (16):*

$$u_{kn}(t,x,\varepsilon) = \exp\left(i(z_0\varepsilon^{-1} + \theta_z + k)x + \lambda_{kn}(\varepsilon)t\right),$$

and hence, the formal set of solutions is

$$u(t,x,\varepsilon) = \sum_{k,n=-\infty}^{\infty} \left(\xi_{kn}u_{kn}(t,x,\varepsilon) + \bar{\xi}_{kn}\bar{u}_{kn}(t,x,\varepsilon)\right), \tag{66}$$

where ξ_{kn} denotes arbitrary complex constants.

This remark applies to Lemmas 5–8.

3.3. Nonlinear Analysis for $\Phi(s) = \Phi_1(s)$

Consider the cases $a^2 > 2$ and $a^2 < 2$ separately.

3.3.1. Case $a^2 > 2$

In this case, we have the equality $\omega_0 = 0, \Omega_0 = 0, p_0 = 1$. The critical case in the stability problem is defined by the equality

$$d_0|\gamma_0| = 1. \tag{67}$$

The following will be based on the representation in (66). Let us write it in a more convenient form:

$$\begin{aligned} u(t,x,\varepsilon) &= E(t,x)\sum_{k,n=-\infty}^{\infty} \xi_{kn}\exp\left(ikx + 2i\pi nc^{-1}(1-\varepsilon c^{-1}a)t + (\lambda_{2kn} + O(\varepsilon))\tau\right) = \\ &= E(t,x)\xi(\tau,x,x_1), \end{aligned} \tag{68}$$

where $\tau = \varepsilon^2 t$ is the "slow" time, $E(t,x) = \exp(i(z_0\varepsilon^{-1} + \theta_z)x + i\pi(2c)^{-1}(1-\varepsilon ac^{-1})t)$, and $\xi_{kn}(\tau) = \xi_{kn}\exp\left((\lambda_{2kn} + O(\varepsilon))\tau\right)$ denotes coefficients of the expansion of $\xi(\tau,x,x_1)$ into a Fourier series with respect to the 2π-periodic argument x and the c-periodic argument $x_1 = (1-\varepsilon c^{-1}a)t$.

Solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u = \varepsilon\left(E(t,x)\xi(\tau,x,x_1) + \overline{cc}\right) + \varepsilon^3 u_3(\tau,x,x_1) + \dots. \tag{69}$$

Here and below, $\overline{cc}$ denotes the term that is complex conjugate to the previous one. The unknown complex function $\xi(\tau,x,x_1)$ is to be defined. Let us substitute (69) into (14) and collect the coefficients of the same powers of ε. Then, at the first power of ε, we obtain an identity. Equating the coefficients of ε^3, we arrive at the equation

$$\begin{aligned} c\frac{\partial\xi}{\partial\tau} =& \left(\frac{1}{2}a^2 - 1\right)\frac{\partial^2\xi}{\partial x_1^2} + (2\gamma_0)^{-1}\gamma''(z_0)\frac{\partial^2\xi}{\partial x^2} - i\gamma_0^{-1}\gamma''(z_0)\theta_z\frac{\partial\xi}{\partial x} + \\ &+ ic^{-1}\left(a^2 - \frac{\pi}{2}\left(1 - \frac{1}{2}a^2\right)\right)\frac{\partial\xi}{\partial x_1} + B_0\xi + 3b_1\xi|\xi|^2, \\ B_0 =& c^{-2}\pi^2\frac{1}{4}\left(1 - \frac{1}{2}a^2\right) + ia^2c^{-2}\frac{1}{2}\pi + \frac{1}{2}\gamma''(z_0)\theta_z^2 + 2d_1c^{-1}\gamma_0 \end{aligned} \tag{70}$$

with the boundary conditions

$$\xi(\tau,x,x_1 + c) \equiv \xi(\tau,x,x_1) \equiv \xi(\tau,x + 2\pi,x_1). \tag{71}$$

Here, the following relations are taken into account:

$$\begin{aligned}\frac{d\xi}{dt} =&\varepsilon^2\frac{\partial\xi}{\partial\tau}+\frac{\partial\xi}{\partial x_1}(1-\varepsilon ac^{-1}),\\ \xi_{t-c} =&\xi(\tau-\varepsilon^2c,x,x_1-c(1-\varepsilon ac^{-1}))=\\ =&\xi(\tau,x,x_1)-\varepsilon^2c\frac{\partial\xi}{\partial\tau}+\varepsilon a\frac{\partial\xi}{\partial x_1}+\frac{1}{2}\varepsilon^2a^2\frac{\partial^2\xi}{\partial x_1^2}+o(\varepsilon^2).\end{aligned}$$

Let us introduce the following notation. We arbitrarily fix the value $\theta_{0z}\in[0,1)$ and let $\varepsilon_n=\varepsilon_n(\theta_{0z})$ denote a sequence for which $\varepsilon_n\to 0$ as $n\to\infty$ and $\theta_z(\varepsilon_n,\theta_{0z})=\theta_{0z}$. The above constructions justify the following result.

Theorem 5. *Let $a^2>2$ and conditions (53) and (67) be satisfied. Let $\theta_{0z}\in[0,1)$ be arbitrarily fixed, and let the boundary value problem (70), (71) at $\theta_z=\theta_{0z}$ have a bounded solution $\xi(\tau,x,x_1)$ as $\tau\to\infty$, $x\in[0,2\pi]$, $x_1\in[0,c]$. Then, the function $u(t,x,\varepsilon)=\varepsilon\big(E(t,x)\xi(\tau,x,x_1)+\overline{cc}\big)+\varepsilon^3u_3(\tau,x,x_1)$ satisfies the boundary value problem (14), (16) with accuracy up to $o(\varepsilon^3)$.*

Thus, the parabolic boundary value problem (70), (71) is a quasinormal form for the boundary value problem (14), (16).

3.3.2. Case $a^2<2$

The dynamical properties in this case are much more complicated. The principal parts of the roots $\lambda_{kn}(\varepsilon)$ of the characteristic equation are close to $i\omega_0\varepsilon^{-1}$: i.e., they are asymptotically large. Therefore, the oscillations in the boundary value problem (14), (16) will be rapid.

Note that, in this case,

$$\omega_0=\left(1-\frac{a^2}{2}\right)^{1/2},\quad p_0=\frac{a^2}{2}(4-a^2)^{1/2},\quad d_0=p_0|\gamma_0|^{-1}. \tag{72}$$

The roots of $\lambda_{kn}(\varepsilon)$ correspond to the Euler solutions of the linear boundary value problem (15), (16):

$$u_{kn}^{\pm}(t,x,\varepsilon)=\exp\big(\pm i(z_0\varepsilon^{-1}+\theta_z+k)x+\lambda_{kn}(\varepsilon)t\big).$$

It is more convenient to write these functions in the form

$$u_{kn}^{\pm}(t,x,\varepsilon)=E^{\pm}(t,x)\exp\big(ikx+2i\pi nx_1+(\lambda_{2kn}+O(\varepsilon))\tau\big),$$

where

$$\begin{aligned}E^{\pm}(t,x)=&\exp\left(i(c^{-1}\omega_0\varepsilon^{-1}+c^{-1}\left(\theta_\omega-\Omega_0+\frac{\pi}{2}\right)+\varepsilon c^{-1}\varkappa^{-1}(2\omega_0-ia)\cdot\right.\\ &\left.\cdot ic^{-1}(\theta_\omega-\Omega_0))t\pm i(z_0\varepsilon^{-1}+\theta_z)x\right),\end{aligned}$$

$$R=\varkappa^{-1}(2\omega_0-ia)\cdot ic^{-1}(\theta_\omega-\Omega_0),\quad \Im R=0,$$

$\tau=\varepsilon^2t$, $x_1=(1-\varepsilon c^{-1}R)t$. Hence, we find that

$$\begin{aligned}\sum_{k,n=-\infty}^{\infty}\xi_{kn}^{\pm}u_{kn}^{\pm}(t,x,\varepsilon)=E^{\pm}(t,x)\sum_{k,n=-\infty}^{\infty}\xi_{kn}^{\pm}(\tau)\exp\big(ikx+2i\pi nx_1\big)=\\ =E^{\pm}(t,x)\xi^{\pm}(\tau,x,x_1).\end{aligned}$$

Here, $\xi_{kn}^{\pm}$ denotes arbitrary complex constants, and $\xi_{kn}^{\pm}(\tau) = \xi_{kn}^{\pm} \exp\big((\lambda_{2kn} + O(\varepsilon))\tau\big)$. The functions $\xi_{kn}^{\pm}(\tau)$ are the Fourier coefficients of the function $\xi^{\pm}(\tau, x, x_1)$, which is 2π-periodic with respect to x and c-periodic with respect to x_1.

Solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u(t,x) = u^{+}(t,x) + u^{-}(t,x), \tag{73}$$

$$u^{\pm}(t,x) = \varepsilon\big(\xi^{\pm}(\tau,x,x_1)E^{\pm}(t,x) + \overline{cc}\big) + \varepsilon^3 u_3(t,\tau,x,x_1) + \dots,$$

where the dependencies on t, x and x_1 are periodic.

Let us substitute (73) into (14) and equate the coefficients of the same powers of ε. In the first step, we obtain an identity, and by collecting the coefficients of ε^3, we obtain the equation for u_3. From its solvability condition in the specified class of functions, we arrive at the relation

$$\frac{\partial \xi^{\pm}}{\partial \tau} = A_1 \frac{\partial^2 \xi^{\pm}}{\partial x_1^2} + A_2 \frac{\partial \xi^{\pm}}{\partial x_1} + A_3 \xi^{\pm} + B_1 \frac{\partial^2 \xi^{\pm}}{\partial x^2} + B_2 \frac{\partial \xi^{\pm}}{\partial x} + c^{-1}\beta \xi^{\pm}(|\xi^{\pm}|^2 + 2|\xi^{\mp}|^2), \tag{74}$$

in which

$$\begin{aligned}
A_1 =& -c^{-3}\left[\varkappa^{-1} - \frac{1}{2}(ia - 2\omega_0)^2 \varkappa^{-2}\right],\\
A_2 =& c^{-3}\Big[-2(\varkappa^{-1} - \frac{1}{2}(ia - 2\omega_0)^2 \varkappa^{-2}(\theta_\omega - \Omega_0)) + c\varkappa^{-2} 2\omega_0(2\omega_0 - ia) +\\
&+ c^2 \varkappa^{-1} a(\theta_\omega - \Omega_0)\Big],\\
A_3 =& c^{-3}\left[\frac{1}{2}(ia - 2\omega_0)\varkappa^2(\theta_\omega - \Omega_0)^2 - \varkappa^{-1}\right] + d_1 c^{-1} p_0^{-1} +\\
&+ \frac{1}{2}\gamma''(z_0) c^{-1} \theta_z^2 (p_0\gamma_0)^{-1} - i(c\varkappa)^{-2} 2i\omega_0(2\omega_0 - ia)\cdot\\
&\cdot(\theta_\omega - \Omega_0) - i(c\varkappa)^{-1} a(\theta_\omega - \Omega_0),\\
B_1 =& \frac{1}{2} c^{-1}\gamma''(z_0)(p_0\gamma_0)^{-1},\\
B_2 =& c^{-1}\gamma''(z_0)\theta_z(p_0\gamma_0)^{-1},\\
\beta =& b_1 + i\omega_0 b_2 - \omega_0^2 b_3 - i\omega_0^3 b_4.
\end{aligned}$$

Recall that the function $\xi(\tau, x, x_1)$ satisfies the boundary conditions

$$\xi(\tau, x, x_1 + c) \equiv \xi(\tau, x, x_1) \equiv \xi(\tau, x + 2\pi, x_1). \tag{75}$$

In order to formulate the final result, we introduce some notations. We arbitrarily fix $\theta_{0\omega} \in [0, 2\pi)$ and let the sequence $\varepsilon_s = \varepsilon_s(\theta_{0\omega})$ be defined by the condition $\theta_\omega(\varepsilon_s(\theta_{0\omega})) = \theta_{0\omega}$ $(s = 1, 2, \dots)$. We let $\Gamma(\theta_{0\omega})$ denote all limit points of the sequence $\theta_z(\varepsilon_s(\theta_{0\omega}))$ from the interval $[0, 1]$. We let θ_{0z} denote the limit element of $\Gamma(\theta_{0\omega})$ and let the subsequence ε_{s_Γ} of the sequence ε_s be such that

$$\lim_{\Gamma\to\infty} \theta_z(\varepsilon_{s_\Gamma}) = \theta_{0z}.$$

Note that it is possible that the set $\Gamma(\theta_{0\omega})$ coincides with the segment $[0, 1]$, and it is possible that this set consists of a single element.

Theorem 6. *Let $0 < a^2 < 2$ and $d_0 = p_0|\gamma_0|^{-1}$. We arbitrarily fix $\theta_{0\omega} \in [0, 2\pi)$ and let $\theta_{0z} \in \Gamma(\theta_{0\omega})$. Let $\xi^{\pm}(\tau, x, x_1)$ be a bounded solution of the boundary value problem (74), (75) as $\tau \to \infty$, $x \in [0, 2\pi]$, $x_1 \in [0, c]$. Then, the function*

$$u(t,x,\varepsilon) = \varepsilon\big(\xi^{+}(\tau,x,x_1)E^{+}(t,x) + \overline{cc} + \xi^{-}(\tau,x,x_1)E^{-}(t,x) + \overline{cc}\big) + \varepsilon^3 u_3(t,\tau,x,x_1)$$

satisfies the boundary value problem (14), (16) *up to* $o(\varepsilon_{s_\Gamma}^3)$ *for* $\tau = \varepsilon^2 t$, $x_1 = (1 - \varepsilon c^{-1} R)t$, *for the sequence* $\varepsilon = \varepsilon_{s_\Gamma}$.

Thus, the boundary value problem (74), (75) is a quasinormal form for the original boundary value problem (14), (16) in this critical case.

3.4. Nonlinear Analysis for $\Phi(s) = \Phi_2(s)$

And here, we consider the cases $a^2 > 2$ and $a^2 < 2$ separately.

3.4.1. Case $a^2 > 2$

In this case, we have the equality $\omega_0 = 0, \Omega_0 = 0, p_0 = 1$. The critical case in the stability problem is defined by the equality

$$d_0|\gamma_0| = p_0. \tag{76}$$

The following will be based on the representation in (66). Let us write it in a more convenient form:

$$u(t,x,\varepsilon) = E(t,x)\sum_{k,n=-\infty}^{\infty}\xi_{kn}\exp\left(ikx + 2i\pi nc^{-1}(1-\varepsilon c^{-1}a)t + (\lambda_{2kn} + O(\varepsilon))\tau\right) = \\ = E(t,x)\xi(\tau,x,x_1), \tag{77}$$

where $\tau = \varepsilon^2 t$ is the "slow" time,

$$E(t,x) = \exp\left(i(\omega_0\varepsilon^{-1} + \theta_\omega - \Omega_0 + \frac{1}{2}(z_0 + \pi + \varepsilon\theta_z)) + i(z_0\varepsilon^{-1} + \theta_z)x + i\pi(2c)^{-1}(1 - \varepsilon a c^{-1})t\right),$$

and $\xi_{kn}(\tau) = \xi_{kn}\exp\left((\lambda_{2kn} + O(\varepsilon))\tau\right)$ denotes coefficients of the expansion of $\xi(\tau,x,x_1)$ into a Fourier series with respect to the 2π-periodic argument x and the c-periodic argument $x_1 = (1 - \varepsilon c^{-1}a)t$.

The solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u = \varepsilon\left(E(t,x)\xi(\tau,x,x_1) + \overline{cc}\right) + \varepsilon^3 u_3(\tau,x,x_1) + \dots. \tag{78}$$

Here and below, $\overline{cc}$ denotes the term that is complex conjugate to the previous one. The unknown complex function $\xi(\tau,x,x_1)$ is to be defined. Let us substitute (78) into (14) and collect the coefficients of the same powers of ε. Then, at the first power of ε, we obtain an identity. Equating the coefficients of ε^3, we arrive at the equation

$$c\frac{\partial\xi}{\partial\tau} = \left(\frac{1}{2}a^2 - 1\right)\frac{\partial^2\xi}{\partial x_1^2} + (2\gamma_0)^{-1}\gamma''(z_0)\frac{\partial^2\xi}{\partial x^2} - i\gamma_0^{-1}\gamma''(z_0)\theta_z\frac{\partial\xi}{\partial x} + \\ + ic^{-1}\left(a^2 - \frac{\pi}{2}\left(1 - \frac{1}{2}a^2\right)\right)\frac{\partial\xi}{\partial x_1} + B_0\xi + 3b_1\xi|\xi|^2, \tag{79}$$

$$B_0 = c^{-2}\pi^2\frac{1}{4}\left(1 - \frac{1}{2}a^2\right) + ia^2c^{-2}\frac{1}{2}\pi + \frac{1}{2}\gamma''(z_0)\theta_z^2 + 2d_1c^{-1}\gamma_0$$

with the boundary conditions

$$\xi(\tau,x,x_1 + c) \equiv \xi(\tau,x,x_1) \equiv \xi(\tau,x + 2\pi,x_1). \tag{80}$$

Let us introduce some notation. We arbitrarily fix the value $\theta_{0z} \in [0,1)$ and let $\varepsilon_n = \varepsilon_n(\theta_{0z})$ denote a sequence for which $\varepsilon_n \to 0$ as $n \to \infty$ and $\theta_z(\varepsilon_n, \theta_{0z}) = \theta_{0z}$. The above constructions justify the following result.

Theorem 7. *Let* $a^2 > 2$ *and conditions* (53) *and* (76) *be satisfied. Let* $\theta_{0z} \in [0,1)$ *be arbitrarily fixed, and let the boundary value problem* (79), (80) *at* $\theta_z = \theta_{0z}$ *have a bounded solution* $\xi(\tau,x,x_1)$

as $\tau \to \infty$, $x \in [0, 2\pi]$, $x_1 \in [0, c]$. *Then, the function* $u(t,x,\varepsilon) = \varepsilon\big(E(t,x)\xi(\tau,x,x_1) + \overline{cc}\big) + \varepsilon^3 u_3(\tau,x,x_1)$ *satisfies the boundary value problem* (14), (16) *with accuracy up to* $o(\varepsilon^3)$.

Thus, the parabolic boundary value problem (79), (80) is a quasinormal form for the boundary value problem (14), (16).

3.4.2. Case $a^2 < 2$

The principal parts of the roots $\lambda_{kn}(\varepsilon)$ of the characteristic equation are close to $i\omega_0\varepsilon^{-1}$: i.e., they are asymptotically large. Therefore, the oscillations in the boundary value problem (14), (16) will be rapid.

Note that, in this case,

$$\omega_0 = \left(1 - \frac{a^2}{2}\right)^{1/2}, \quad p_0 = \frac{a^2}{2}(4 - a^2)^{1/2}, \quad d_0 = p_0|\gamma_0|^{-1}. \tag{81}$$

The roots of $\lambda_{kn}(\varepsilon)$ correspond to the Euler solutions of the linear boundary value problem (15), (16):

$$u^{\pm}_{kn}(t,x,\varepsilon) = \exp\big(\pm i(z_0\varepsilon^{-1} + \theta_z + k)x + \lambda_{kn}(\varepsilon)t\big).$$

It is more convenient to write these functions in the form

$$u^{\pm}_{kn}(t,x,\varepsilon) = E^{\pm}(t,x)\exp\big(ikx + 2i\pi n x_1 + (\lambda_{2kn} + O(\varepsilon))\tau\big),$$

where

$$E^{\pm}(t,x) = \exp\left(i(c^{-1}\omega_0\varepsilon^{-1} + c^{-1}\left(\theta_\omega - \Omega_0 + \frac{\pi}{2}\right) + \varepsilon c^{-1}\varkappa^{-1}(2\omega_0 - ia)\cdot \right.$$
$$\left. \cdot\, ic^{-1}(\theta_\omega - \Omega_0))t \pm i(z_0\varepsilon^{-1} + \theta_z)x\right),$$

$$R = \varkappa^{-1}(2\omega_0 - ia)\cdot ic^{-1}(\theta_\omega - \Omega_0) = \frac{2}{a}, \quad \Im R = 0,$$

$\tau = \varepsilon^2 t$, $x_1 = (1 - \varepsilon c^{-1}R)t$. Hence we find that

$$\sum_{k,n=-\infty}^{\infty} \xi^{\pm}_{kn} u^{\pm}_{kn}(t,x,\varepsilon) = E^{\pm}(t,x)\sum_{k,n=-\infty}^{\infty} \xi^{\pm}_{kn}(\tau)\exp\big(ikx + 2i\pi n x_1\big) =$$
$$= E^{\pm}(t,x)\xi^{\pm}(\tau,x,x_1).$$

Here, $\xi^{\pm}_{kn}$ denotes arbitrary complex constants, and $\xi^{\pm}_{kn}(\tau) = \xi^{\pm}_{kn}\exp\big((\lambda_{2kn} + O(\varepsilon))\tau\big)$. The functions $\xi^{\pm}_{kn}(\tau)$ are the Fourier coefficients of the function $\xi^{\pm}(\tau,x,x_1)$, which is 2π-periodic with respect to x and c-periodic with respect to x_1.

The solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u(t,x) = u^{+}(t,x) + u^{-}(t,x), \tag{82}$$

$$u^{\pm}(t,x) = \varepsilon\big(\xi^{\pm}(\tau,x,x_1)E^{\pm}(t,x) + \overline{cc}\big) + \varepsilon^3 u_3(t,\tau,x,x_1) + \dots,$$

where the dependencies on t, x and x_1 are periodic.

Let us substitute (82) into (14) and equate the coefficients of the same powers of ε. In the first step, we obtain an identity, whereas, by collecting the coefficients of ε^3, we obtain the equation for u_3. From its solvability condition in the specified class of functions, we arrive at the relation

$$c\frac{\partial\xi^{\pm}}{\partial\tau} = A_1\frac{\partial^2\xi^{\pm}}{\partial x_1^2} + A_2\frac{\partial\xi^{\pm}}{\partial x_1} + A_3\frac{\partial^2\xi^{\pm}}{\partial x^2} + A_4\frac{\partial\xi^{\pm}}{\partial x} + A_5\xi + \beta\xi^{\pm}(|\xi^{\pm}|^2 + 2|\xi^{\mp}|^2), \tag{83}$$

in which

$$A_1 = 2a^2 - \left(p_0 \exp(i\Omega_0)\right)^{-1}, \quad \Re A_1 > 0,$$
$$A_2 = 2iA_1 lc^{-1} + 4a^{-2},$$
$$A_3 = d_0\gamma_0^{-1}\gamma''(z_0),$$
$$A_4 = -2i\theta_z d_0\gamma_0^{-1}\gamma''(z_0) - 2(a^2c)^{-1},$$
$$A_5 = A_1 l^2 + d_1 d_0^{-1} + d_0\gamma_0^{-1}\gamma''(z_0)\theta_z^2 - 4i(a^2c)^{-1}l + a^{-1}\theta_z,$$
$$l = \theta_\omega - \Omega_0 + \frac{1}{2}(\pi + z_0),$$
$$\beta = b_1 + i\omega_0 b_2 - \omega_0^2 b_3 - i\omega_0^3 b_4.$$

Recall that the function $\xi(\tau, x, x_1)$ satisfies the boundary conditions

$$\xi(\tau, x, x_1 + c) \equiv \xi(\tau, x, x_1) \equiv \xi(\tau, x + 2\pi, x_1). \tag{84}$$

In order to formulate the final result, we introduce some notations. We arbitrarily fix $\theta_{0\omega} \in [0, 2\pi)$ and let the sequence $\varepsilon_s = \varepsilon_s(\theta_{0\omega})$ be defined by the condition $\theta_\omega(\varepsilon_s(\theta_{0\omega})) = \theta_{0\omega}$ $(s = 1, 2, \ldots)$. Let $\Gamma(\theta_{0\omega})$ denote all limit points of the sequence $\theta_z(\varepsilon_s(\theta_{0\omega}))$ from the interval $[0, 1]$. Let θ_{0z} denote the limit element of $\Gamma(\theta_{0\omega})$, and let the subsequence ε_{s_Γ} of the sequence ε_s be such that

$$\lim_{\Gamma\to\infty} \theta_z(\varepsilon_{s_\Gamma}) = \theta_{0z}.$$

We formulate the final result.

Theorem 8. *Let $0 < a^2 < 2$ and $d_0 = p_0|\gamma_0|^{-1}$. We arbitrarily fix $\theta_{0\omega} \in [0, 2\pi)$ and let $\theta_{0z} \in \Gamma(\theta_{0\omega})$. Let $\xi^{\pm}(\tau, x, x_1)$ be a bounded solution of the boundary value problem (74), (75) as $\tau \to \infty$, $x \in [0, 2\pi]$, $x_1 \in [0, c]$. Then, the function*

$$u(t, x, \varepsilon) = \varepsilon\left(\xi^+(\tau, x, x_1)E^+(t, x) + \overline{c}\overline{c} + \xi^-(\tau, x, x_1)E^-(t, x) + \overline{c}\overline{c}\right) + \varepsilon^3 u_3(t, \tau, x, x_1)$$

satisfies the boundary value problem (14), (16) up to $o(\varepsilon_{s_\Gamma}^3)$ for $\tau = \varepsilon^2 t$, $x_1 = (1 - \varepsilon c^{-1}R)t$, for the sequence $\varepsilon = \varepsilon_{s_\Gamma}$.

Thus, the boundary value problem (74), (75) is a quasinormal form for the original boundary value problem (14), (16) in this critical case.

3.5. Quasinormal Forms in the Case of Small Values of the Parameter σ

Here, we assume that for each fixed $\sigma_1 > 0$, the following condition is satisfied:

$$\sigma = \varepsilon\sigma_1. \tag{85}$$

Let us separately consider the cases where $\Phi(s) = \Phi_1(s)$ and $\Phi(s) = \Phi_2(s)$.

3.5.1. Building a Quasinormal Form under the Condition $\Phi(s) = \Phi_1(s)$ and $A^2 > 2$

Under condition (85) for the function $\gamma(z)$, we have the following:

$$\gamma(z) = 2i(\sin z) \cdot \exp\left(-\frac{1}{2}\varepsilon^2\sigma_1^2 z^2\right).$$

Set

$$\gamma_0(z) = 2i \sin z.$$

Then,

$$\gamma(z) = \gamma_0(z)\left(1 - \frac{1}{2}\varepsilon^2\sigma_1^2 z^2 + O(\varepsilon^4)\right).$$

The largest value $|\gamma_0(z)| = 2$, and for all values $z_m^{\pm}$ at which this value is reached, we have the equations

$$z_m^{\pm} = \pi\left(2m \pm \frac{1}{2}\right), \quad m = 0, \pm1, \pm2, \ldots.$$

Recall that $\varepsilon = 2\pi N^{-1}$. Consider the sets of integers $\pi(2m \pm 1/2)\varepsilon^{-1} + k$, $k, m = 0, \pm1, \pm2, \ldots$. We let $u_{kmn}(t, x)$ denote the Euler solutions of the linear problem (15), (16):

$$u_{kmn}^{\pm}(t, x) = \exp\left[i\left(\pi\left(2m + \frac{1}{2}\right)\varepsilon^{-1} + \theta_{zm} + k\right)x + \lambda_{kmn}^{\pm}(\varepsilon)t\right].$$

Here, $\lambda_{kmn}^{\pm}(\varepsilon)$ denotes the roots of the characteristic Equation (17) whose real parts tend to zero as $\varepsilon \to 0$. Note that

$$\theta_{zm} = \begin{cases} 0, & \text{N = 4P} \\ 3/4, & \text{N = 4P + 1} \\ 1/2, & \text{N = 4P + 2} \\ 1/4, & \text{N = 4P + 3.} \end{cases}$$

Recall that, for $a^2 > 2$, we have $\omega_0 = \Omega_0 = 0$, $p_0 = 1, |\gamma_0| = 2, d_0 = 1/2$. Let us first consider the asymptotics of $\lambda_{kmn}^{\pm}(\varepsilon)$.

Lemma 9. *Let conditions* (53), (54) *and* (85) *be satisfied. Then, there are the asymptotic relations*

$$\lambda_{kmn}^{\pm}(\varepsilon) = c^{-1}i\pi\left(2n \pm \frac{1}{2}\right) + \varepsilon\lambda_{1kmn}^{\pm} + \varepsilon^2\lambda_{2kmn}^{\pm} + \ldots,$$

where

$$\begin{aligned} \lambda_{1kmn}^{\pm} =& iac^{-2}\pi\left(2n \pm \frac{1}{2}\right), \\ c\lambda_{2kmn}^{\pm} =& \frac{1}{2}(2 - a^2)c^{-2}\pi^2\left(2n \pm \frac{1}{2}\right)^2 - ia^2c^{-2}\pi\left(2n \pm \frac{1}{2}\right) + 4d_1 - \frac{1}{2}\sigma_1^2\pi^2\left(2m \pm \frac{1}{2}\right)^2 - \\ & - \frac{1}{4}(k \pm \theta_{zm})^2. \end{aligned}$$

The set of Euler solutions of the linear boundary value problem (15), (16)

$$u^{\pm}(t, x, \varepsilon) = \sum_{k,m,n=-\infty}^{\infty} \xi_{kmn}^{\pm} \exp\left[i\left(\pi\left(2m \pm \frac{1}{2}\right)\varepsilon^{-1} \pm \theta_{zm} + k\right)x + \lambda_{kmn}^{\pm}(\varepsilon)t\right]$$

can be written in the form

$$\begin{aligned} u^{\pm}(t, x, \varepsilon) =& E^{\pm}(t, x) \sum_{k,m,n=-\infty}^{\infty} \xi_{kmn}^{\pm}(\tau) \exp\left[ikx + 2i\pi nc^{-1}x_1 + 2i\pi my\right] = \\ & = \xi^{\pm}(\tau, x, x_1, y). \end{aligned} \tag{86}$$

Here,

$$E^{\pm}(t, x) = \exp\left[\pm i\frac{\pi}{2}\left(c^{-1}(1 - \varepsilon ac^{-1})t + (\varepsilon^{-1} + \theta_{zm})x\right)\right],$$

$$\xi_{kmn}^{\pm}(\tau) = \xi_{kmn}^{\pm} \exp\left(\lambda_{2kmn}^{\pm} + O(\varepsilon)\tau\right), \quad y = x\varepsilon^{-1}, \quad x_1 = (1 + \varepsilon ac^{-1})t.$$

Given that $E^-(t,x) = \overline{E}^+(t,x)$, we will look for solutions of the nonlinear boundary value problem (14), (16) in the form of

$$u(t,x) = \varepsilon\left(E(t,x)\xi(\tau,x,x_1,y) + \overline{c}\overline{c}\right) + \varepsilon^3 u_3(\tau,x,x_1,y) + \dots. \tag{87}$$

Substituting (87) into (14) and performing the standard steps, we arrive at the boundary value problem for determining the unknown function $\xi(\tau,x,y,x_1)$:

$$c\frac{\partial\xi}{\partial\tau} = -\left(1-\frac{1}{2}a^2\right)\frac{\partial^2\xi}{\partial x_1^2} + \frac{\partial\xi}{\partial x_1}\left(\frac{a^2}{c} - \frac{i\pi}{c}\left(1-\frac{1}{2}a^2\right)\right) + \frac{\partial^2\xi}{\partial x^2} + i\theta_z\cdot\frac{\partial\xi}{\partial x} - \frac{\sigma_1^2}{2}\cdot\frac{\partial^2\xi}{\partial y^2} +$$
$$+ i\sigma_1^2\pi^2\cdot\frac{\partial\xi}{\partial y} + \left(2d_1 + \frac{\pi^2}{4c^2}\left(1-\frac{1}{2}a^2\right) + i\frac{a^2\pi^2}{c^2} - \frac{1}{2}\theta_z^2 - \frac{1}{8}\sigma_1^2\pi^2\right)\xi + b_1\xi|\xi|^2, \tag{88}$$

$$\xi(\tau,x+2\pi,x_1,y) \equiv \xi(\tau,x,x_1,y) \equiv \xi(\tau,x,x_1+c,y) \equiv \xi(\tau,x,x_1,y+1). \tag{89}$$

As a result of the above constructions, we come to the justification of the following result.

Theorem 9. *Let conditions* (53), (54) *and* (85) *be satisfied. Let* $\theta_{z0} \in [0,1)$ *be arbitrarily fixed, and let* $\xi(\tau,x,x_1,y)$ *be a bounded solution of the boundary value problem* (88)–(89) *as* $\tau\to\infty$, $x\in[0,2\pi]$, $y\in[0,1]$, $x_1\in[0,c]$. *Then, for the sequence* $\varepsilon_s(\theta_z(\varepsilon_s(\theta_{z0}) = \theta_{z0})$, *the function*

$$u(t,x) = \varepsilon\left(E(t,x)\xi(\tau,x,x_1,y) + \overline{c}\overline{c}\right) + \varepsilon^3 u_3(\tau,x,x_1,y)$$

satisfies the boundary value problem of (14), (16) *up to* $o(\varepsilon_s^3)$ *at* $\theta_z = \theta_{z0}$.

3.5.2. Building Quasinormal Forms under the Conditions $\Phi(s) = \Phi_2(s)$, $\sigma = \varepsilon\sigma_1$ and $A^2 > 2$

The values of the parameter z for which the critical cases are realized are determined by the following relation:

$$z_m = \pi(2m+1); \quad m = 0, \pm1, \pm2, \dots.$$

Thus, $\gamma_0(z_m) = -2$ and $p_0 = 1$, $\omega_0 = \Omega_0 = 0$, $d_0 = 1/2$. It follows from the condition $\varepsilon = 2\pi N^{-1}$ that

$$\theta_z = \theta_{zm} = \begin{cases} 0, & \text{if N } - \text{ even,} \\ \dfrac{1}{2}, & \text{if N } - \text{ odd.} \end{cases}$$

Below, we separately consider the cases where $\theta_z = 0$ and where $\theta_z = 1/2$.

3.5.3. Building Quasinormal Forms for $\theta_z = 0$

For the roots $\lambda_{kmn}(\varepsilon)$ $(k,m,n = 0,\pm1,\pm2,\dots)$ of the characteristic Equation (17) whose real parts tend to zero as $\varepsilon\to0$, the following asymptotic equality takes place:

$$\lambda_{kmn}(\varepsilon) = i\pi(2n+1) + \varepsilon\lambda_{1kmn} + \varepsilon^2\lambda_{2kmn} + \dots, \tag{90}$$
$$\lambda_{1kmn} = -iac^{-2}\pi(2n+1) + \frac{1}{2}ic^{-1}k,$$
$$c\lambda_{2kmn} = \left(1-\frac{1}{2}a^2\right)(\pi(2n+1)c^{-1})^2 - \frac{1}{8}k^2 - \frac{1}{2}\sigma_1^2(\pi(2m+1))^2 - \frac{1}{2}a\pi(2n+1)k +$$
$$+ ia^2c^{-1}\pi(2n+1) - ia(2c)^{-1}k + 2d_1.$$

The solutions of the linear boundary value problem (15), (16) can then be written in the form

$$\begin{aligned} u(t,x) = &\sum_{k,m,n=-\infty}^{\infty} \xi_{kmn} \exp\left[i\pi(2m+1)\varepsilon^{-1}x + kx + \left(i\pi(2n+1)(1-\varepsilon ac^{-1}) - \right.\right. \\ &\left.\left. - \varepsilon a(2c)^{-1}k\right)t + (\lambda_{2kmn} + O(\varepsilon))\tau\right] = \\ &= \sum_{k,m,n=-\infty}^{\infty} \xi_{kmn}(\tau) \exp\left[i\pi(2m+1)y + i\pi(2n+1)x_1 + ikx_2\right] = \\ &= \xi(\tau, x_1, x_2, y), \end{aligned} \tag{91}$$

where

$$\begin{aligned} &\xi_{kmn}(\tau) = \xi_{kmn} \cdot \exp[(\lambda_{2kmn} + O(\varepsilon))\tau], \\ &x_1 = (1 - \varepsilon ac^{-1})t, \quad x_2 = x - \varepsilon a(2c)^{-1}t, \quad y = x\varepsilon^{-1}. \end{aligned} \tag{92}$$

Based on equality (91), we seek solutions to the nonlinear boundary value problem (14), (16) of the form

$$u(t,x,\varepsilon) = \varepsilon\xi(\tau, x_1, x_2, y) + \varepsilon u_3(\tau, x_1, x_2, y) + \dots.$$

Substituting this expression into (14) and performing the standard steps, we arrive at the parabolic boundary value problem for finding a real function $\xi(\tau, x_1, x_2, y)$:

$$\begin{aligned} c\frac{\partial\xi}{\partial\tau} = &\left(\frac{1}{2}a^2 - 1\right)\frac{\partial^2\xi}{\partial x_1^2} + \frac{1}{8}\frac{\partial^2\xi}{\partial x_2^2} + \frac{\sigma_1^2}{2}\cdot\frac{\partial^2\xi}{\partial y^2} - \frac{1}{2}\frac{\partial^2\xi}{\partial x_1\partial x_2} + ac^{-1}\frac{\partial\xi}{\partial x_1} - \\ &- a(2c)^{-1}\frac{\partial\xi}{\partial x_2} + 2d_1\xi + b_1\xi|\xi|^2, \end{aligned} \tag{93}$$

with the boundary conditions

$$-\xi(\tau, x_1 + c, x_2, y) \equiv \xi(\tau, x_1, x_2, y) \equiv \xi(\tau, x_1, x_2 + 2\pi, y), \tag{94}$$

$$-\xi(\tau, x_1, x_2, y + 1) \equiv \xi(\tau, x_1, x_2, y). \tag{95}$$

This boundary value problem is a quasinormal form in the considered case.

3.5.4. Quasinormal Form for $\theta_z = 1/2$

In this case, let us give the following formulas for the elements of λ_{1kmn} and λ_{2kmn} :

$$\begin{aligned} &\lambda_{1kmn} = -iac^{-2}\pi(2n+1) + \frac{1}{2}ic^{-1}\left(\frac{1}{2} + k\right), \\ &c\lambda_{2kmn} = \left(1 - \frac{1}{2}a^2\right)(\pi(2n+1)c^{-1})^2 - \frac{1}{8}k^2 - \frac{1}{2}\sigma_1^2(\pi(2m+1))^2 - \\ &- \frac{1}{2}a\pi(2n+1)c^{-1}\left(\frac{1}{2} + k\right) + ia^2c^{-2}\pi(2n+1) - ia(2c)^{-1}\left(\frac{1}{2} + k\right) + 2d_1. \end{aligned}$$

The "critical" solutions of the linear boundary value problem (15), (16) can be written in the form

$$\begin{aligned} u(t,x) = &E(t,x)\sum_{k,m,n=-\infty}^{\infty} \xi_{kmn} \exp\left[i\pi(2m+1)y + i\pi(2n+1)x_1 + ikx_2\right] = \\ &= E(t,x)\xi(\tau, x_1, x_2, y), \end{aligned}$$

where

$$E(t,x) = \exp\left[i\frac{1}{2}(x + \varepsilon(2c)^{-1}t)\right] = \exp\left[i\frac{1}{2}x_2\right].$$

Therefore, solutions of the nonlinear boundary value problem (14), (16) are sought in the form

$$u(t,x,\varepsilon) = \varepsilon\big(E(t,x)\xi(\tau,x_1,x_2,y) + \overline{c}\overline{c}\big) + \varepsilon^3 u_3(t,\tau,x_1,x_2,y) + \dots.$$

Let us substitute this expression into (14). After straightforward calculations, we obtain a parabolic boundary value problem, namely, a quasinormal form, for finding the complex function $\xi(\tau,x_1,x_2,y)$:

$$\begin{aligned} c\frac{\partial\xi}{\partial\tau} =& \left(\frac{1}{2}a^2 - 1\right)\frac{\partial^2\xi}{\partial x_1^2} + \frac{1}{8}\frac{\partial^2\xi}{\partial x_2^2} + \frac{\sigma_1^2}{2}\cdot\frac{\partial^2\xi}{\partial y^2} - \frac{1}{2}\frac{\partial^2\xi}{\partial x_1\partial x_2} + \left(i\frac{a}{2} + ac^{-1}\right)\frac{\partial\xi}{\partial x_1} + \\ &+ \left(\frac{i}{8} - a(2c)^{-1}\right)\frac{\partial\xi}{\partial x_2} + \left(2d_1 - \frac{1}{32} - \frac{ia}{4c}\right)\xi + 3b_1\xi|\xi|^2 \end{aligned} \tag{96}$$

with boundary conditions (94) and (95).

Let us make one remark. In the right part of (96), there is no term of the form $Const\cdot E^2(t,x)\xi^3$. This is due to the fact that

$$E(t,x) = \exp\left[i\frac{1}{2}(x + (2c\varepsilon)^{-1}\tau)\right].$$

As a result of the principle of averaging over a rapidly oscillating periodic argument τ (see, e.g., [16,17]), the corresponding term in the principal term vanishes.

3.6. Building a Quasinormal Form under the Conditions $\Phi(s) = \Phi_2(s)$, $\sigma = \varepsilon\sigma_1$, $0 < a^2 < 2$

We first give the values of the coefficients $\lambda_{1,2,kmn}$ in formula (90) for the asymptotic representation of the roots $\lambda_{kmn}(\varepsilon)$ $(k,m,n = 0,\pm1,\pm2,\dots)$ of the characteristic Equation (17):

$$\begin{aligned} &\lambda_{1kmn} = -2i(ac)^{-1}R + i(2c)^{-1}(\theta_z + k),\\ &c\lambda_{2kmn} = -\frac{1}{2}\big(2a^{-1}R + \frac{1}{2}(\theta_z + k)\big)^2 + \frac{1}{2}\big(2a^{-1}R - \frac{1}{2}(\theta_z + k)\big)(\theta_z + k) -\\ &- c^{-1}(\theta_z + k)^2 + (p_0\exp(i\Omega_0))^{-1}R^2 + 2\omega_0\big(2a^{-1}R - \frac{1}{2}(\theta_z + k)\big) -\\ &- 2ic^{-1}R + ia(2c)^{-1}(\theta_z + k) - \frac{1}{2}\sigma_1^2\big(\pi(2m+1)\big)^2 + 2d_1p_0^{-1}, \end{aligned}$$

where $R = (\theta_\omega - \Omega_0 + \pi(2n+1))c^{-1}$.

Let us write the "critical" solutions of the linear boundary value problem (15), (16) in the form

$$\begin{aligned} u(t,x) =& E(t,x)\sum_{k,m,n=-\infty}^{\infty}\xi_{kmn}\exp\left[i\pi(2n+1)x_1 + ikx_2 + i\pi(2m+1)y\right] =\\ &= E(t,x)\xi(\tau,x_1,x_2,y), \end{aligned}$$

where

$$E(t,x) = \exp\left[i(\omega_0\varepsilon^{-1} + (\theta_\omega - \Omega_0)c^{-1}(1 - 2\varepsilon ac^{-1}) + \varepsilon c^{-1}\theta_z)t + i\theta_z x\right],$$

and for $x_{1,2}$ and y, the relations in (92) hold. Then, the solutions of the nonlinear boundary value problem (14), (16) are found in the form

$$u(t,x,\varepsilon) = \varepsilon\big(E(t,x)\xi(\tau,x_1,x_2,y) + \overline{c}\overline{c}\big) + \varepsilon^3 u_3(t,\tau,x_1,x_2,y) + \dots, \tag{97}$$

and the dependence on t, x_1, x_2 and y is periodic. Let us substitute (97) into (14), and in the resulting formal identity, we will successively equate the coefficients of the same powers of ε. As a result, we arrive at an equation for u_3, from the solvability condition of which we

obtain a boundary value problem for determining the unknown amplitude $\xi(\tau, x_1, x_2, y)$ in the specified class of functions:

$$c\frac{\partial \xi}{\partial \tau} = H_1\frac{\partial^2 \xi}{\partial x_1^2} + H_2\frac{\partial \xi}{\partial x_1} - \frac{1}{8}\frac{\partial^2 \xi}{\partial x_2^2} + H_3\frac{\partial \xi}{\partial x_2} + \left(\frac{1}{4} - a^{-1}\right)\frac{\partial^2 \xi}{\partial x_1 \partial x_2} + \\ + \frac{1}{2}\sigma_1^2\frac{\partial^2 \xi}{\partial y^2} + H_4\xi + 3\beta\xi|\xi|^2 \tag{98}$$

with boundary conditions (94) and (95), where

$$H_1 = -\varkappa^{-1} + \frac{1}{2}(ia - 2\omega_0)^2\varkappa^{-2},$$

$$H_2 = c^{-1}\Big[-2(\varkappa^{-1} - \frac{1}{2}(ia - 2\omega_0)^2\varkappa^{-2}(\theta_\omega - \Omega_0)) + c\varkappa^{-2}2\omega_0(2\omega_0 - ia) + \\ + c^2\varkappa^{-1}a(\theta_\omega - \Omega_0)\Big],$$

$$H_3 = i\big[2\theta_z - \big(-(2a)^{-1}(\theta_\omega - \Omega_0) + a^{-1} - \omega_0 + ia(2c)^{-1}\big)\big],$$

$$H_4 = -\frac{1}{2}\left(2(\theta_\omega - \Omega_0) + \frac{1}{2}\theta_z\right)^2 + \frac{1}{2}\left(2a^{-1}(\theta_\omega - \Omega_0) - \frac{1}{2}\theta_z\right)\theta_z - c^{-1}\theta_z^2 + \\ + \left(p_0\exp(i\Omega_0)^{-1}(\theta_\omega - \Omega_0) + 2\omega_0(2a^{-1}(\theta_\omega - \Omega_0) - \frac{1}{2}\theta_z\right) + 2ic^{-1}(\theta_\omega - \Omega_0) + \\ + ia(2c)^{-1}\theta_z + 2d_1p_0^{-1},$$

$$\beta = b_1 + i\omega_0 b_2 - \omega_0^2 b_3 - i\omega_0^3 b_4.$$

Recall that, depending on the evenness or oddness of N, the value of θ_z takes a value of 0 or $1/2$.

The main result is that the boundary value problem (94), (95), (98) obtained here plays the role of a quasinormal form for the boundary value problem (14), (16) in the above critical case.

4. Conclusions

The local dynamics of a system of coupled identical oscillators are considered. The large number of oscillators gave grounds for the transition to the consideration of the boundary value problem with a continuous spatial variable. The presence of a large delay in the couplings made it possible to use special asymptotic methods [20,21].

Critical cases in the problem of the stability of the zero equilibrium state were singled out. It was shown that all of them have infinite dimensionality, so the known methods of local analysis based on the use of methods of invariant integral manifolds and methods of normal forms [22,23] are not directly applicable. This research is based on special infinite normalization methods [24,25]. The main results include the construction of the analogs of normal forms—quasinormal forms—nonlinear equations of the parabolic type containing no small parameters. Their nonlocal dynamics determine the local dynamics of the original problem. The corresponding quasinormal forms contain two or three spatial variables, so we can conclude that the dynamics of the problems under consideration are, in general, complex. Asymptotic formulas linking the solutions of quasinormal forms and solutions of the original equation were given.

We emphasize that asymptotic approximations were constructed on an infinite time interval. Therefore, a quasinormal form requires the existence of a bounded solution on the entire axis. Most often, "quasinormal forms" are boundary value problems of the parabolic type, which have the property of local solvability. Based on the known results of the numerical analysis of such problems (see, e.g., [26]), one can often conclude that solutions bounded on the entire axis exist. However, in the present paper, we do not talk

about the asymptotics of exact solutions of the original system, but about the asymptotic approximation of functions satisfying the original system with a certain degree of accuracy. Of course, one can formulate conclusions about determining the asymptotics of solutions by means of solutions of a quasinormal form on a finite $O(\varepsilon^{-1})$-order time-varying interval, especially since the dependence on the time variable $x_1 = c^{-1}(1 - \varepsilon c^{-1}a)t$ is periodic.

It is interesting to note that, in the case of $a^2 > 2$, the quasinormal forms contain a coefficient at nonlinearity b_1 and do not contain the coefficients of b_2, b_3 or b_4. In the case of $a^2 < 2$, the quasinormal forms contain all coefficients of the function f.

The parameter a plays an important role in the dynamics of quasinormal forms. The structure of solutions in the case $a^2 < 2$ is much more complicated than in the case $a^2 > 2$, because quasinormal forms at $a^2 < 2$ are complex boundary value problems of the Ginzburg–Landau type, and the solutions contain rapidly oscillating t components. Explicit formulas are obtained that allow us to trace the role of the parameter c, included in the delay coefficient (13).

Quasinormal forms do not explicitly contain the parameter ε but depend essentially on ε through θ_ω and θ_z. As $\varepsilon \to 0$, these quantities run indefinitely from 0 to $c\omega_0$ and from 0 to 1, respectively. At the same time, unlimited alternations of forward and backward bifurcations can be observed in quasinormal forms. This indicates the high sensitivity of the dynamical properties to changes in the parameter ε and, hence, to changes in the values of N and T. In particular, even changing a large value of N to 1 can significantly affect the dynamics of the problem.

Cases where the parameter σ is small enough were considered. It was shown that quasinormal forms become even more complicated, since there appears a third spatial variable, and the dimensionality of the diffusion operator increases. It entails the complication of the dynamics of the initial problem. It is important to note that the condition $\sigma \ll 1$ is of special interest: the couplings between elements are more "close" to those that arise at standard approximations of the diffusion and advection operators (see (11), (12)).

It is interesting to note that, under the condition $T \gg 1$, we were able to obtain explicit formulas for all parameters defining the critical cases.

Let us focus on the most interesting differences in the structure of the solutions for the cases $\Phi(s) = \Phi_1(s)$ and $\Phi(s) = \Phi_2(s)$. The "critical" modes are adjacent to the values $z_0\varepsilon^{-1} + \theta_z$, and these values are determined by relations (52) and (61). When σ is small, these values are also different. In the first case, $z_m = \pi(m + 1/2)$, and in the second, $\pi(2m + 1)$ $(m = 0, \pm1, \pm2, \ldots)$. Not only are the coefficients and even the number of equations in the corresponding quasinormal forms different, but the boundary conditions (89) and (94), (95) are also different. Thus, the dynamics, even in the case of different advective-type couplings, can be essentially different.

The obtained results can be extended to other systems with diffusive, advective or other couplings (see, for example, [27]). We note that accounting for quadratic nonlinearities in (14) does not lead to additional difficulties.

It is important to emphasize that the principal terms of the asymptotics of the solutions of the original equation are determined by the solutions of the (nonlocal) quasinormal forms.

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