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Multivalued Contraction Fixed-Point Theorem in *b*-Metric Spaces

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Abstract: The authors explore fixed-point theory in b-metric spaces and strong b-metric spaces. They wish to prove some new extensions of the Covitz and Nadler fixed-point theorem in b-metric spaces. In so doing, they wish to answer a question proposed by Kirk and Shahzad about Nadler's theorem holding in strong b-metric spaces. In addition, they offer an improvement to the fixed-point theorem proven by Dontchev and Hager.

Keywords: b-metric spaces; fixed-point theorems; multivalued maps; contractions

MSC: 47H04; 47H10

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1. Introduction

Fixed-point theory is a major and important tool in the study of nonlinear phenomena. This theory has been applied in such diverse fields as topology, differential equations and inclusions, economics, game theory, engineering, physics, optimal control, and nonlinear functional analyses. Many authors are interested in fixed-point theorems in metric spaces. The concept of a b-metric space is an old notion that is used in many areas of mathematics. In 1970, Coifman and Guzfian [1] introduced a weaker notion of a metric space called a quasi-metric space; some researchers have used the notion of the b-distance in an attempt to include b-distance functions such as

$$d(x,y) = |x-y|^n, \quad x,\ y \in \mathbb{R}^n,$$

to resolve some central questions in harmonic analyses (see also [2–4]). The actual definition of a b-metric was introduced in 1979 by Madas and Segovia [5]. The notion of a b-metric was first used in fixed-point theory by Bakhtin [6] and extended by Czerwik [7]. Chapter 12, and in particular Section 12.1, of the monograph by Kirk and Shahzad [8] presents a nice introduction to the origin and history of this type of metric space as well as some elementary examples of such spaces.

Our aim in this work is to prove some new versions of the Covitz and Nadler fixed-point theorem [9,10] and to answer a question proposed by Kirk and Shahzad [8], namely, does Nadler's theorem hold in strong b-metric spaces [8] (page 128) (see Theorem 3 below)?

2. Preliminaries

We begin with some essential concepts and results. In what follows, $\mathcal{P}(X)$ denotes the set of all nonempty subsets of X so that $\mathcal{P}_{cl,b}(X)$ is the set of all nonempty closed and bounded subsets of X, and $\mathcal{P}_{cp}(X)$ is the set of all nonempty compact subsets of X.

Definition 1. *Let A, $B \in \mathcal{P}(X)$ and define:*

- $H^*_d(A,B) = \sup\{d(a,B) : a \in A\}$;

- $H_d^*(B,A) = \sup\{d(A,b) : b \in B\}$;
- $H_d(A,B) = \max(H_d^*(A,B), H_d^*(B,A))$ *(the b-Hausdorff distance between A and B).*

Remark 1. *For* $\epsilon > 0$, *let*

$$A_\epsilon = \{x \in X : d(x,A) < \epsilon\}.$$

Then,

$$H_d^*(A,B) = \inf\{\epsilon > 0 : A \subset B_\epsilon\}, \ \ H_d^*(B,A) = \inf\{\epsilon > 0 : B \subset A_\epsilon\}.$$

Next, we define what is meant by a b-metric space and a strong b-metric space.

Definition 2. *Let X be a nonempty set and $s \geq 1$. By a b-metric on X, we mean a map d: $X \times X \to [0,\infty)$ with the following properties for all x, y, $z \in X$:*

(i) $d(x,y) = 0$ *if and only if* $x = y$;
(ii) *(Symmetry)* $d(x,y) = d(y,x)$;
(iii) *(s-relaxed triangle inequality)* $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The triple (X,d,s) is called a b-metric space.

Definition 3. *Let X be a nonempty set and $s \geq 1$. By a strong b-metric on X, we mean a map $d : X \times X \to [0,\infty)$ with the following properties for all x, y, $z \in X$:*

(i) $d(x,y) = 0$ *if and only if* $x = y$;
(ii) *(Symmetry)* $d(x,y) = d(y,x)$;
(iii) *(s-relaxed triangle inequality)* $d(x,y) \leq d(x,z) + sd(z,y)$.

The triple (X,d,s) is called a strong b-metric space.

A useful generalization of the s-relaxed triangle inequality is given in the following lemma.

Lemma 1. *Let (X,d,s) be a strong b-metric space. Then, for $x_0, x_1, \ldots, x_n \in X$, we have*

$$d(x_0,x_n) \leq \sum_{i=0}^{n-2} s^{i+1} d(x_i, x_{i+1}) + s^{n-1} d(x_{n-1}, x_n).$$

The next two lemmas will be used in our proofs.

Lemma 2. *Let (X,d,s) be a strong b-metric space. Then, d is a continuous mapping.*

Proof. For any x, y, x_0, $y_0 \in X$,

$$\begin{aligned} d(x,y) &\leq sd(x,x_0) + d(x_0,y) \\ &\leq sd(x,x_0) + d(x_0,y_0) + sd(y_0,y). \end{aligned}$$

Hence,

$$d(x,y) - d(x_0,y_0) \leq sd(x,x_0) + sd(y_0,y).$$

Similarly,

$$d(x_0,y_0) - d(x,y) \leq sd(x,x_0) + sd(y_0,y). \tag{1}$$

This implies that

$$|d(x,y) - d(x_0,y_0)| \leq s[d(x,x_0) + d(y_0,y)],$$

and therefore d is continuous. □

Lemma 3 ([11]). *Let (X,d,s) be a b-metric space. Then, every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ for which there exists $\gamma \in (0,1)$ such that*

$$d(x_n, x_{n+1}) \leq \gamma d(x_n, x_{n-1}), \quad n \in \mathbb{N},$$

is a Cauchy sequence.

Lemma 4. *Let (X,d,s) be a b-metric space and A, $B \in \mathcal{P}_{cp}(X)$, which is the set of all nonempty compact subsets of X. If d is a continuous b-metric, then for any $x \in A$, there exists $y \in B$ such that*

$$d(x,y) \leq d(x,B).$$

Proof. Let $x \in A$; then, for every $n \in \mathbb{N}$, there exists $y_n \in B$ with

$$d(x, y_n) \leq d(x,B) + \frac{1}{n}. \tag{2}$$

Since B is compact, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ converging to $y \in B$. Since d is continuous, letting $n \to \infty$ in (2), we obtain

$$d(x,y) \leq d(x,B) \leq H_d(A,B),$$

which proves the lemma. □

3. Covitz–Nadler-Type Fixed-Point Theorems

In this section, we give versions of the Covitz and Nadler fixed-point theorem in b-metric spaces. They proved their classical fixed-point theorem in metric spaces for contraction multi-valued operators in 1970 (see [9,10]) (also see Deimling [12] (Theorem 11.1)).

Definition 4. *A mapping $F : X \to \mathcal{P}(X)$ is a multivalued map if for each $x \in X$, $F(x) \in \mathcal{P}(X)$. The point p is a fixed point of a multivalued map F if $p \in F(p)$. We will denote the set of fixed points of the mapping F by Fix F.*

We also have the notion of a contraction for multivalued maps.

Definition 5. *If the mapping F has a Lipschitz constant $c < 1$, then f is called a multivalued contraction mapping.*

The following lemma is referred to as the Covitz and Nadler fixed-point theorem [9].

Lemma 5. *Let (X,d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then Fix $X \neq \emptyset$.*

Our first result is contained in the following theorem.

Theorem 1. *Let (X,d,s) be a complete b-metric space and d be continuous. If $F : X \to \mathcal{P}_{cp}(X)$ is a contraction, then Fix $F \neq \emptyset$.*

Proof. Assume that $H_d(F(x), F(y)) \leq Ld(x,y)$ for every $x, y \in X$, where $L \in [0,1)$, and let $x \in X$. Since $F(x)$ is compact, by Lemma 4, we can choose $x_1 \in F(x)$ such that

$$d(x, x_1) \leq d(x, F(x)).$$

Then, we may choose $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) \leq d(x_1, F(x_1)) \text{ implies } d(x_1, x_2) \leq H_d(F(x), F(x_1)).$$

This means that

$$d(x_1, x_2) \leq Ld(x, F(x)).$$

Continuing this way, we can find a sequence $\{x_n : n \in \mathbb{N}\} \subset X$ with

$$d(x_n, x_{n+1}) \leq d(x_n, F(x_n)).$$

Hence,

$$\begin{aligned} d(x_n, x_{n+1}) \leq d(x_n, F(x_n)) &\leq H_d(F(x_{n-1}), x_n) \\ &\leq Ld(x_{n-1}, x_n) \leq L^n d(x, F(x)). \end{aligned}$$

By Lemma 3, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, we let $\tilde{x} = \lim_{n\to\infty} x_n$. Then, $x_{n+1} \in F(x_n)$ for every $n \in \mathbb{N}$, and

$$0 \leq d(\tilde{x}, F(\tilde{x})) \leq s[d(x_{n+1}, \tilde{x}) + d(x_{n+1}, F(\tilde{x}))] \leq s[d(x_{n+1}, \tilde{x}) + Ld(x_n, \tilde{x})].$$

Letting $n \to +\infty$ gives $\tilde{x} \in F(\tilde{x})$ as claimed, and this proves the theorem. □

As a direct consequence of Theorem 1, we are able to obtain the following generalization of Nadler's fixed-point theorem to strong b-metric spaces.

Corollary 1. *Let (X, d, s) be a complete strong b-metric space. If $F : X \to \mathcal{P}_{cp}(X)$ is an L-contraction, then Fix $F \neq \emptyset$.*

Proof. Since (X, d, s) is a complete strong b-metric space, it is complete. By Lemma 2, d is continuous. By Theorem 1, F has at least one fixed point, and this completes the proof. □

Our next result on the existence of a fixed point is contained in the following theorem.

Theorem 2. *Let (X, d, s) be a complete b-metric space and $F : X \to \mathcal{P}_{cl,b}(X)$ be an L-contraction multi-valued mapping. Then, F has a fixed point in X.*

Proof. We will employ a standard iterative procedure for contracting mappings. Let $L \in (0, 1)$ be such that

$$H_d(F(x), F(y)) \leq Ld(x, y) \quad \text{for all } x,\, y \in X.$$

Let $x_0 \in X$ be fixed and choose $x_1 \in F(x_0)$ such that

$$d(x_1, x_0) \leq d(x_0, F(x_0)) + L.$$

From the definition of the Hausdorff distance, we can find $x_2 \in F(x_1)$ with

$$d(x_1, x_2) \leq d(x_1, F(x_1)) + L, \text{ which implies } d(x_1, x_2) \leq H_d(F(x_0), F(x_1)) + L.$$

Similarly, we can find $x_3 \in F(x_2)$, with

$$d(x_3, x_2) \leq H_d(F(x_2), F(x_1)) + L^2.$$

Continuing this process, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_{i+1} \in (x_n)$ and

$$d(x_{i+1}, x_i) \leq H_d(F(x_i), F(x_{i-1})) + L^i.$$

For fixed $m \in \mathbb{N}$,

$$\begin{aligned}
d(x_m, x_{m+1}) &\leq H_d(F(x_m), F(x_{m-1})) + L^m \\
&\leq Ld(x_m, x_{m-1}) + L^m \\
&\leq LH_d(F(x_{m-1}), F(x_{m-2})) + 2L^m \\
&\leq L^2 d(x_{m-1}, x_{m-2}) + 2L^m \\
&\leq L^2 (H_d(F(x_{m-2}), F(x_{m-3})) + L^{m-2}) + 2L^m \\
&\leq L^3 d(x_{m-2}, x_{m-3}) + 3L^m \\
&\ \ \vdots \\
&\leq L^m d(x_1, x_0) + mL^m.
\end{aligned}$$

By the s-relaxed triangle inequality in b-metric spaces, for every $p \in \mathbb{N}$ and $q = [\log_2 p]$,

$$\begin{aligned}
d(x_{m+1}, x_{m+p}) &\leq sd(x_{m+1}, x_{m+2}) + sd(x_{m+2}, x_{m+p}) \\
&\leq sd(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+2^2}) + s^2 d(x_{m+2^2}, x_{m+p}) \\
&\ \ \vdots \\
&\leq \sum_{n=1}^{q} s^n d(x_{m+2^{n-1}}, x_{m+2^n}) + s^{q+1} d(x_{m+2^q}, x_{m+p}).
\end{aligned}$$

By Lemma 1, we obtain

$$\begin{aligned}
d(x_{m+1}, x_{m+p}) \leq \sum_{n=1}^{q} s^{2n} \sum_{i=m}^{m+2^{n-1}-1} d(x_{2^{n-1}+i}, x_{m+2^{n-1}+i+1}) \\
+ s^{2(q+1)} \sum_{i=m}^{m+p-2^q-1} d(x_{2^q+i}, x_{2^q+i+1}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
d(x_{m+1}, x_{m+p}) &\leq \sum_{n=1}^{q} s^{2n} \sum_{i=m}^{m+2^{n-1}-1} (L^{2^{n-1}+i} d(x_0, x_1) + (2^{n-1}+i) L^{2^{n-1}+i}) \\
&\quad + s^{2(q+1)} \sum_{i=m}^{m+p-2^q-1} (L^{2^q+i} d(x_0, x_1) + (2^q+i) L^{2^q+i}) \\
&\leq \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} [L^{2^{n-1}+i+m} d(x_0, x_1) \\
&\quad + (2^{n-1}+i+m) L^{2^{n-1}+i+m}] \\
&\leq L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} L^{2^{n-1}+i} d(x_0, x_1) \\
&\quad + L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} (2^{n-1}+i) L^{2^{n-1}+i} \\
&\quad + mL^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} (2^{n-1}+i) L^{2^{n-1}+i}.
\end{aligned}$$

Using simple calculations, we can see that

$$L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} L^{2^{n-1}+i} d(x_0, x_1) \leq \frac{L^m d(x_0, x_1)}{1-L} \sum_{n=1}^{q+1} L^{2n \log_L s + 2^{n-1}}, \tag{3}$$

and

$$L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} (2^{n-1}+i) L^{2^{n-1}+i}$$
$$\leq L^m \sum_{n=1}^{q+1} s^{2n} 2^{n-1} L^{2^{n-1}} \sum_{i=0}^{2^{n-1}-1} L^i + L^m \sum_{n=1}^{q+1} s^{2n} L^{2^{n-1}} \sum_{i=0}^{2^{n-1}-1} i L^i$$
$$\leq \frac{2L^m}{1-L} \sum_{n=1}^{q+1} (2s)^{2n} L^{2^{n-1}} + L^m \sum_{i=0}^{\infty} i L^i \sum_{n=1}^{q+1} s^{2n} L^{2^{n-1}}.$$

Then,

$$L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} (2^{n-1}+i) L^{2^{n-1}+i}$$
$$\leq \frac{2L^m}{1-L} \sum_{n=1}^{q+1} L^{2n \log_L 2s + 2^{n-1}} + L^m \sum_{i=0}^{\infty} i L^i \sum_{n=1}^{q+1} L^{2n \log_L s + 2^{n-1}}.$$

Hence,

$$L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} 2^{n-1} L^{2^{n-1}+i} \leq \frac{2L^m}{1-L} \sum_{n=1}^{q+1} L^{2n \log_L 2s + 2^{n-1}}, \tag{4}$$

and

$$L^m \sum_{n=1}^{q+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} i L^{2^{n-1}+i} \leq L^m \sum_{i=0}^{\infty} i L^i \sum_{n=1}^{q+1} L^{2n \log_L s + 2^{n-1}}. \tag{5}$$

We observe that

$$\lim_{n\to\infty} (2n \log_L s + 2^{n-1} - n) = \lim_{n\to\infty} (2n \log_L 2s + 2^{n-1} - n) = \infty.$$

For a fixed $M > 0$, there exist $n_0 \in \mathbb{N}$ such that

$$2n \log_L s + 2^{n-1} - n \geq M, \text{ and } 2n \log_L 2s + 2^{n-1} - n \geq M, \quad \text{for all } n \geq n_0.$$

Then,

$$L^{2n \log_L s + 2^{n-1}} \leq L^M L^n \text{ and } L^{2n \log_L 2s + 2^{n-1}} \leq L^n L^M,$$

and since $\lim_{n\to\infty} \frac{(n+1)L^{n+1}}{nL^n} = L \in (0,1)$, we conclude that

$$L_1 := \sum_{n=1}^{\infty} L^{2n \log_L s + 2^{n-1}}, \; L_2 := \sum_{n=1}^{\infty} L^{2n \log_L 2s + 2^{n-1}}, \; L_3 := \sum_{n=1}^{\infty} n L^n \tag{6}$$

are convergent series. Using (3)–(6), we obtain

$$d(x_{m+1}, x_{m+p}) \leq \frac{L^m L_1 d(x_1, x_0)}{1-L} + \frac{(2+(1-L)L_1)L_2(1+m)L^m}{1-L}.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and so $x_n \to x$ for some $x \in X$.
Next, we prove that $x \in F(x)$. For all $n \in \mathbb{N}$,

$$0 \leq d(x, F(x)) \leq s[d(x, x_n) + d(x_n, F(x))]$$
$$\leq s[d(x, x_n) + H_d(F(x_{n-1}), F(x))]$$
$$\leq s[d(x, x_n) + L d(x_{n-1}, x)].$$

Letting $n \to \infty$, we see that

$$d(x, F(x)) = 0,$$

which implies $x \in F(x)$, and so x is a fixed point of F. This proves the theorem. □

Remark 2. *In [13], Czerwik obtained the result in Theorem 2 for b-metric spaces, but with the more restrictive condition that $sL \in (0,1)$. Kirk and Shahzad [8] (Theorem 12.5) relaxed the result for strong b-metric spaces with $L \in (0,1)$. Theorem 2 is an extension of the results of Czerwik and Kirk and Shahzad.*

4. Local Version of the Covitz–Nadler Theorem

For the next result, we give a version of the fixed-point theorems proved by Beer and Dontchev [14] (see Theorem 4) and Dontchev and Hager [15] in a strong b-metric space. Hence, we obtain a partial answer to the question raised by Kirk and Shahzad [8] (p. 128).

Theorem 3. *Let (X, d, s) be a complete strong b-metric space and $F : X \to \mathcal{P}_{cp}(X)$. Assume there exist $x_0 \in X$, $r > 0$, and $sL \in (0, 1)$ such that*

(i) $d(x_0, F(x_0)) < r(1 - sL)$;

(ii) $H_d^*(F(x) \cap \bar{B}(x_0, r), F(y)) \leq Ld(x, y)$ *for all* $x, y \in \bar{B}(x_0, r)$.

Then, F has a fixed point in $\bar{B}(x_0, r)$.

Proof. Since $F(x_0) \in \mathcal{P}_{cp}(X)$, there exists $x_1 \in F(x_0)$ with $x_1 \in B(x_0, r)$ such that

$$d(x_1, x_0) < r(1 - sL) \tag{7}$$

and

$$H_d^*(F(x_0) \cap \bar{B}(x_0, r), F(x_1)) \leq Ld(x_1, x_0).$$

Since $x_1 \in F(x_0) \cap \bar{B}(x_0, r)$,

$$d(x_1, F(x_1)) \leq H_d^*(F(x_0) \cap \bar{B}(x_0, r), F(x_1)) \leq Ld(x_1, x_0) < r(1 - sL)L.$$

Then, there exists $x_2 \in F(x_1)$ with

$$d(x_1, x_2) < r(1 - sL)L,$$

so we have

$$d(x_0, x_2) \leq d(x_0, x_1) + sd(x_1, x_2) < r(1 - sL) + sr(1 - sL)L,$$

that is,

$$d(x_1, x_2) < r(1 - sL)L, \ \ d(x_0, x_2) < r(1 - (sL)^2), \ \ \text{and } x_2 \in \bar{B}(x_0, r). \tag{8}$$

Hence,

$$d(x_2, F(x_2)) \leq H_d^*(F(x_1) \cap \bar{B}(x_0, r), F(x_2)) \leq Ld(x_1, x_2) < rL^2(1 - sL).$$

Then, there exists $x_3 \in F(x_2)$ such that

$$d(x_2, x_3) < rL^2(1 - sL),$$

and so

$$\begin{aligned} d(x_0, x_3) \leq d(x_0, x_2) + sd(x_2, x_3) &\leq r(1 - (sL)^2) + srL^2(1 - sL) \\ &\leq r(1 - (sL)^2) + srL^2(1 - (sL)^2) \end{aligned}$$

since $sL < 1$. We then have

$$d(x_2, x_3) < rL^2(1 - sL), \ \ d(x_0, x_3) < r(1 - (sL)^4), \ \ \text{and } x_3 \in \bar{B}(x_0, r). \tag{9}$$

From (7)–(9), we can proceed by induction, so that there exist $(x_n)_{n\in\mathbb{N}} \subset \bar{B}(x_0, r)$ with $x_n \in F(x_{n-1})$, $n \in \mathbb{N}$, such that

$$d(x_n, x_{n+1}) < rL^n(1 - sL), \quad n \in \mathbb{N}_0.$$

By the s-relaxed triangular inequality, for $n \geq m$, we have

$$d(x_m, x_n) \leq s\sum_{i=m}^{n-1} d(x_i, x_{i+1}) \leq rs(1 - sL)\sum_{i=m}^{n-1} L^i \leq rs(1 - sL)L^m \sum_{i=0}^{\infty} L^i.$$

Therefore,

$$d(x_m, x_n) \leq \frac{rs(1 - sL)L^m}{1 - L} \to 0 \text{ as } m \to \infty,$$

which implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x \in \bar{B}(x_0, r)$. By condition (ii),

$$d(x_n, F(x)) \leq H_d^*(F(x_{n-1}) \cap \bar{B}(x_0, r), F(x)) \leq Ld(x_{n-1}, x).$$

The s-relaxed triangle inequality implies that

$$d(x, F(x)) \leq sd(x, x_n) + d(x_n, F(x)) \leq sd(x, x_n) + Ld(x_{n-1}, x) \to 0$$

as $n \to \infty$. Therefore, $d(x, F(x)) = 0$, and hence, x is a fixed point of F. This proves the theorem. □

A second result in the same direction is contained in the following theorem.

Theorem 4. *Let (X, d, s) be a complete strong b-metric space and $F : X \to \mathcal{P}_{cl}(X)$. Assume there exist $x_0 \in X$, $r > 0$, and $L \in (0, 1)$ such that*

(i) $d(x_0, F(x_0)) < \frac{r}{s}(1 - L)$;

(ii) $H_d^*(F(x) \cap \bar{B}(x_0, r), F(y)) \leq Ld(x, y)$ *for all* $x, y \in \bar{B}(x_0, r)$.

Then, F has a fixed point in $\bar{B}(x_0, r)$.

Proof. Since $F(x_0) \in \mathcal{P}_{cp}(X)$, there exists $x_1 \in F(x_0)$ with $x_1 \in \bar{B}(x_0, r)$ such that

$$d(x_1, x_0) < \frac{r}{s}(1 - L)$$

and

$$H_d^*(F(x_0) \cap \bar{B}(x_0, r), F(x_1)) \leq Ld(x_1, x_0).$$

Since $x_1 \in F(x_0) \cap \bar{B}(x_0, r)$,

$$d(x_1, F(x_1)) \leq H_d^*(F(x_0) \cap \bar{B}(x_0, r), F(x_1)) \leq Ld(x_1, x_0) < \frac{r}{s}(1 - L)L,$$

and so there exists $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) < \frac{r}{s}(1 - L)L.$$

Hence, we have

$$d(x_0, x_2) \leq s[d(x_0, x_1) + d(x_1, x_2)] < r(1 - L) + r(1 - L)L = r(1 - L^2),$$

which means

$$d(x_1, x_2) < \frac{r}{s}(1 - L)L, \;\; d(x_0, x_2) < r(1 - L^2), \;\; \text{and } x_2 \in \bar{B}(x_0, r).$$

Thus,

$$d(x_2, F(x_2)) \le H_d^*(F(x_1) \cap \bar{B}(x_0, r), F(x_2)) \le Ld(x_1, x_2) < \frac{r}{s}L^2(1-L),$$

and so then there exists $x_3 \in F(x_2)$ such that

$$d(x_2, x_3) < \frac{r}{s}L^2(1-L).$$

This implies

$$\begin{aligned} d(x_0, x_3) \le d(x_0, x_2) + sd(x_2, x_3) &\le r(1-L^2) + rL^2(1-L) \\ &\le r(1-L^2) + rL^2(1-L^2) \end{aligned}$$

since $L < 1$. Thus, we have

$$d(x_2, x_3) < \frac{r}{s}L^2(1-L), \;\; d(x_0, x_3) < r(1-L^4), \;\; \text{and } x_3 \in \bar{B}(x_0, r).$$

Proceeding by induction, there exists $(x_n)_{n\in\mathbb{N}} \subset \bar{B}(x_0, r)$ with $x_n \in F(x_{n-1})$, $n \in \mathbb{N}$, such that

$$d(x_n, x_{n+1}) < \frac{r}{s}L^n(1-L), \quad n \in \mathbb{N}_0.$$

As in the proof of Theorem 3, we again see that $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since X is complete, there exists $x \in \bar{B}(x_0, r)$ such that $\lim_{n\to\infty} x_n = x$ and $x \in F(x)$, which proves the theorem. □

The next result is our improvement of Dontchev and Hager's [15] (Lemma) fixed-point theorem.

Theorem 5. *Let (X, d, s) be a complete strong b-metric space and $F : X \to \mathcal{P}(X)$. Assume there exist $x_0 \in X$, $r > 0$, and $L \in (0, 1)$ such that*

(i) The set $\mathcal{G}r(F) \cap \bar{B}(x_0, r) \times \bar{B}(x_0, r)$ is a closed set;
(ii) $d(x_0, F(x_0)) < \frac{r}{s}(1-L)$;
(iii) $H_d^*(F(x) \cap \bar{B}(x_0, r), F(y)) \le Ld(x, y)$ for all $x, y \in \bar{B}(x_0, r)$.

Then, F has a fixed point in $\bar{B}(x_0, r)$.

Proof. Since $d(x_0, F(x_0)) < \frac{r}{s}(1-L)$, there exists $x_1 \in F(x_0)$ with $x_1 \in \bar{B}(x_0, r)$ such that

$$d(x_1, x_0) < \frac{r}{s}(1-L) \tag{10}$$

and

$$H_d^*(F(x_0) \cap \bar{B}(x_0, r), F(x_1)) \le Ld(x_1, x_0).$$

Since $x_1 \in F(x_0) \cap \bar{B}(x_0, r)$,

$$d(x_1, F(x_1)) \le H_d^*(F(x_0) \cap \bar{B}(x_0, r), F(x_1)) \le Ld(x_1, x_0) < \frac{r}{s}(1-L)L,$$

and so there exists $x_2 \in F(x_1)$ such that

$$d(x_1, x_2) < \frac{r}{s}(1-L)L$$

and

$$d(x_0, x_2) \le s[d(x_0, x_1) + d(x_1, x_2)] < r(1-L) + r(1-L)L = r(1-L^2).$$

That is,

$$d(x_1, x_2) < \frac{r}{s}(1-L)L, \ \ d(x_0, x_2) < r(1-L^2), \ \text{ and } \ x_2 \in \bar{B}(x_0, r). \tag{11}$$

Hence,

$$d(x_2, F(x_2)) \leq H_d^*(F(x_1) \cap \bar{B}(x_0, r), F(x_2)) \leq Ld(x_1, x_2) < \frac{r}{s}L^2(1-L),$$

so there exists $x_3 \in F(x_2)$ such that

$$d(x_2, x_3) < \frac{r}{s}L^2(1-L).$$

It follows that

$$\begin{aligned} d(x_0, x_3) \leq d(x_0, x_2) + sd(x_2, x_3) &\leq r(1-L^2) + rL^2(1-L) \\ &\leq r(1-L^2) + rL^2(1-L^2), \end{aligned}$$

that is,

$$d(x_2, x_3) < \frac{r}{s}L^2(1-L), \ \ d(x_0, x_3) < r(1-L^4), \ \text{ and } \ x_3 \in \bar{B}(x_0, r). \tag{12}$$

By induction, there exists

$$(x_n)_{n\in\mathbb{N}} \subset \bar{B}(x_0, r), \ \ x_n \in F(x_{n-1}), \ n \in \mathbb{N}, \tag{13}$$

with

$$d(x_n, x_{n+1}) < \frac{r}{s}L^n(1-L), \quad n \in \mathbb{N}_0.$$

As in the proof of Theorem 3, $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence, and since X is complete, there exists $x \in \bar{B}(x_0, r)$ such that $\lim_{n\to\infty} x_n = x$. Hence, $(x_{n-1}, x_n) \to (x, x)$ as $n \to \infty$. From (13) and condition (i), we have

$$\{(x_{n-1}, x_n)\}_{n\in\mathbb{N}} \subset \mathcal{G}r(F) \cap \bar{B}(x_0, r) \times \bar{B}(x_0, r),$$

and so

$$(x, x) \in \mathcal{G}r(F) \cap \bar{B}(x_0, r) \times \bar{B}(x_0, r).$$

Therefore, $x \in F(x)$ and this completes the proof of the theorem. □

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References

1. Coifman, R.; de Guzman, M. Singular integrals and multipliers on homogeneous spaces. *Rev. Union Mat. Argent.* **1970**, *35*, 137–144.
2. Coifman, R.; Weiss, G. *Analyse Harmonique Non-Commutative sur Certains Spaces Homogenes, Etude de Certaines Integmles Singulieres*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 1971; Volume 242.
3. Donggao, D.; Yongsheng, H. *Harmonic Analysis on Spaces of Homogeneous Type*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 2009; Volume 1966.
4. Mitrea, D.; Mitrea, I.; Mitrea, M.; Monniaux, S. *Groupoid Metrization Theory, with Applications to Analysis on Quasi-Metric Spaces and Functional Analysis*; Applied and Numerical Harmonic Analysis; Birkhäuser: New York, NY, USA, 2013.

5. Macias, R.A.; Segovia, C. Lipschitz functions on spaces of homogeneous type. *Adv. Math.* **1979**, *33*, 257–270. [CrossRef]
6. Bakhtin, I.A. The contraction principle in quasimetric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
7. Czerwik, S. Contraction mappings in *b*-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
8. Kirk, W.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer: New York, NY, USA, 2014.
9. Covitz, H.; Nadler, S.B., Jr. Multi-valued contraction mappings in generalized metric spaces. *Isr. J. Math.* **1970**, *8*, 5–11. [CrossRef]
10. Nadler, S.B., Jr. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]
11. Miculescu, R.; Mihail, A. New fixed point theorems for set-valued contractions in *b*-metric spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 2153–2163. [CrossRef]
12. Deimling, K. *Multi-Valued Differential Equations*; De Gruyter: Berlin, Germany; New York, NY, USA, 1992.
13. Czerwik, S. Nonlinear set-valued contraction mappings in *b*-metric spaces. *Atti Semin. Mat. Fis. Univ. Modena* **1998**, *46*, 263–276.
14. Beer, G.; Dontchev, A.L. The weak Ekeland variational principle and fixed points. *Nonlinear Anal.* **2014**, *102*, 91–96. [CrossRef]
15. Dontchev, A.L.; Hager, W.W. An inverse mapping theorem for setvalued maps. *Proc. Am. Math. Soc.* **1994**, *121*, 481–489. [CrossRef]