mathematics

Article

Local Second Order Sobolev Regularity for p-Laplacian Equation in Semi-Simple Lie Group

Chengwei Yu [1] and Yue Zeng [2,*]

[1] Department of Basic, China Fire and Rescue Institue, 4 Nanyan Road, Changping District, Beijing 102202, China; yushengwei@cfri.edu.cn
[2] School of Mathematical Sciences, Beihang University, Haidian District, Beijing 100191, China
* Correspondence: sy1809102@buaa.edu.cn

Abstract: In this paper, we establish a structural inequality of the ∞-subLaplacian $\triangle_{0,\infty}$ in a class of the semi-simple Lie group endowed with the horizontal vector fields $X_1, \ldots, X_{2n}$. When $1 < p \leq 4$ with $n = 1$ and $1 < p < 3 + \frac{1}{n-1}$ with $n \geq 2$, we apply the structural inequality to obtain the local horizontal $W^{2,2}$-regularity of weak solutions to p-Laplacian equation in the semi-simple Lie group. Compared to Euclidean spaces $\mathbb{R}^{2n}$ with $n \geq 2$, the range of this p obtained is already optimal.

Keywords: structural inequality; $W^{2,2}$-regularity; weak solutions; p-Laplacian equation; semi-simple Lie group; range of p

MSC: 35H20; 35B65

check for updates

Citation: Yu, C.; Zeng, Y. Local Second Order Sobolev Regularity for p-Laplacian Equation in Semi-Simple Lie Group. *Mathematics* **2024**, *12*, 601. https://doi.org/10.3390/math12040601

Academic Editor: Adolfo Ballester-Bolinches

Received: 15 January 2024
Revised: 12 February 2024
Accepted: 16 February 2024
Published: 17 February 2024

Copyright: © 2024 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

In this research article, we consider the compact, connected, semi-simple Lie group $\mathbb{LG}$ endowed with the horizontal vector fields $X_1, X_2, \ldots, X_{2n}$; see Section 2 for details. We denote by $\triangle_{0,\infty} v = \sum_{i,j=1}^{2n} X_i v X_i X_j v X_j v$ the ∞-subLaplacian of a function v. For any function $v \in C^\infty$, we establish a structural inequality of $\triangle_{0,\infty} v$ (see Lemma 3 below), that is,

$$\begin{aligned}&\left| |D_0^2 v \nabla_{\mathcal{H}} v|^2 - \triangle_0 v \triangle_{0,\infty} v - \frac{1}{2}[|D_0^2 v|^2 - (\triangle_0 v)^2] |\nabla_{\mathcal{H}} v|^2 \right| \\ &\quad \leq (n-1)[|D_0^2 v|^2 |\nabla_{\mathcal{H}} v|^2 - |D_0^2 v \nabla_{\mathcal{H}} v|^2].\end{aligned}$$

Here, for any function $v \in C^1$, we notate $\nabla_{\mathcal{H}} v = (X_1 v, X_2 v, \ldots, X_{2n} v)$ as the horizontal gradient of v, and

$$D_0^2 v = \left(\frac{X_i X_j v + X_j X_i v}{2} \right)_{1 \leq i,j \leq 2n}$$

as the symmetrization of $\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v = (X_i X_j v)_{1 \leq i,j \leq 2n}$, and $\triangle_0 v = \sum_{i=1}^{2n} X_i X_i v$ as the 2-subLaplacian. Here, see [1,2] for the definitions. Based on this structural inequality, we aim to obtain the local horizontal $W^{2,2}$-regularity of weak solutions to the p-Laplacian equation in the semi-simple Lie group.

Let $1 < p < \infty$. For a given domain $\Omega \subset \mathbb{LG}$, we define that the functions $u : \Omega \to \mathbb{R}$ are p-harmonic functions in Ω if $u \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ are weak solutions to the degenerate p-Laplacian equation

$$\triangle_{\mathcal{H},p} u(x) = \sum_{i=1}^{2n} X_i^* (|\nabla_{\mathcal{H}} u|^{p-2} X_i u) = 0 \quad \forall x \in \Omega, \tag{1}$$

that is,

$$\int_{\Omega}\sum_{i=1}^{2n}|\nabla_{\mathcal{H}}u|^{p-2}X_iuX_i\varrho dx=0 \quad \forall\varrho\in C_0^{\infty}(\Omega),$$

where X_i^* is the formal adjoint of X_i, and $W^{1,p}_{\mathcal{H},\mathrm{loc}}(\Omega)$ is the first-order p-th integrable local horizontal Sobolev space defined in Section 2. From [3], for left-invariant vector fields, it holds true that $X_i^*=-X_i, i=1,2,\ldots,2n$. Then, Equation (1) becomes

$$\triangle_{\mathcal{H},p}u(x)=-\sum_{i=1}^{2n}X_i(|\nabla_{\mathcal{H}}u|^{p-2}X_iu)=0 \quad \forall x\in\Omega. \tag{2}$$

In the ordinary case $p=2$, we habitually refer to the 2-harmonic functions in the semi-simple Lie group as harmonic functions, and Hörmander [4] proved their C^{∞}-regularity. In the non-ordinary case $p\neq 2$, if the horizontal gradient $\nabla_{\mathcal{H}}u$ of the p-harmonic function u in the semi-simple Lie group has the boundary $0<B^{-1}\leq|\nabla_{\mathcal{H}}u|\leq B$, Domokos–Manfredi [5] proved $u\in C^{\infty}$. But we cannot expect the C^{∞}-regularity for u if the assumption is not satisfied. For general p-harmonic functions in the semi-simple Lie group, Domokos–Manfredi [3] established the $C^{0,1}$-regularity with $1<p<\infty$ and the $C^{1,\alpha}$-regularity with $2\leq p<\infty$.

We denote by $W^{2,2}_{\mathcal{H},\mathrm{loc}}$-regularity the local horizontal $W^{2,2}$-regularity, and also call it the local second-order horizontal Sobolev regularity. Here, for any given domain $\Omega\subset\mathbb{G}$, we define that the function $v:\Omega\to\mathbb{R}$ belongs to $W^{2,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$ if the function v belongs to $W^{1,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$ and its second-order horizontal derivative $\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}v$ belongs to $L^2_{\mathrm{loc}}(\Omega)$. We notate

$$K_{\varrho}=1+\|\nabla_{\mathcal{H}}\varrho\|^2_{L^{\infty}(\Omega)}+\|\varrho\nabla_{\mathcal{R}}\varrho\|_{L^{\infty}(\Omega)} \tag{3}$$

for any function $\varrho\in C_0^{\infty}(\Omega)$. Here, we denote by $\nabla_{\mathcal{R}}v=(R_1v,R_2v,\ldots,R_vv)$ the vertical gradient of a function v. In this paper, we will prove the following local horizontal $W^{2,2}$-regularity of p-harmonic functions u in the semi-simple Lie group.

Theorem 1. *Any p-harmonic function u in a domain $\Omega\subset\mathbb{LG}$ belongs to $W^{2,2}_{\mathcal{H},\mathrm{loc}}(\Omega)$ for*

$$1<p\leq 4 \quad \textit{with } n=1,$$

and

$$1<p<3+\frac{1}{n-1} \quad \textit{with } n\geq 2. \tag{4}$$

In addition, when $1<p\leq 2$, for any function $\varrho\in C_0^{\infty}(\Omega)$, we have

$$\int_{\Omega}\varrho^2|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2dx\leq cK_{\varrho}\left(\int_{spt(\varrho)}|\nabla_{\mathcal{H}}u|^{2-p}dx\right)^{\frac{1}{2}}\left(\int_{spt(\varrho)}|\nabla_{\mathcal{H}}u|^{p+2}dx\right)^{\frac{1}{2}}; \tag{5}$$

when $2<p<\infty$ and p satisfies (4), for any $\varrho\in C_0^{\infty}(\Omega)$, we have

$$\begin{aligned}\int_{\Omega}\varrho^6|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u|^2dx\leq& cK_{\varrho}\left(\int_{spt(\varrho)}|\nabla_{\mathcal{H}}u|^{p+2}dx\right)^{\frac{1}{4}}\left(\int_{spt(\varrho)}|\nabla_{\mathcal{H}}u|^{p-2}dx\right)^{\frac{1}{4}}\\ &\times\left(\int_{\Omega}\varrho^6|\nabla_{\mathcal{H}}u|^{4-p}dx\right)^{\frac{1}{2}},\end{aligned} \tag{6}$$

where K_{ϱ} is notated in (3), $c=c(n,p)>0$, and the support of ϱ is notated as $spt(\varrho)$.

When $1<p<\infty$, for p-harmonic functions in Euclidean spaces $\mathbb{R}^n$, we refer to [6–10] for their $C^{0,1}$ and $C^{1,\alpha}$-regularities. When $1<p<3+\frac{2}{n-2}$, their local horizontal $W^{2,2}$-regularity was proved by Manfredi–Weitsman [11], and Dong et al. [1] also gave a new proof. For p-harmonic functions in Heisenberg group $\mathbb{H}^n$, grounding on the work of [12–16], Zhong [17] established their $C^{0,1}$-regularity with $1 < p < \infty$ and $C^{1,\alpha}$-regularity with $2 \leq p < \infty$. Mukherjee–Zhong [18] increased the range of p to $1 < p < \infty$. Domokos–Manfredi [13] established their local horizontal $W^{2,2}$-regularity. Liu et al. [2] improved the range of p, that is, $1 < p \leq 4$ with $n = 1$ and $1 < p < 3 + \frac{1}{n-1}$ with $n \geq 2$. Citti–Mukherjee [19] extended Zhong's method to Hörmander vector fields of step two and established $C^{0,1}$ and $C^{1,\alpha}$-regularities with $1 < p < \infty$. In addition, Muhkerjee–Sire [20] established $C^{1,\alpha}$-regularity for inhomogeneous quasi-linear equations on the Heisenberg group $\mathbb{H}^n$ with $2 \leq p < \infty$, and Yu [21] increased the range of p to $2 - \frac{1}{2n+2} < p < \infty$. For further research on the second-order Sobolev regularity, Domokos–Manfredi [13,22] first established the Cordes condition and applied it to obtain the $HW^{2,2}_{\text{loc}}$-regularity in the Heisenberg group $\mathbb{H}^1$ for $\frac{\sqrt{17}-1}{2} \leq p < \frac{5+\sqrt{5}}{2}$, Fazio et al. [23] established the $W^{2,2}_{X,\text{loc}}$-regularity in the Grušin plane for p near 2, and Domokos–Manfredi ([5], Theorem 4.1) obtained a main inequality for studying the $W^{2,2}_{X,\text{loc}}$-regularity on more general vector fields when $2 \leq p < \frac{2\nu}{\nu-1}$. Recently, Yu [24,25] established the $W^{2,2}_{\mathcal{H},\text{loc}}$-regularity on SU(3) with $1 < p < \frac{7}{2}$ and $W^{2,2}_{X,\text{loc}}$-regularity on the first-order Grušin plane with $1 < p \leq 4$. For further study on the quasilinear equations, Yu. G. Reshetnyak proved that the mappings with the bounded distortion are continuous, open and discrete [26]. The key point of this proof is based on a deep connection between the mappings with the bounded distortion and the solutions of quasilinear equations of the elliptic type and non-linear potential theory. Further development of quasi-conformal analysis and related functional classes on Carnot groups and more general metric spaces, see [27–30], initiated the study of the relationship between the mappings with bounded distortion and solutions of subelliptic equations in the geometry of vector fields, satisfying the Hörmander condition; see [31] for an example.

The proof of Theorem 1 depends on the study of the regularized equation of Equation (2). Let u be the weak solution to Equation (2). For any given smooth domain $U \Subset \Omega$ and any positive constant $\varepsilon \in (0,1]$, we consider the regularized equation

$$\sum_{i=1}^{2n} X_i[(\varepsilon + |\nabla_{\mathcal{H}} v|^2)^{\frac{p-2}{2}} X_i v] = 0 \quad \forall x \in U; \quad v - u \in W^{1,p}_{\mathcal{H},0}(U). \tag{7}$$

We notate $u^\varepsilon \in W^{1,p}_{\mathcal{H}}(U)$ as weak solutions to Equation (7). The existence, uniqueness and smoothness of solutions were studied in [3,5]. Domokos–Manfredi [3] proved that their horizontal gradients $\{\nabla_{\mathcal{H}} u^\varepsilon\}_{\varepsilon\in(0,1]}$ have the unifrom $L^\infty_{\text{loc}}(U)$-regularity in $\varepsilon \in (0,1]$. In addition, it is given in [3] that as $\varepsilon \to 0$, $u^\varepsilon \to u$ in $C^0(\bar{U})$ (see Proposition 1).

By studying Equation (7), we establish the following theorem, which gives that $u^\varepsilon \in W^{2,2}_{\mathcal{H},\text{loc}}(U)$ uniformly in $\varepsilon \in (0,1]$. From this, by letting $\varepsilon \to 0$, we can apply the same method as ([2], Section 5) to infer Theorem 1.

Theorem 2. *When p satisfies the condition (4), weak solutions $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ to Equation (7) have the uniform $W^{2,2}_{\mathcal{H},\text{loc}}(U)$-regularity in $\varepsilon \in (0,1]$. In addition, in the case $1 < p \leq 2$, for any function $\varrho \in C^\infty_0(U)$, we have*

$$\begin{aligned}&\int_U \varrho^2 |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\varepsilon|^2 dx \\ &\leq cK_\varrho \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{2-p}{2}} dx \right)^{\frac{1}{2}} \left(\int_{spt(\varrho)} (\varepsilon + \nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{2}} dx \right)^{\frac{1}{2}};\end{aligned} \tag{8}$$

in the case that $2 < p < \infty$ and p satisfies (4), for any function $\varrho \in C^\infty_0(U)$, we have

$$\int_U \varrho^6 |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\varepsilon|^2 dx$$
$$\leq cK_\phi \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{2}} dx \right)^{\frac{1}{4}} \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} dx \right)^{\frac{1}{4}}$$
$$\times \left(\int_U \varrho^6 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{4-p}{2}} dx \right)^{\frac{1}{2}}, \tag{9}$$

where K_ϱ is notated in (3), $c = c(n, p) > 0$, and the support of ϱ is notated as $spt(\varrho)$.

Before we specify the idea of the proof, we denote by

$$Bv := \nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v - D_0^2 v = \left(\frac{X_i X_j v - X_j X_i v}{2} \right)_{1 \leq i,j \leq 2n} = \left(\frac{[X_i, X_j] v}{2} \right)_{1 \leq i,j \leq 2n}$$

the difference between $\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v$ and $D_0^2 v$. Noting that $B = (b_{i,j})_{1 \leq i,j \leq 2n}$ is an anti-symmetric matrix ($b_{i,j} = -b_{j,i}$), we have $|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v|^2 = |D_0^2 v|^2 + |Bv|^2$. Thus, we can obtain the estimate of $|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\varepsilon|$ by estimating $|D_0^2 u^\varepsilon|^2$ and $|Bu^\varepsilon|^2$.

The proof of Theorem 2 relies on some prior estimates and some Cacciopoli-type inequalities of u^ε built up by Domokos–Manfredi [3] (see Lemma 1). To be specific, we divide the proof into the case $1 < p < 2$ and the case $2 < p < \infty$. When $1 < p < 2$, we deduce (8) from Lemma 1 directly. When $2 < p < \infty$ and p satisfies (4), we use some ideas from [1] to establish a structural inequality (see Lemma 3). Applying the structural inequality to break down the horizontal Hessian matrix, we obtain a decomposition inequality for $D_0^2 u^\varepsilon$ (see Lemma 4). Applying some priori estimates and some Cacciopoli-type inequalities in Lemma 1, we obtain the estimate of $|Bu^\varepsilon|^2$ (see Lemma 7) and estimates of all decomposition terms (see Lemmas 5 and 6). Finally, based on the decomposition inequality (18), combining all estimates, we obtain (9). The proof is shown in Section 5.

Consequently, our new results improve the range of p in [23] (p near 2 for the Grušin plane), Ref. [13] ($\frac{\sqrt{17}-1}{2} \leq p < \frac{5+\sqrt{5}}{2}$ for the Heisenberg group $\mathbb{H}^1$) and ([5], Theorem 4.1) ($2 \leq p < \frac{2\nu}{\nu-1}$). Compared to Euclidean spaces $\mathbb{R}^{2n}$ with $n \geq 2$, the range of this p obtained is already optimal. Our method can also be applied to more general vector fields to generalize and improve some known results in the literature.

2. Preliminaries

We consider a special class of semi-simple Lie group $\mathbb{LG}$, which was first proposed by Domokos–Manfredi [3]. The semi-simple Lie group $\mathbb{LG}$ is connected and compact. We notate $\mathcal{LG}$ as its Lie algebra. The inner product on $\mathcal{LG}$ satisfies the properties

$$\langle gXg^{-1}, gYg^{-1} \rangle = \langle X, Y \rangle, \quad \forall\, g \in \mathbb{LG}, \text{ and } X, Y \in \mathcal{LG},$$

and

$$\langle [X, Y], Z \rangle = -\langle Y, [X, Z] \rangle, \quad \forall\, X, Y, Z \in \mathcal{LG}.$$

Let $\mathbb{LS}$ be the maximal torus of $\mathbb{LG}$. We notate $\mathcal{LS}$ as its Lie algebra. Owing to the fact that $\mathcal{LS}$ is a maximal commutative subalgebra of $\mathcal{LG}$, we call it Cartan subalgebra. We denote by $\mathcal{R}$ the set of all roots, where we say that $R \in \mathcal{LS}$ is a root if $R \neq 0$ with the root space $\mathcal{LG}_R \neq \{0\}$. Here, $\mathcal{LG}_R = \{Z \in \mathcal{LG}_{\mathbb{C}} : [S, Z] = i\langle R, S \rangle Z, \quad \forall S \in \mathcal{LS}\}$.

According to ([3], Section 5), we can define the orthogonal complement of $\mathcal{LS}$ denoted by $\mathcal{H}$, and we can choose its orthonormal basis satisfying Property 1. We notate $\mathcal{B}_{\mathcal{H}} = \{X_1, X_2, \ldots, X_{2n}\}$ as the orthonormal basis of $\mathcal{H}$.

Property 1.

(i) $\forall 1 \leq k \leq n$, $\exists R_k \in \mathcal{R}^+$ *s.t.* $\text{span}\{X_{2k-1}, X_{2k}\} = \mathcal{H}_{R_k}$.

(ii) $[X_{2k-1}, X_{2k}] = -R_k, \quad [X_{2k}, R_k] = -\|R_k\|^2 X_{2k-1}, \quad [R_k, X_{2k-1}] = \|R_k\|^2 X_{2k}$.

(iii) $[X_l, X_m] \in \mathcal{H}$ *when* $(l, m) \neq (2k-1, 2k)$.

(iv) $\{[X_{2k-1}, S], [X_{2k}, S]\} \subset \mathcal{H}_{R_k}$ *when* $S \in \mathcal{LS}$.

Based on the properties of $\mathcal{B}_{\mathcal{H}}$, a basis of $\mathcal{LS}$ can be selected, that is, $\{R_1, R_2, \dots, R_v\}$. For any function v, we denote by

$$\nabla_{\mathcal{H}} v = (X_1 v, X_2 v, \dots, X_{2n} v), \quad \nabla_{\mathcal{R}} v = (R_1 v, R_2 v, \dots, R_v v)$$

the horizontal and vertical gradients. Here, the homogeneous dimension of $\mathbb{LG}$ is $2n + 2v$; see ([3], Section 5) for the definitions of the horizontal and vertical gradients. Moreover, from Property 1, we draw the conclusion

$$[X_i, X_j] = \sum_{k=1}^{2n} \lambda_{i,j}^{(k)} X_k + \sum_{l=1}^{v} \theta_{i,j}^{(l)} R_l, \quad [X_i, R_j] = \sum_{k=1}^{2n} \vartheta_{i,j}^{(k)} X_k. \tag{10}$$

Here, $\lambda_{i,j}^{(k)}$, $\theta_{i,j}^{(l)}$ and $\vartheta_{i,j}^{(k)}$ are constants.

Given a domain $\Omega \subset \mathbb{LG}$, we notate $W_{\mathcal{H}}^{1,p}(\Omega)$ as the horizontal Sobolev space for $1 < p < \infty$. We define that a function v belongs to $W_{\mathcal{H}}^{1,p}(\Omega)$ if it belongs to $L^p(\Omega)$ and its horizontal gradient $\nabla_{\mathcal{H}} v$ belongs to $L^p(\Omega)$. Here, we define the norm of v as

$$\|v\|_{W_{\mathcal{H}}^{1,p}(\Omega)} = \left(\|v\|_{L^p(\Omega)}^p + \|\nabla_{\mathcal{H}} v\|_{L^p(\Omega)}^p \right)^{1/p}.$$

Moreover, we notate $W_{\mathcal{H}}^{k,p}(\Omega, \mathbb{R})$ as the k-order horizontal Sobolev space for any $k \geq 2$. For any function v, we say $v \in W_{\mathcal{H}}^{k,p}(\Omega, \mathbb{R})$ if $\nabla_{\mathcal{H}} v \in W_{\mathcal{H}}^{k-1,p}(\Omega)$, and define its norm in a similar method. For any index $k \geq 1$ and $1 < p < \infty$, we notate $W_{\mathcal{H},\mathrm{loc}}^{k,p}(\Omega)$ as the collection of all functions $v : \Omega \to \mathbb{R}$ satisfying $v \in W_X^{k,p}(U)$ for all $U \Subset \Omega$. We notate $W_{\mathcal{H},0}^{k,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W_{\mathcal{H}}^{1,p}(\Omega)$ endowed with the $\|\cdot\|_{W_{\mathcal{H}}^{k,p}(\Omega)}$-norm.

Let $1 < p < \infty$ and u be a p-harmonic function in Ω. In the rest of this section, for any $\varepsilon \in (0, 1]$ and any smooth domain $U \Subset \Omega$, we list several priori uniform estimates for u^ε established by Domokos–Manfredi [3], where $u^\varepsilon \in W_{\mathcal{H}}^{1,p}(U)$ is the weak solution to the regularized Equation (7).

According to ([3], Theorem 5.1), we have the following uniform estimate for $\nabla_{\mathcal{H}} u^\varepsilon$ and convergence.

Proposition 1 ([3], Theorem 5.1)**.** *For any $0 \leq \varepsilon < 1$, if $u^\varepsilon \in W_{\mathcal{H},\mathrm{loc}}^{1,p}(U)$ is the weak solution to Equation (7) with $1 < p < \infty$, then its horizontal gradient $\nabla_{\mathcal{H}} u^\varepsilon$ has the uniform $L_{\mathrm{loc}}^\infty(U)$-regularity in $\varepsilon \in [0, 1)$, and for any Carnot–Carathéodory ball $B_r \subset U$, the following holds:*

$$\|\nabla_{\mathcal{H}} u^\varepsilon\|_{L^\infty(B_{r/2})} \leq c(p) \left(\fint_{B_r} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \tag{11}$$

where we notate $u^0 = u$ for $\varepsilon = 0$. In addition, $u^\varepsilon \to u$ in $C^0(\bar{U})$.

Here, we notate $\fint_M \psi dx = |M|^{-1} \int_M \psi dx$ as the average of integrable function ψ over the measurable set M. According to ([5], Theorem 1.1), we obtain the following corollary from Proposition 1 in a direct method.

Corollary 1. *When $1 < p < \infty$, for any $\varepsilon > 0$, the weak solution $u^\varepsilon \in W_{\mathcal{H},\mathrm{loc}}^{1,p}(U)$ to Equation (7) has the $C^\infty(U)$-regularity.*

Based on ([3], Section 5), we can obtain the same conclusion as [3] (Corollary 4.1) in almost the same method. Here, we omit the proof.

Lemma 1. *If any function $\varrho \in C_0^\infty(U)$ satisfies $0 \le \varrho \le 1$, then there are the following conclusions:*

(i) *For any $\beta \ge 0$, the inequality holds:*

$$\begin{aligned}&\int_U \varrho^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{R}}u^\varepsilon|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{R}}u^\varepsilon|^2dx\\ &\quad\le c\int_U|\nabla_{\mathcal{H}}\varrho|^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{R}}u^\varepsilon|^{2\beta+2}dx\\ &\qquad+c(\beta+1)^2\int_U\varrho^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p}{2}}|\nabla_{\mathcal{R}}u^\varepsilon|^{2\beta}dx.\end{aligned}\tag{12}$$

(ii) *For any $\beta \ge 0$, the inequality holds:*

$$\begin{aligned}&\int_U \varrho^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}+\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\varepsilon|^2dx\\ &\quad\le c(\beta+1)^4\int_U\varrho^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}+\beta}|\nabla_{\mathcal{R}}u^\varepsilon|^2dx\\ &\qquad+c(\beta+1)^2K_\varrho\int_{spt(\varrho)}(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p}{2}+\beta}dx,\end{aligned}\tag{13}$$

(iii) *For any $\beta \ge 1$, the inequality holds:*

$$\begin{aligned}&\int_U \varrho^{2\beta+2}(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{R}}u^\varepsilon|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\varepsilon|^2dx\\ &\quad\le c^\beta(\beta+1)^{4\beta}\|\nabla_{\mathcal{H}}\varrho\|^{2\beta}_{L^\infty(U)}\int_U\varrho^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}+\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\varepsilon|^2dx.\end{aligned}\tag{14}$$

(iv) *For any $\beta \ge 1$, the inequality holds:*

$$\begin{aligned}&\int_U \varrho^2(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}+\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\varepsilon|^2dx\\ &\quad\le c(\beta+1)^{12}K_\varrho\int_{spt(\varrho)}(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p}{2}+\beta}dx.\end{aligned}\tag{15}$$

Here, K_ϱ is notated in (3), all of the above $c = c(p)$ are positive constants, and the support of ϱ is notated as $spt(\varrho)$.

The following result is directly derived from (14) and (15).

Lemma 2. *If any $\beta \ge 1$ and any function $\varrho \in C_0^\infty(U)$ satisfies $0 \le \varrho \le 1$, then the following holds:*

$$\begin{aligned}&\int_U \varrho^{2\beta+2}(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{R}}u^\varepsilon|^{2\beta}|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}u^\varepsilon|^2dx\\ &\quad\le c^\beta(\beta+1)^{12+4\beta}K_\varrho^{\beta+1}\int_{spt(\varrho)}(\delta+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p}{2}+\beta}dx.\end{aligned}\tag{16}$$

Here, K_ϱ is notated in (3), $c = c(p) > 0$, and the support of ϱ is notated as $spt(\varrho)$.

3. A Decomposition Inequality for $D_0^2u^\varepsilon$

Firstly, we introduce the structural inequality of $\triangle_{0,\infty}v$. Here, we define $\triangle_{0,\infty}v = \sum_{i,j=1}^{2n} X_ivX_iX_jvX_jv$ as the ∞-subLaplacian of a function v. Its proof is placed at the end of this section. Applying the idea for proving ([1], Lemma 2.1), we establish the following structural inequality. For simplicity, the ∞-subLaplacian $\triangle_{0,\infty}v$ of $v \in C^\infty$ is written as

$$\triangle_{0,\infty}v = (\nabla_{\mathcal{H}}v)^T \nabla_{\mathcal{H}}\nabla_{\mathcal{H}}v \nabla_{\mathcal{H}}v = (\nabla_{\mathcal{H}}v)^T D_0^2 v \nabla_{\mathcal{H}}v.$$

Lemma 3. *The following holds for any function $v \in C^\infty(U)$,*

$$\begin{aligned}\Big| |D_0^2v\nabla_{\mathcal{H}}v|^2 - \triangle_0 v \triangle_{0,\infty}v - \frac{1}{2}[|D_0^2v|^2 - (\triangle_0 v)^2]|\nabla_{\mathcal{H}}v|^2 \Big| \\ \leq (n-1)[|D_0^2v|^2|\nabla_{\mathcal{H}}v|^2 - |D_0^2v\nabla_{\mathcal{H}}v|^2] \quad \forall x \in U.\end{aligned} \tag{17}$$

Proof. We fix any point $\bar{x} \in U$. Apparently, the inequality (17) holds for $\nabla_{\mathcal{H}}v(\bar{x}) = 0$. Below, we consider the case $\nabla_{\mathcal{H}}v(\bar{x}) \neq 0$. We can divide both sides of the inequality (17) by $|\nabla_{\mathcal{H}}v(x)|^2$. Therefore, we only need to consider the case $|\nabla_{\mathcal{H}}v(\bar{x})| = 1$.

At the point $\bar{x}$, noting that D_0^2v is a symmetric matrix, we can derive its eigenvalues $\{\zeta_i\}_{i=1}^{2n} \subset \mathbb{R}$ and obtain an orthogonal matrix $A \in \mathbf{O}(2n)$ ($A^{-1} = A^T$) such that

$$A^T D_0^2 v A = \mathrm{diag}\{\zeta_1, \zeta_2, \ldots, \zeta_{2n}\}.$$

Thus,

$$|D_0^2v|^2 = |A^T D_0^2 v A|^2 = \sum_{i=1}^{2n} (\zeta_i)^2 \quad \text{and} \quad \Delta_0 v = \sum_{i=1}^{2n} \zeta_i.$$

Notating $A^T\nabla_{\mathcal{H}}v = \sum_{i=1}^{2n} \mu_i \mathbf{e}_i =: \vec{\mu}$, we obtain

$$\Delta_{0,\infty}v = (\nabla_{\mathcal{H}}v)^T D_0^2 v \nabla_{\mathcal{H}}v = (A^T\nabla_{\mathcal{H}}v)^T (A^T D_0^2 v A)(A^T \nabla_{\mathcal{H}}v) = \sum_{i=1}^{2n} \zeta_i(\mu_i)^2$$

and

$$|D_0^2v\nabla_{\mathcal{H}}v|^2 = |(A^T D_0^2 v A)(A^T\nabla_{\mathcal{H}}v)|^2 = \sum_{i=1}^{2n} (\zeta_i)^2(\mu_i)^2.$$

Based on ([1], Lemma 2.2) with $\vec{\zeta} := (\zeta_1, \zeta_2, \ldots, \zeta_{2n})$ and $\vec{\mu}$, we deduce

$$\begin{aligned}&\Big| |D_0^2v\nabla_{\mathcal{H}}v|^2 - \Delta_0 v \Delta_{0,\infty}v - \frac{1}{2}[|D_0^2v|^2 - (\Delta_0 v)^2]|\nabla_{\mathcal{H}}v|^2 \Big| \\ &= \Big| \sum_{i=1}^{2n} (\zeta_i)^2(\mu_i)^2 - (\sum_{i=1}^{2n} \zeta_i)[\sum_{j=1}^{2n} \zeta_j(\mu_j)^2] - \frac{1}{2}[\sum_{i=1}^{2n}(\zeta_i)^2 - (\sum_{i=1}^{2n}\zeta_i)^2] \Big| \\ &\leq (n-1)[\sum_{i=1}^{2n}(\zeta_i)^2 - \sum_{i=1}^{2n}(\zeta_i)^2(\mu_i)^2] \\ &= (n-1)[|D_0^2v|^2|\nabla_{\mathcal{H}}v|^2 - |D_0^2v\nabla_{\mathcal{H}}v|^2].\end{aligned}$$

□

Now, we use the structural inequality to obtain the following decomposition inequality for $D_0^2u^\varepsilon$, which is a pointwise estimate.

Lemma 4. *Let $u^\varepsilon \in W^{1,p}_{\mathcal{H},\mathrm{loc}}(U)$ with $1 < p < \infty$ be the weak solution to Equation (7), then*

$$\begin{aligned}&\Big[2n + 2(p-2) + (2-2n)(p-2)^2\Big]|D_0^2u^\varepsilon|^2 \\ &\leq \Big[2n + 2(p-2) + (p-2)^2\Big][|D_0^2u^\varepsilon|^2 - (\triangle_0 u^\varepsilon)^2] \quad \forall x \in U.\end{aligned} \tag{18}$$

Here, note that (4) implies

$$2n + 2(p-2) + (2-2n)(p-2)^2 = 2(p-1)[(1-n)(p-2) + n] > 0,$$

and thus the coefficient in (18) is positive. This ensures that the decomposition inequality (18) is valuable.

Proof. Owing to the $C^\infty(U)$-regularity of u^ε, Equation (7) gives

$$(p-2)\Delta_{0,\infty}u^\varepsilon + (\varepsilon + |\nabla_{\mathcal{H}}u^\varepsilon|^2)\Delta_0 u^\varepsilon = 0 \quad \forall x \in U. \tag{19}$$

We fix any point $\bar{x} \in U$. It is easy to prove that the inequality (18) holds for $\nabla_{\mathcal{H}}u^\varepsilon(\bar{x}) = 0$ since we can deduce $\Delta_0 u^\varepsilon(\bar{x}) = 0$ from (19) in a direct method. Below, we consider the case $\nabla_{\mathcal{H}}u^\varepsilon(\bar{x}) \neq 0$. Letting $v = u^\varepsilon$ in Lemma 3 and multiplying both sides by $2(p-2)^2$, then from (19), at $\bar{x}$, we infer

$$\begin{aligned}
&2(p-2)^2|D_0^2u^\varepsilon\nabla_{\mathcal{H}}u^\varepsilon|^2 + 2(p-2)(\Delta_0 u^\varepsilon)^2[|\nabla_{\mathcal{H}}u^\varepsilon|^2 + \varepsilon] \\
&\quad - (p-2)^2[|D_0^2u^\varepsilon|^2 - (\Delta_0 u^\varepsilon)^2]|\nabla_{\mathcal{H}}u^\varepsilon|^2 \\
&\quad \le (p-2)^2(2n-2)[|D_0^2u^\varepsilon|^2|\nabla_{\mathcal{H}}u^\varepsilon|^2 - |D_0^2u^\varepsilon\nabla_{\mathcal{H}}u^\varepsilon|^2].
\end{aligned}$$

We divide both sides of the above inequality by $|\nabla_{\mathcal{H}}u^\varepsilon(\bar{x})|^2$, then, at $\bar{x}$,

$$\begin{aligned}
&2(p-2)^2 n\frac{|D_0^2u^\varepsilon\nabla_{\mathcal{H}}u^\varepsilon|^2}{|\nabla_{\mathcal{H}}u^\varepsilon|^2} + 2(p-2)\frac{(\Delta_0 u^\varepsilon)^2}{|\nabla_{\mathcal{H}}u^\varepsilon|^2}[|\nabla_{\mathcal{H}}u^\varepsilon|^2 + \varepsilon] \\
&\quad \le (p-2)^2[|D_0^2u^\varepsilon|^2 - (\Delta_0 u^\varepsilon)^2] + (p-2)^2(2n-2)|D_0^2u^\varepsilon|^2
\end{aligned} \tag{20}$$

We apply (19) again, and then apply Hölder's inequality to derive, at $\bar{x}$,

$$(p-2)^2\frac{|D_0^2u^\varepsilon\nabla_{\mathcal{H}}u^\varepsilon|^2}{|\nabla_{\mathcal{H}}u^\varepsilon|^2} \ge (p-2)^2\frac{|\Delta_{0,\infty}u^\varepsilon|^2}{|\nabla_{\mathcal{H}}u^\varepsilon|^4} \ge \frac{(\Delta_0 u^\varepsilon)^2}{|\nabla_{\mathcal{H}}u^\varepsilon|^2}[|\nabla_{\mathcal{H}}u^\varepsilon|^2 + \varepsilon].$$

This and (20) imply

$$\begin{aligned}
&[2n + 2(p-2)]\left(\frac{\Delta_0 u^\varepsilon}{|\nabla_{\mathcal{H}}u^\varepsilon|}\right)^2[|\nabla_{\mathcal{H}}u^\varepsilon|^2 + \varepsilon] \\
&\quad \le (p-2)^2[|D_0^2u^\varepsilon|^2 - (\Delta_0 u^\varepsilon)^2] + (p-2)^2(2n-2)|D_0^2u^\varepsilon|^2.
\end{aligned} \tag{21}$$

Noting that $2n + 2(p-2) = 2(p-1) + 2(n-1) > 0$, from (21), we obtain

$$[2n + 2(p-2)](\Delta_0 u^\varepsilon)^2 \le (p-2)^2[|D_0^2u^\varepsilon|^2 - (\Delta_0 u^\varepsilon)^2] + (p-2)^2(2n-2)|D_0^2u^\varepsilon|^2.$$

Subtracting $\left([2n + 2(p-2)](\Delta_0 u^\varepsilon)^2 - \left[2n + 2(p-2) - (2n-2)(p-2)^2\right]|D_0^2u^\varepsilon|^2\right)$ from both sides of the above inequality, we obtain

$$\begin{aligned}
&\left[2n + 2(p-2) + (2-2n)(p-2)^2\right]|D_0^2u^\varepsilon|^2 \\
&\quad \le \left[2n + 2(p-2) + (p-2)^2\right][|D_0^2u^\varepsilon|^2 - (\Delta_0 u^\varepsilon)^2],
\end{aligned}$$

that is, (18) is valid. □

4. Estimates of All Decomposition Terms and the Estimate of $|Mu^\varepsilon|^2$

Firstly, we give the estimate of the right term in (18).

Lemma 5. *For any function $v \in C^\infty(U)$ and any function $\varrho \in C_0^\infty(U)$, the following holds:*

$$\left|\int_U \varrho^6[|D_0^2 v|^2 - (\triangle_0 v)^2]dx\right| \le c\int_U \varrho^6|\nabla_{\mathcal{H}} v|^2 dx + c\int_U \varrho^6|\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{R}} v|dx + c\int_U |\varrho|^5[|\nabla_{\mathcal{H}}\varrho| + |\varrho|]|\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} v|dx, \tag{22}$$

where $c = c(n) > 0$.

Proof. Recall that

$$D_0^2 v = \left(\frac{X_iX_jv + X_jX_iv}{2}\right)_{1\le i,j\le 2n} \quad \text{and} \quad \triangle_0 v = \sum_{i=1}^{2n} X_iX_iv.$$

Then,

$$\begin{aligned}
&[|D_0^2 v|^2 - (\triangle_0 v)^2] \\
&= \sum_{i,j=1}^{2n}\left(\frac{X_iX_jv + X_jX_iv}{2}\right)^2 - \left(\sum_{i=1}^{2n} X_iX_iv\right)^2 \\
&= \sum_{i,j=1}^{2n}\left[\frac{1}{4}[(X_iX_jv)^2 + (X_jX_iv)^2 + 2X_iX_jvX_jX_iv] - X_iX_ivX_jX_jv\right] \\
&= \frac{1}{4}\sum_{i,j=1}^{2n}[(X_iX_jv)^2 - X_iX_ivX_jX_jv] + \frac{1}{4}\sum_{i,j=1}^{2n}[(X_jX_iv)^2 - X_iX_ivX_jX_jv] \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{2n}[X_iX_jvX_jX_iv - X_iX_ivX_jX_jv] \\
&= \frac{1}{2}\sum_{i,j=1}^{2n}[(X_iX_jv)^2 - X_iX_ivX_jX_jv] + \frac{1}{2}\sum_{i,j=1}^{2n}[X_iX_jvX_jX_iv - X_iX_ivX_jX_jv].
\end{aligned} \tag{23}$$

Firstly, we estimate the integral term $\int_U \varrho^6(X_iX_jv)^2dx$. We apply integration by parts to obtain

$$\begin{aligned}
&\int_U \varrho^6(X_iX_jv)^2dx \\
&= -\int_U \varrho^6X_jvX_iX_iX_jvdx - 6\int_U \varrho^5X_i\varrho X_jvX_iX_jvdx \\
&= -\int_U \varrho^6X_jvX_iX_jX_ivdx - \int_U \varrho^6X_jvX_i[X_i,X_j]vdx - 6\int_U \varrho^5X_i\varrho X_jvX_iX_jvdx \\
&= -\int_U \varrho^6X_jvX_jX_iX_ivdx - 6\int_U \varrho^5X_i\varrho X_jvX_iX_jvdx \\
&\quad - \int_U \varrho^6X_jv[X_i,X_j]X_ivdx - \int_U \varrho^6X_jvX_i[X_i,X_j]vdx \\
&= \int_U \varrho^6X_jX_jvX_iX_ivdx + 6\int_U \varrho^5X_j\varrho X_jvX_iX_ivdx - 6\int_U \varrho^5X_i\varrho X_jvX_iX_jvdx \\
&\quad - \int_U \varrho^6X_jv[X_i,X_j]X_ivdx - \int_U \varrho^6X_jvX_i[X_i,X_j]vdx.
\end{aligned} \tag{24}$$

Secondly, we estimate the integral term $\int_U \varrho^6X_iX_jvX_jX_ivdx$. We apply integration by parts to obtain

$$
\begin{aligned}
&\int_U \varrho^6 X_i X_j v X_j X_i v dx \\
&\quad = -\int_U \varrho^6 X_j v X_i X_j X_i v dx - 6\int_U \varrho^5 X_i \varrho X_j v X_j X_i v dx \\
&\quad = -\int_U \varrho^6 X_j v X_j X_i X_i v dx - \int_U \varrho^6 X_j v [X_i, X_j] X_i v dx - 6\int_U \varrho^5 X_i \varrho X_j v X_j X_i v dx \\
&\quad = \int_U \varrho^6 X_j X_j v X_i X_i v dx - \int_U \varrho^6 X_j v [X_i, X_j] X_i v dx \\
&\qquad + 6\int_U \varrho^5 X_j \varrho X_j v X_i X_i v dx - 6\int_U \varrho^5 X_i \varrho X_j v X_j X_i v dx.
\end{aligned} \tag{25}
$$

Thirdly, we control the term $[X_i, X_j]X_i$. By (10), we have

$$
\begin{aligned}
|[X_i, X_j]X_i v| &\le c(n,v)(|\nabla_{\mathcal{R}} \nabla_{\mathcal{H}} v| + |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v|) \\
&\le c(n,v)(|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v| + |\nabla_{\mathcal{H}} \nabla_{\mathcal{R}} v| + |\nabla_{\mathcal{H}} v| + |\nabla_{\mathcal{R}} v|).
\end{aligned} \tag{26}
$$

Finally, combining (23)–(26), we conclude that (22) holds by $|\nabla_{\mathcal{R}} v| \le c(n,v)|\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} v|$. □

As regards the integral term $\int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon||\nabla_{\mathcal{H}} \nabla_{\mathcal{R}} u^\varepsilon| dx$ in (22), we establish the following upper bound by applying some Caccoippoli-type inequalities established in Lemma 1.

Lemma 6. *Let $u^\varepsilon \in W^{1,p}_{\mathcal{H},\mathrm{loc}}$ be the weak solution to Equation (7). When $2 < p \le 4$, the following holds for any function $\varrho \in C_0^\infty(U)$:*

$$
\begin{aligned}
&\int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon||\nabla_{\mathcal{H}} \nabla_{\mathcal{R}} u^\varepsilon| dx \\
&\quad \le cK_\varrho \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{2}} dx \right)^{\frac{1}{4}} \\
&\qquad \times \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} dx \right)^{\frac{1}{4}} \left(\int_U \varrho^6 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{4-p}{2}} dx \right)^{\frac{1}{2}},
\end{aligned} \tag{27}
$$

where K_ϱ is notated in (3), $c = c(n,p) > 0$, and the support of ϱ is notated as $spt(\varrho)$.

Proof. By (12) in Lemma 1 with $\beta = 0$ and ϱ replaced by ϱ^3, we derive

$$
\begin{aligned}
\int_U \varrho^6 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{H}} \nabla_{\mathcal{R}} u^\varepsilon|^2 dx \le & cK_\varrho \int_U \varrho^4 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{R}} u^\varepsilon|^2 dx \\
& + c\int_U \varrho^6 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p}{2}} dx.
\end{aligned} \tag{28}
$$

Applying Hölder's inequality, and then letting $\beta = 1$ in Lemma 2, we derive

$$
\begin{aligned}
&\int_U \varrho^4 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{R}} u^\varepsilon|^2 dx \\
&\quad \le \left(\int_U \varrho^4 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{R}} u^\varepsilon|^4 dx \right)^{\frac{1}{2}} \left(\int_U \varrho^4 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} dx \right)^{\frac{1}{2}} \\
&\quad \le K_\varrho \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{2}} dx \right)^{\frac{1}{2}} \left(\int_U \varrho^4 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{29}
$$

Combining (28) and (29), and then applying Hölder's inequality, we derive

$$\int_U \varrho^6(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}\nabla_{\mathcal{R}}u^\varepsilon|^2dx$$
$$\leq cK_\varrho^2\left(\int_{spt(\varrho)}(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p+2}{2}}dx\right)^{\frac{1}{2}}\left(\int_U \varrho^4(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}dx\right)^{\frac{1}{2}}. \tag{30}$$

By Hölder's inequality, we have

$$\int_U \varrho^6|\nabla_{\mathcal{H}}u^\varepsilon||\nabla_{\mathcal{H}}\nabla_{\mathcal{R}}u^\varepsilon|dx \leq \int_U \varrho^6(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{1}{2}}|\nabla_{\mathcal{H}}\nabla_{\mathcal{R}}u^\varepsilon|dx$$
$$\leq\left(\int_U \varrho^6(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{p-2}{2}}|\nabla_{\mathcal{H}}\nabla_{\mathcal{R}}u^\varepsilon|^2dx\right)^{\frac{1}{2}}\left(\int_U \varrho^6(\varepsilon+|\nabla_{\mathcal{H}}u^\varepsilon|^2)^{\frac{4-p}{2}}dx\right)^{\frac{1}{2}}. \tag{31}$$

Combining (30) and (31), we conclude that (27) holds.
□

Finally, we bound the integral term involving $|Mv|^2$ as below.

Lemma 7. *For any function $v\in C^\infty(U)$ and any function $\varrho\in C_0^\infty(U)$, the following holds:*

$$\int_U \varrho^6|Mv|^2dx \leq c\int_U \varrho^6|\nabla_{\mathcal{H}}v||\nabla_{\mathcal{H}}\nabla_{\mathcal{R}}v|dx+c\int_U|\varrho|^5|\nabla_{\mathcal{H}}\varrho||\nabla_{\mathcal{H}}v||\nabla_{\mathcal{H}}v|dx$$
$$+c\int_U \varrho^6|\nabla_{\mathcal{H}}v|^2dx. \tag{32}$$

Proof. Recall that

$$Mv=\nabla_{\mathcal{H}}\nabla_{\mathcal{H}}v-D_0^2v=\left(\frac{[X_i,X_j]v}{2}\right)_{1\leq i,j\leq 2n}.$$

Then, from (10), we have

$$|Mv|^2=\frac{1}{4}\sum_{i,j=1}^{2n}([X_i,X_j]v)^2$$
$$=\frac{1}{4}\sum_{i,j=1}^{2n}\left(\sum_{k=1}^{2n}\lambda_{i,j}^{(k)}X_k+\sum_{l=1}^{v}\theta_{i,j}^{(l)}R_l\right)^2\leq c(n,v)[|\nabla_{\mathcal{H}}v|^2+|\nabla_{\mathcal{R}}v|^2].$$

Thus,

$$\int_U \varrho^6|Mv|^2dx\leq c(n,v)\int_U \varrho^6|\nabla_{\mathcal{R}}v|^2dx+c(n,v)\int_U \varrho^6|\nabla_{\mathcal{H}}v|^2dx. \tag{33}$$

We estimate the integral of $|\nabla_{\mathcal{R}}v|^2$. By Property 1, we note that

$$R_k=-[X_{k1},X_{k2}],\quad k1,k2\in\{1,2,\dots,2n\}.$$

Then, we have

$$\int_U \varrho^6|\nabla_{\mathcal{R}}v|^2dx=\sum_{k=1}^{v}\int_U \varrho^6(R_kv)^2dx=-\sum_{k=1}^{v}\int_U \varrho^6[X_{k1},X_{k2}]vR_kvdx.$$

Since $[X_{k1}, X_{k2}] = X_{k1}X_{k2} - X_{k2}X_{k1}$, by integration by parts, we have

$$\begin{aligned}\int_U \varrho^6 [X_{k1}, X_{k2}] v R_k v dx &= \int_U \varrho^6 X_{k1}X_{k2} v R_k v dx - \int_U \varrho^6 X_{k2}X_{k1} v R_k v dx \\ &= -\int_U \varrho^6 X_{k2} v X_{k1} R_k v dx - 6\int_U \varrho^5 X_{k1}\varrho X_{k2} v R_k v dx \\ &\quad + \int_U \varrho^6 X_{k1} v X_{k2} R_k v dx + 6\int_U \varrho^5 X_{k2}\varrho X_{k1} v R_k v dx.\end{aligned}$$

Combining the above two equations, we have

$$\begin{aligned}\int_U \varrho^6 |\nabla_{\mathcal{R}} v|^2 dx \le& c(n,v)\int_U \varrho^6 |\nabla_{\mathcal{H}} v||\nabla_{\mathcal{H}}\nabla_{\mathcal{R}} v| dx \\ &+ c(n,v)\int_U |\varrho|^5 |\nabla_{\mathcal{H}}\varrho||\nabla_{\mathcal{H}} v||\nabla_{\mathcal{R}} v| dx.\end{aligned} \tag{34}$$

Combining (33) and (34), we conclude that (32) holds.
□

5. Proof of Theorem 2

Now, the proof of Theorem 2 is shown in this section.

Proof of Theorem 2. We prove (8) and (9) in turn. When $1 < p \le 2$, by (13) with $\beta = \frac{2-p}{2}$ in Lemma 1, we have

$$\int_U \varrho^2 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\varepsilon|^2 dx \le c\int_U \varrho^2 |\nabla_{\mathcal{R}} u^\varepsilon|^2 dx + cK_\varrho \int_U (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2) dx. \tag{35}$$

Applying Hölder's inequality and the fact that $|\nabla_{\mathcal{R}} u^\varepsilon| \le c(n,v)|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\varepsilon|$, and then letting $\beta = 1$ in Lemma 2, we derive

$$\begin{aligned}&\int_U \varrho^2 |\nabla_{\mathcal{R}} u^\varepsilon|^2 dx \\ &= \int_U \varrho^2 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{2-p}{4}} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{4}} |\nabla_{\mathcal{R}} u^\varepsilon|^2 dx \\ &\le \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{2-p}{2}} dx\right)^{\frac{1}{2}} \left(\int_U \varrho^4 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{R}} u^\varepsilon|^4 dx\right)^{\frac{1}{2}} \\ &\le cK_\varrho \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{2-p}{2}} dx\right)^{\frac{1}{2}} \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{2}} dx\right)^{\frac{1}{2}}.\end{aligned} \tag{36}$$

Finally, we combine (35) and (36), and then apply Hölder's inequality to conclude (8). When $2 < p \le 4$, note that (4) implies

$$2n + 2(p-2) + (2-2n)(p-2)^2 = 2(p-1)[(1-n)(p-2) + n] > 0.$$

Recall that

$$|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\varepsilon|^2 = |D_0^2 u^\varepsilon|^2 + |Mu^\varepsilon|^2.$$

From this, by Lemmas 4, 5 and 7, and the fact $|\nabla_{\mathcal{R}} u^\varepsilon| \le c(n,v)|\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\varepsilon|$, when $2 < p \le 4$ and p satisfies (4), we have

$$\begin{aligned}\int_U \varrho^6 |\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\varepsilon|^2 dx \le& c\int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon|^2 dx + c\int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon||\nabla_{\mathcal{H}}\nabla_{\mathcal{R}} u^\varepsilon| dx \\ &+ c\int_U |\varrho|^5 [|\nabla_{\mathcal{H}}\varrho| + |\varrho|]|\nabla_{\mathcal{H}} u^\varepsilon||\nabla_{\mathcal{H}}\nabla_{\mathcal{H}} u^\varepsilon| dx,\end{aligned}$$

where $c = c(n, p) > 0$. From this, we apply Young's inequality to derive

$$\int_U \varrho^6 |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\varepsilon|^2 dx \leq c \int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon|^2 dx + c \int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon| |\nabla_{\mathcal{H}} \nabla_{\mathcal{R}} u^\varepsilon| dx \\ + c \int_U \varrho^4 [|\nabla_{\mathcal{H}} \varrho|^2 + \varrho^2] |\nabla_{\mathcal{H}} u^\varepsilon|^2 dx. \tag{37}$$

We use Lemma 6 to estimate the term $\int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon| |\nabla_{\mathcal{H}} \nabla_{\mathcal{R}} u^\varepsilon| dx$ in (37), then

$$\int_U \varrho^6 |\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u^\varepsilon|^2 dx \leq c \int_U \varrho^6 |\nabla_{\mathcal{H}} u^\varepsilon|^2 dx + c \int_U \varrho^4 [|\nabla_{\mathcal{H}} \varrho|^2 + \varrho^2] |\nabla_{\mathcal{H}} u^\varepsilon|^2 dx \\ + c K_\varrho \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{2}} dx \right)^{\frac{1}{4}} \\ \times \left(\int_{spt(\varrho)} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{2}} dx \right)^{\frac{1}{4}} \left(\int_U \varrho^6 (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{4-p}{2}} dx \right)^{\frac{1}{2}}.$$

From this, noting that

$$|\nabla_{\mathcal{H}} u^\varepsilon|^2 \leq (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p}{4}} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{4-p}{4}},$$

and

$$(\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p}{4}} = (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p-2}{4}} (\varepsilon + |\nabla_{\mathcal{H}} u^\varepsilon|^2)^{\frac{p+2}{4}},$$

then we apply Hölder's inequality to conclude (9). □

6. Conclusions

In this paper, on the semi-simple Lie group endowed with the horizontal vector fields $X_1, \ldots, X_{2n}$, we establish a structural inequality of the ∞-subLaplacian $\triangle_{0,\infty}$, that is,

$$\left| |D_0^2 v \nabla_{\mathcal{H}} v|^2 - \triangle_0 v \triangle_{0,\infty} v - \frac{1}{2} [|D_0^2 v|^2 - (\triangle_0 v)^2] |\nabla_{\mathcal{H}} v|^2 \right| \\ \leq (n-1) [|D_0^2 v|^2 |\nabla_{\mathcal{H}} v|^2 - |D_0^2 v \nabla_{\mathcal{H}} v|^2].$$

This structural inequality is stronger and more precise than the Cordes condition first established by Domokos–Manfredi [13,22]. When $1 < p \leq 4$ with $n = 1$ and $1 < p < 3 + \frac{1}{n-1}$ with $n \geq 2$, we apply the structural inequality to obtain the local horizontal $W^{2,2}$-regularity of weak solutions u to the p-Laplacian equation in the semi-simple Lie group, that is, $\nabla_{\mathcal{H}} \nabla_{\mathcal{H}} u \in L^2_{\mathrm{loc}}$. This regularity result improves the range of p in [23] (p near 2 for the Grušin plane), Ref. [13] ($\frac{\sqrt{17}-1}{2} \leq p < \frac{5+\sqrt{5}}{2}$ for the Heisenberg group $\mathbb{H}^1$) and [5] (Theorem 4.1) ($2 \leq p < \frac{2\nu}{\nu-1}$). Compared to Euclidean spaces $\mathbb{R}^{2n}$ with $n \geq 2$, the range of this p obtained is already optimal. Our method can also be applied to more general vector fields.

In summary, these results established in this paper are original. We are convinced that our results will be broadly applicable to study the regularity for p-Laplacian-type equations and other fields of applied sciences.

Author Contributions: Writing—original draft, C.Y. and Y.Z.; writing—review and editing, C.Y. and Y.Z. Both authors contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: Chengwei Yu is partially supported by the National Natural Science Foundation of China (No. 12025102).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors wish to express their hearty thanks to the anonymous referees for their valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Dong, H.; Peng, F.; Zhang, Y.; Zhou, Y. Hessian estimates for equations involving p-Laplacian via a fundamental inequality. *Adv. Math.* **2020**, *370*, 107212. [CrossRef]
2. Liu, J.; Peng, F.; Zhou, Y. $HW^{2,2}_{\mathrm{loc}}$-regularity for p-harmonic functions in Heisenberg groups. *arXiv* **2021**, arXiv:2112.07908.
3. Domokos, A.; Manfredi, J.J. $C^{1,\alpha}$-subelliptic regularity on $\mathrm{SU}(3)$ and compact, semi-simple Lie groups. *Anal. Math. Phys.* **2020**, *10*, 1664–2368. [CrossRef]
4. Hörmander, L. Hypoelliptic second order differential equations. *Acta Math.* **1967**, *119*, 147–171. [CrossRef]
5. Domokos, A.; Manfredi, J.J. Nonlinear subelliptic equations. *Manuscripta Math.* **2009**, *130*, 251–271. [CrossRef]
6. Evans, L.C. A new proof of local $C^{1,\alpha}$-regularity for solutions of certain degenerate elliptic p.d.e. *J. Differ. Equ.* **1982**, *45*, 356–373. [CrossRef]
7. Lewis, J.L. Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.* **1983**, *32*, 849–858. [CrossRef]
8. Tolksdorf, P. Regularity for a more general class of quasilinear elliptic equations. *J. Differ. Equ.* **1984**, *51*, 126–150. [CrossRef]
9. Uhlenbeck, K.K. Regularity for a class of non-linear elliptic systems. *Acta Math.* **1977**, *138*, 219–240. [CrossRef]
10. Ural'ceva, N.N. Degenerate quasilinear elliptic systems. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **1968**, *7*, 184–222.
11. Manfredi, J.J.; Weitsman, A. On the Fatou theorem for p-harmonic functions. *Commun. Partial Differ. Equ.* **1988**, *13*, 651–668. [CrossRef]
12. Domokos, A.; Manfredi, J.J. $C^{1,\alpha}$-regularity for p-harmonic functions in the Heisenberg group for p near 2. *Contemp. Math.* **2005**, *370*, 17–23.
13. Domokos, A.; Manfredi, J.J. Subelliptic cordes estimates. *Proc. Am. Math. Soc.* **2005**, *133*, 1047–1056. [CrossRef]
14. Manfredi, J.J.; Mingione, G. Regularity results for quasilinear elliptic equations in the Heisenberg group. *Math. Ann.* **2007**, *339*, 485–544. [CrossRef]
15. Mingione, G.; Zatorska-Goldstein, A.; Zhong, X. Gradient regularity for elliptic equations in the Heisenberg group. *Adv. Math.* **2009**, *222*, 62–129. [CrossRef]
16. Ricciotti, D. *p-Laplace Equation in the Heisenberg Group*; Springer International Publishing: Berlin/Heidelberg, Germany, 2015. [CrossRef]
17. Zhong, X. Regularity for variational problems in the Heisenberg group. *arXiv* **2017**. [CrossRef]
18. Mukherjee, S.; Zhong, X. $C^{1,\alpha}$-regularity for variational problems in the Heisenberg group. *Anal. PDE* **2021**, *14*, 567–594. [CrossRef]
19. Citti, G.; Mukherjee, S. Regularity of quasi-linear equations with Hörmander vector fields of step two. *Adv. Math.* **2022**, *408*, 108593. [CrossRef]
20. Mukherjee, S.; Sire, Y. Regularity of inhomogeneous quasi-linear equations on the Heisenberg group. *Anal. Theory Appl.* **2021**, *37*, 1672–4070.
21. Yu, C. Regularity for quasi-linear p-Laplacian type non-homogeneous equations in the Heisenberg Group. *Mathematics* **2022**, *10*, 4129. [CrossRef]
22. Domokos, A. $W^{2,2}$ estimates for solutions to non-uniformly elliptic PDE's with measurable coefficients. *J. Inequal. Pure Appl. Math.* **2005**, *3*, 1443–5756.
23. Di Fazio, G.; Domokos, A.; Fanciullo, M.S.; Manfredi, J.J. $C^{1,\alpha}_{\mathrm{loc}}$ regularity for subelliptic p-harmonic functions in Grušin plane. *Matematiche.* **2005**, 469–473.
24. Yu, C. Second order Sobolev regularity for p-harmonic functions in $\mathrm{SU}(3)$. *Electron. J. Differ. Equ.* **2022**, *2022*, 27. [CrossRef]
25. Yu, C. Regularity for p-harmonic functions in the Grušin plane. *J. Math. Study.* **2023**, *3*, 2096–9856. [CrossRef]
26. Reshetnyak, Y.G. *Space Mappings with Bounded Distortion*; Springer: Naukavosibirsk, Russia, 1982. (In Russian)
27. Hajlasz, P.; Koskela, P. Sobolev Met Poincaré. *Mem. Amer. Math. Soc.* **2000**, *145*, 688. [CrossRef]
28. Heinonen, J.; Koskela, P. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* **1998**, *181*, 1–61. [CrossRef]
29. Vodop'yanov, S.K.; Greshnov, A.V. On extension of functions of bounded mean oscillation from domains in a space of homogeneous type with intrinsic metric. *Sib. Math. J.* **1995**, *36*, 873–901. [CrossRef]
30. Vodop'yanov, S.K.; Greshnov, A.V. Analytic properties of quasiconformal mappings on Carnot groups. *Sib. Math. J.* **1995**, *36*, 1142–1151. [CrossRef]
31. Vodop'yanov, S.K. Topological and geometric properties of mappings with summable jacobian in Sobolev classes. I. *Sib. Math. J.* **2000**, *41*, 23–48.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.