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Article

Generalized Almost Periodicity in Measure

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Abstract: This paper investigates diverse classes of multidimensional Weyl and Doss ρ-almost periodic functions in a general measure setting. This study establishes the fundamental structural properties of these generalized ρ-almost periodic functions, extending previous classes such as m-almost periodic and (equi-)Weyl-p-almost periodic functions. Notably, a new class of (equi-)Weyl-p-almost periodic functions is introduced, where the exponent $p > 0$ is general. This paper delves into the abstract Volterra integro-differential inclusions, showcasing the practical implications of the derived results. This work builds upon the extensions made in the realm of Levitan N-almost periodic functions, contributing to the broader understanding of mathematical functions in diverse measure spaces.

Keywords: Weyl ρ-almost periodic functions; Doss ρ-almost periodic functions; general measure; convolution products; Volterra integro-differential inclusions

MSC: 42A75; 43A60; 47D99

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1. Introduction

The class of almost periodic functions was introduced by H. Bohr around 1925 and later generalized by many other mathematicians, including V. Stepanov, H. Weyl, and A. S. Besicovitch. Suppose that $(X, \|\cdot\|)$ is a complex Banach space and $F : \mathbb{R}^n \to X$ is a continuous function ($n \in \mathbb{N}$). Then, we say that the function $F(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$ such that

$$\left\|F(\mathbf{t} + \tau) - F(\mathbf{t})\right\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n;$$

here, $|t - t_0|$ denotes the Euclidean distance in $\mathbb{R}^n$ between t and t_0. Any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic, and it is well known that a continuous function $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$ which converges uniformly to $F(\cdot)$. Almost periodic functions are incredibly important in the qualitative analysis of solutions to the ordinary differential equations, the partial differential equations, and the abstract (nonlinear) Volterra integro-differential equations. These functions exhibit unique properties that make them indispensable for understanding the behavior of dynamic systems. For a comprehensive exploration of almost periodic functions and their diverse applications, readers are encouraged to delve into prominent research monographs such as those authored by A. S. Besicovitch [1], A. M. Fink [2], G. M. N'Guérékata [3], and S. Zaidman [4]. Furthermore, the hierarchy and classification of almost-periodic function spaces, as discussed by J. Andres, A. M. Bersani, and R. F. Grande in [5], offer valuable insights into the structural aspects of this mathematical concept. Additionally, M. Kostić's extensive contributions in [6,7] and T. Diagana's work [8] on almost-automorphic-type and almost-periodic-type functions provides a contemporary perspective on the subject. M. Levitan's foundational text [9] and S. Stoiński's comprehensive treatment [10] are also essential references, offering historical and language-specific insights into almost periodic functions. This diverse array of references forms a solid foundation for readers seeking a nuanced understanding of almost periodic functions and their multifaceted applications.

In physics and crystallography, modulated crystals and almost periodic measures, as explored in works like Lee et al. [11], provide crucial mathematical frameworks for understanding the intricate structures and behaviors of materials. Additionally, their applications extend into medicine, exemplified by E. Alvarez, S. Castillo, M. Pinto [12] work on (ω, c)-pseudoperiodic functions, addressing phenomena in the Lasota–Wazewska model, particularly relevant in medical contexts involving red cell production. Furthermore, the versatility of almost periodic functions finds applications in engineering, as demonstrated by M. F. Hasler and G. M. N'Guérékata's [13] exploration of Bloch-periodic functions, showcasing their utility in addressing engineering challenges with periodic characteristics.

If the function $F : \mathbb{R}^n \to X$ is locally p-integrable, where $1 \leq p < \infty$, then we say that $F(\cdot)$ is Stepanov-p-almost periodic if and only if for every $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \mathbb{R}^n$ with

$$\left\| F(\mathbf{t} + \tau + \mathbf{u}) - F(\mathbf{t} + \mathbf{u}) \right\|_{L^p([0,1]^n : X)} \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n.$$

The class of Weyl almost periodic functions was introduced by H. Weyl in 1927 as a generalization of the class of Stepanov-p-almost periodic functions, and this class was later extended by A. S. Kovanko in 1944: If the function $F : \mathbb{R}^n \to X$ is locally p-integrable, where $1 \leq p < \infty$, then we say that $F(\cdot)$ is:

(i) equi-Weyl-p-almost periodic if and only if, for every $\epsilon > 0$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, L)$ with

$$\sup_{\mathbf{t} \in \mathbb{R}^n} \left[l^{-\frac{n}{p}} \left\| F(\tau + \cdot) - F(\cdot) \right\|_{L^p(\mathbf{t} + l[0,1]^n : X)} \right] < \epsilon. \tag{1}$$

(ii) Weyl-p-almost periodic if and only if, for every $\epsilon > 0$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, L)$ with

$$\limsup_{l \to +\infty} \sup_{\mathbf{t} \in \mathbb{R}^n} \left[l^{-\frac{n}{p}} \left\| F(\tau + \cdot) - F(\cdot) \right\|_{L^p(\mathbf{t} + l[0,1]^n : X)} \right] < \epsilon. \tag{2}$$

(iii) Doss-p-almost periodic if and only if, for every $\epsilon > 0$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, L)$ with

$$\limsup_{l \to +\infty} \left[l^{-\frac{n}{p}} \left\| F(\tau + \cdot) - F(\cdot) \right\|_{L^p(B(0,l) : X)} \right] < \epsilon. \tag{3}$$

Any equi-Weyl-p-almost periodic function is Weyl-p-almost periodic, while the converse statement is not generally true. Furthermore, any Weyl-p-almost periodic function is Doss-p-almost periodic, while the converse statement is not generally true. By (e)-$W^p AP(\mathbb{R}^n : X)$, we denote the space of all (equi-)Weyl-p-almost periodic functions $F : \mathbb{R}^n \to X$.

On the other hand, the class of one-dimensional almost periodic functions in view of the Lebesgue measure (m-almost periodic functions, in short) was introduced by S. Stoiński in 1994 [14] and later reconsidered in a series of important research papers by Polish mathematicians. Notable works include the research by D. Bugajewski and A. Nawrocki on the asymptotic properties and convolutions of m-almost periodic functions, with applications to linear differential equations [15,16]. Additionally, the integration of almost periodic functions into the context of integrate-and-fire models was explored by P. Kasprzak, A. Nawrocki, and J. Signerska-Rynkowska [17]. Furthermore, the application of convolution to linear differential equations with Levitan almost periodic coefficients was studied by A. Nawrocki [18]. These works collectively contribute to a deeper understanding of the rich properties and applications of m-almost periodic functions. In our recent research article [19], we have introduced and analyzed the class of multidimensional almost periodic functions in general measure: a Lebesgue measurable function $F : \mathbb{R}^n \rightarrow X$ is said to be m-almost periodic if and only if for each $\epsilon > 0$ the set

$$\left\{\tau \in \mathbb{R}^n : \sup_{\mathbf{t}\in\mathbb{R}^n} m\Big(\big\{\mathbf{s} \in \mathbf{t} + [0,1]^n : \|F(\mathbf{s}+\tau) - F(\mathbf{s})\| \geq \epsilon\big\}\Big) \leq \epsilon\right\}$$

is relatively dense in $\mathbb{R}^n$. Any Stepanov-p-almost periodic function $F : \mathbb{R}^n \rightarrow X$ is m-almost periodic ($p \geq 1$), and we also know that any bounded, m-almost periodic function $F : \mathbb{R}^n \rightarrow X$ is Stepanov-p-almost periodic ($p \geq 1$). In [19], we observed that the characteristic function of any compact subset of $\mathbb{R}^n$ is not m-almost periodic; on the other hand, we already know that this function is equi-Weyl-p-almost periodic for any finite exponent $p \geq 1$.

In this paper, we further generalize the class of m-almost periodic functions and the class of (equi-)Weyl-p-almost periodic functions (Doss-p-almost periodic functions) by considering a new class of (equi-)Weyl-p-almost periodic functions (Doss-p-almost periodic functions) in general measure, with a general exponent $p > 0$. At this place, it should be also worthwhile to mention that A. Michalowicz and S. Stoiński [20] have extended the class of Levitan N-almost periodic functions in a similar manner, by considering the class of Levitan N-almost periodic functions in the Lebesgue measure.

This paper is structurally organized as follows. After explaining the notation used in this paper, we recall the basic definitions and results of multidimensional Weyl ρ-almost-periodic-type functions in general metrics (Section 2). In Section 3, we introduce and analyze the class of (equi-)Weyl $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic functions; see Definition 4. Our first structural result is given in Proposition 1. In particular, these results show that any (equi)-Weyl-p-almost periodic function $F : \mathbb{R}^n \rightarrow X$ is (equi)-Weyl-m-almost periodic, as well as that any bounded, (equi)-Weyl-m-almost periodic function $F : \mathbb{R}^n \rightarrow X$ is (equi)-Weyl-p-almost periodic ($p \geq 1$). Section 4 investigates the multidimensional Doss ρ-almost periodic type functions in general measure. The class of Doss $(\mathrm{F}, \mathcal{B}, \Lambda', \rho, m', \nu)$-almost periodic functions is introduced in Definition 5 and our first structural result is given in Proposition 2. In Section 5, we present some applications to the abstract Volterra integro-differential inclusions in Banach spaces. We divide this section into two separate subsections. In the final section, we provide several useful remarks, observations and perspectives for the further explorations of generalized almost periodicity and generalized almost automorphy in measure.

2. Mathematical Preliminaries and Notations

We assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces. I denotes the identity operator on Y, $L(Y)$ denotes the Banach space of all bounded linear operators from Y into Y, $n \in \mathbb{N}$ is a fixed integer, and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$ ($s \in \mathbb{R}$). By $\mathcal{B}$, we denote a certain collection of nonempty subsets of X which satisfies that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$, $m(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}^n$ and $P(A)$ stands for the power set of A. Here, the abbreviation a.e. will be used almost everywhere. The vector space $C_b(I : Y)$, where $\emptyset \neq I \subseteq \mathbb{R}^n$, consists of all continuous functions $u : I \rightarrow Y$ satisfying that $\sup_{\mathbf{t}\in I} \|u(\mathbf{t})\|_Y < +\infty$; equipped with the sup-norm $\|\cdot\|_\infty := \sup_{\mathbf{t}\in I} \|\cdot(\mathbf{t})\|_Y$, $C_b(I : Y)$ becomes a Banach space. We will deal henceforth with the space $L_\nu^p(\Omega : Y) := \{u : \Omega \to Y\,;\, u(\cdot)$ is Lebesgue measurable and $||u||_p < \infty\}$, where $p > 0$, $\|\cdot\|_p := \|\nu(\mathbf{t}) \cdot (\mathbf{t})\|_{L^p(\Omega:Y)}$ and $\nu : \Omega \to (0,\infty)$ is a Lebesgue measurable function.

Generalized ρ-almost Periodic Type Functions and Their Metrical Generalizations

In this subsection, we recall the basic definitions and facts about generalized ρ-almost periodic type functions and their metrical generalizations. Assume now that the following conditions hold true:

(WM1-1): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Lambda)$, the collection of all Lebesgue measurable functions from Λ into $[1,+\infty]$, $\Lambda' + \Lambda \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty) \times \Lambda \to (0,\infty)$.

(WM1-2): For every $\mathbf{t} \in \Lambda$ and $l > 0$, $\mathcal{P}_{\mathbf{t},l} = (P_{\mathbf{t},l}, d_{\mathbf{t},l})$ is a pseudometric space of functions from $\mathbb{C}^{\mathbf{t}+l\Omega}$ containing the zero function. Define $\|f\|_{P_{\mathbf{t},l}} := d_{\mathbf{t},l}(f,0)$ for all $f \in P_{\mathbf{t},l}$; $\mathcal{P} = (P,d)$ is a pseudometric space of functions from $\mathbb{C}^{\Lambda}$ containing the zero function and $\|f\|_P := d(f,0)$ for all $f \in P$. The argument from Λ will be denoted by $\cdot\cdot$ and the argument from $\mathbf{t} + l\Omega$ will be denoted by $\cdot$.

In Definition 2 of Ref. [21], we introduced the following notion:

Definition 1. *(i)* *By $e - W^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t},l},\mathcal{P})_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u};x))$, $\mathbf{u} \in \bigcup_{l>0;\mathbf{t}\in\Lambda}(\mathbf{t}+l\Omega)$ is well-defined, and*

$$\sup_{x\in B}\left\|\mathbb{F}(l,\cdot\cdot)\phi\Big(\big\|F(\tau+\cdot;x) - G_x(\cdot)\big\|_{P_{\cdot\cdot,l}}\Big)\right\|_P < \epsilon. \tag{4}$$

(ii) *By $W^{(\phi,\mathbb{F},\rho,\mathcal{P}_{\mathbf{t},l},\mathcal{P})_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto G_x(\mathbf{u}) \in \rho(F(\mathbf{u};x))$, $\mathbf{u} \in \bigcup_{l>0;\mathbf{t}\in\Lambda}(\mathbf{t}+l\Omega)$ is well defined, and*

$$\limsup_{l\to+\infty}\sup_{x\in B}\left\|\mathbb{F}(l,\cdot\cdot)\phi\Big(\big\|F(\tau+\cdot;x) - G_x(\cdot)\big\|_{P_{\cdot\cdot,l}}\Big)\right\|_P < \epsilon. \tag{5}$$

The multidimensional Weyl ρ-almost periodic functions are special cases of the above introduced classes of functions, with $P_{\mathbf{t},l} = L^{p(\cdot)}(\mathbf{t}+l\Omega : \mathbb{C})$, the metric $d_{\mathbf{t},l}$ induced by the norm of this Banach space ($\mathbf{t} \in \Lambda$, $l > 0$), $P = L^\infty(\Lambda : \mathbb{C})$ and the metric d induced by the norm of this Banach space. If $P_{\mathbf{t},l} = L_\nu^{p(\cdot)}(\mathbf{t}+l\Omega : \mathbb{C})$ ($\mathbf{t} \in \Lambda$, $l > 0$) and $P = L^\infty(\Lambda : \mathbb{C})$, then the corresponding space will be denoted by (e-)$W^{p(\mathbf{u}),\phi,\mathbb{F},\nu}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$; similarly, if $p \in (0,1)$, $P_{\mathbf{t},l} = L_\nu^p(\mathbf{t}+l\Omega : \mathbb{C})$ ($\mathbf{t} \in \Lambda$, $l > 0$) and $P = L^\infty(\Lambda : \mathbb{C})$, then the corresponding space will be denoted by (e-)$W^{p,\phi,\mathbb{F},\nu}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$. For more details about the Lebesgue spaces with variable exponent $L^{p(x)}$, we refer the reader to the monograph authored by L. Diening et al. [22].

We need the following notion (see Definition 3.2.1(ii)-(b) of the Ref. [7] with $\phi(x) \equiv x$, and Definition 6.2.11(ii)-(b) of the Ref. [7]. Here and hereafter, $\Lambda_l := \Lambda \cap B(0,l)$):

Definition 2. *Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\nu : \Lambda \to [0,+\infty)$, $p \in \mathcal{P}(\Lambda)$ [$0 < p < 1$] and the function $F : \Lambda \times X \to Y$ satisfies that $\|F(\cdot+\tau;x) - y_{\cdot;x}\| \cdot \nu(\cdot) \in L^{p(\cdot)}(\Lambda_l)$ [$L^p(\Lambda_l)$] for all $l > 0$, $x \in X$, $\tau \in \Lambda'$ and $y_{\cdot;x} \in \rho(F(\cdot;x))$. Then, it is said that $F(\cdot;\cdot)$ is Doss-$(p(\cdot), \mathrm{F}, \mathcal{B}, \Lambda', \rho, \nu)$-almost periodic (Doss-$(p, \mathrm{F}, \mathcal{B}, \Lambda', \rho, \nu)$-almost periodic) if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists a point*

$\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $l > 0$, $x \in B$ and $\cdot \in \Lambda_l$, we have the existence of an element $y_{\cdot;x} \in \rho(F(\cdot;x))$ such that

$$\limsup_{l \to +\infty} \mathrm{F}(l) \sup_{x \in B} \Bigl[\|F(\cdot + \tau; x) - y_{\cdot;x}\|_Y \cdot \nu(\cdot) \Bigr]_P < \epsilon, \tag{6}$$

where $P = L^{p(\cdot)}(\Lambda_l)$ [$P = L^p(\Lambda_l)$].

The usual class of Doss-p-almost periodic functions is obtained by plugging $\Lambda = \Lambda' = \mathbb{R}^n$, $\rho = \mathrm{I}$, $\nu(\cdot) \equiv 1$, $\Omega = [-1,1]^n$ and $\mathrm{F}(l) \equiv l^{-n/p}$ ($p > 0$). A very simple argumentation shows that a p-locally integrable function $F : \mathbb{R}^n \to Y$ is Doss-p-almost periodic if and only if, for every $\epsilon > 0$, there exists $L > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists a point $\tau \in B(\mathbf{t}_0, L)$ such that, for every $\mathbf{t} \in \mathbb{R}^n$, there exists $l_{\mathbf{t}} > 0$ such that, for every $l \geq l_{\mathbf{t}}$, we have

$$\left[(2l)^{-n} \int_{\mathbf{t}+l[-1,1]^n} \|F(\mathbf{s} + \tau) - F(\mathbf{s})\|_Y^p \, d\mathbf{s} \right] \leq \epsilon.$$

3. Multidimensional Weyl ρ-Almost-Periodic-Type Functions in General Measure

We will always assume henceforth that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\nu : \Lambda \to [0, \infty)$, $m' : P(\mathbb{R}^n) \to [0, \infty]$, $m'(\emptyset) = 0$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a nonempty compact set, $\mathbb{F} : (0, \infty) \times \Lambda \to (0, \infty)$ and $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$. For every $\epsilon > 0$, $l > 0$ and for every two functions $f : \Lambda \to Y$ and $g : \Lambda \to Y$, we define

$$d_{\epsilon,l,\mathbb{F},\nu}(f, g) := \sup_{\mathbf{t} \in \Lambda} \left[\mathbb{F}(l, \mathbf{t}) \cdot m'\Bigl(\bigl\{ \mathbf{s} \in \mathbf{t} + l\Omega : \|f(\mathbf{s}) - g(\mathbf{s})\|_Y \cdot \nu(\mathbf{s}) \geq \epsilon \bigr\} \Bigr) \right] \tag{7}$$

and $\|f\|_{P_{\epsilon,l,\mathbb{F},\nu}} := d_{\epsilon,l,\mathbb{F},\nu}(0, f)$. Then, we have $0 \leq d_{\epsilon,l,\mathbb{F},\nu}(f,g) \leq +\infty$, $d_{\epsilon,l,\mathbb{F},\nu}(f,f) = 0$, $d_{\epsilon,l,\mathbb{F},\nu}(f,g) = d_{\epsilon,l,\mathbb{F},\nu}(g,f)$ and $d_{\epsilon,l,\mathbb{F},\nu}(f,g) = d_{\epsilon,l,\mathbb{F},\nu}(f+h, g+h)$ so that $d_{\epsilon,l,\mathbb{F},\nu}(\cdot;\cdot)$ is a translation invariant pseudo-semimetric on the space of all functions from Λ into Y, provided that for each $l > 0$ we have $\sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) < +\infty$. Moreover, the following holds:

(i) If $f : \Lambda \to Y$, $g : \Lambda \to Y$, $h : \Lambda \to Y$ and the assumptions A, B, $C \subseteq \mathbb{R}^n$ and $A \subseteq B \cup C$ imply $m'(A) \leq m'(B) + m'(C)$, then we have

$$d_{\epsilon,l,\mathbb{F},\nu}(f, h) \leq d_{\epsilon/2,l,\mathbb{F},\nu}(f, g) + d_{\epsilon/2,l,\mathbb{F},\nu}(g, h), \quad \epsilon > 0,\ l > 0. \tag{8}$$

(ii) Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\tau \in \Lambda'$, $M > 0$, the assumption $\mathbf{v} \in \Lambda + l\Omega + \tau$ implies $\nu(\mathbf{v} - \tau) \leq M\nu(\mathbf{v})$ and the assumption $A \subseteq B \subseteq \mathbb{R}^n$ implies $m'(A) \leq m'(B)$. Then, we have

$$d_{\epsilon,l,\mathbb{F},\nu}(f(\cdot + \tau), g(\cdot + \tau)) \leq d_{\epsilon/M,l,\mathbb{F},\nu}(f, g), \tag{9}$$

for any two functions $f : \Lambda \to Y$ and $g : \Lambda \to Y$.

(iii) Suppose that $T \in L(Y)$, $f : \Lambda \to Y$ and $g : \Lambda \to Y$. Then we have

$$d_{\epsilon,l,\mathbb{F},\nu}(Tf, Tg) \leq d_{\epsilon/\|T\|,l,\mathbb{F},\nu}(f, g), \tag{10}$$

where $d_{\epsilon/\|T\|,l,\mathbb{F}}(f,g) = 0$ for $T = 0$.

(iv) Suppose that $f : \Lambda \to Y$ and $g : \Lambda \to Y$. If the assumption $A \subseteq B \subseteq \mathbb{R}^n$ implies $m'(A) \leq m'(B)$, then for each $\epsilon' \in (0, \epsilon)$ we have

$$d_{\epsilon,l,\mathbb{F},\nu}(f, g) \leq d_{\epsilon',l,\mathbb{F},\nu}(f, g) \quad \text{and} \quad \|f\|_{P_{\epsilon,l,\mathbb{F},\nu}} \leq \|f\|_{P_{\epsilon',l,\mathbb{F},\nu}}.$$

(v) The triangle inequality

$$d_{\epsilon,l,\mathbb{F},\nu}(f, h) \leq d_{\epsilon,l,\mathbb{F},\nu}(f, g) + d_{\epsilon,l,\mathbb{F},\nu}(g, h)$$

does not hold, in general, and the assumption $d_{\epsilon,l,\mathbb{F},\nu}(f,g)=0$ does not imply $f=g$ a.e., in general.

In particular, the property (v) shows that $d_{\epsilon,l,\mathbb{F},\nu}(\cdot;\cdot)$ is not a pseudo-metric on the space of all functions from Λ into Y; therefore, we cannot apply Theorem 2.1 of the Ref. [21] in order to see that $\lim_{l\to+\infty} d_{\epsilon,l,\mathbb{F},\nu}(f,g)$ exists for fixed $\epsilon>0$, $\mathbb{F}(\cdot;\cdot)$ and $f(\cdot)$, $g(\cdot)$. Concerning this issue, we will state and prove the following result:

Theorem 1. *Suppose that $\Lambda=[0,\infty)^n$ or $\Lambda=\mathbb{R}^n$, $\Omega=[0,1]^n$, $F:\Lambda\times X\to Y$ and $G:\Lambda\times X\to Y$. If $B\subseteq X$ is an arbitrary nonempty set, then we define*

$$d^B_{\epsilon,l,\mathbb{F},\nu}(F,G):=\sup_{x\in B;\mathbf{t}\in\Lambda}\Biggl[\mathbb{F}(l)\cdot m'\Bigl(\bigl\{\mathbf{s}\in\mathbf{t}+l\Omega:\|F(\mathbf{s};x)-G(\mathbf{s};x)\|_Y\cdot\nu(\mathbf{s})\geq\epsilon\bigr\}\Bigr)\Biggr].$$

Then $\lim_{l\to+\infty} d^B_{\epsilon,l,\mathbb{F},\nu}(F,G)$ exists in $[0,\infty]$, provided that the following conditions hold:

(i) If $A,\ B\subseteq\mathbb{R}^n$, then $m'(A\cup B)\leq m'(A)+m'(B)$.

(ii) For every $l_1>0$, we have $\limsup_{l_2\to+\infty}\Bigl[\frac{\mathbb{F}(l_2)}{\mathbb{F}(l_1)}\cdot\bigl\lceil\frac{l_2}{l_1}\bigr\rceil^n\Bigr]\leq 1$.

In particular, (ii) holds with $\mathbb{F}(l)\equiv l^{-n}$.

Proof. If $l_2>l_1>0$, then (i) easily implies

$$\begin{aligned}&\sup_{x\in B;\mathbf{t}\in\Lambda}\Biggl[\mathbb{F}(l_2)\cdot m'\Bigl(\bigl\{\mathbf{s}\in\mathbf{t}+l_2\Omega:\|F(\mathbf{s};x)-G(\mathbf{s};x)\|_Y\cdot\nu(\mathbf{s})\geq\epsilon\bigr\}\Bigr)\Biggr]\\&\leq\Biggl[\frac{\mathbb{F}(l_2)}{\mathbb{F}(l_1)}\cdot\Bigl\lceil\frac{l_2}{l_1}\Bigr\rceil^n\Biggr]\\&\cdot\sup_{x\in B;\mathbf{t}\in\Lambda}\Biggl[\mathbb{F}(l_1)\cdot m'\Bigl(\bigl\{\mathbf{s}\in\mathbf{t}+l_1\Omega:\|F(\mathbf{s};x)-G(\mathbf{s};x)\|_Y\cdot\nu(\mathbf{s})\geq\epsilon\bigr\}\Bigr)\Biggr].\end{aligned}$$

Applying (ii), we obtain

$$\limsup_{l_2\to+\infty} d^B_{\epsilon,l_2,\mathbb{F},\nu}(F,G)\leq d^B_{\epsilon,l_1,\mathbb{F},\nu}(F,G)$$

and

$$\limsup_{l_2\to+\infty} d^B_{\epsilon,l_2,\mathbb{F},\nu}(F,G)\leq\liminf_{l_1\to+\infty} d^B_{\epsilon,l_1,\mathbb{F},\nu}(F,G),$$

which simply yields the final conclusion. □

In Definition 3.1 of Ref. [19], we recently introduced the following notion:

Definition 3. *Suppose that $\emptyset\neq\Lambda'\subseteq\mathbb{R}^n$, $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$, $F:\Lambda\times X\to Y$ is a given function, ρ is a binary relation on Y, $R(F)\subseteq D(\rho)$ and $\Lambda+\Lambda'\subseteq\Lambda$. Then, we say that $F(\cdot;\cdot)$ is Bohr $(\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-almost periodic if and only if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exists $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that, for every $\mathbf{t}\in\Lambda$, $x\in B$ and $\mathbf{s}\in\mathbf{t}+\Omega$, there exists an element $y_{\mathbf{s};x}\in\rho(F(\mathbf{s};x))$ such that*

$$\sup_{x\in B}\Bigl\|F(\cdot+\tau;x)-y_{\cdot;x}\Bigr\|_{P_{\epsilon,1,1,\nu}}\leq\epsilon. \tag{11}$$

Now, we would like to introduce the following notion:

Definition 4. *Suppose that* $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ *is a given function,* ρ *is a binary relation on* Y, $R(F) \subseteq D(\rho)$, $\mathbb{F} : (0, \infty) \times \Lambda \to (0, \infty)$, $\Lambda + l\Omega \subseteq \Lambda$ *for all* $l > 0$ *and* $\Lambda + \Lambda' \subseteq \Lambda$. *Then, we say that:*

(i) $F(\cdot;\cdot)$ *is equi-Weyl* $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$*-almost periodic if and only if for every* $B \in \mathcal{B}$ *and* $\epsilon > 0$ *there exist* $l > 0$ *and* $L > 0$ *such that for each* $\mathbf{t}_0 \in \Lambda'$ *there exists* $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ *such that, for every* $\mathbf{t} \in \Lambda$, $x \in B$ *and* $\mathbf{s} \in \mathbf{t} + l\Omega$, *there exists an element* $y_{\mathbf{s};x} \in \rho(F(\mathbf{s}; x))$ *such that*

$$\sup_{x \in B} \Big\| F(\cdot + \tau; x) - y_{\cdot;x} \Big\|_{P_{\epsilon,l,\mathbb{F},\nu}} \leq \epsilon. \tag{12}$$

(ii) $F(\cdot;\cdot)$ *is Weyl* $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$*-almost periodic if and only if for every* $B \in \mathcal{B}$ *and* $\epsilon > 0$ *there exists* $L > 0$ *such that for each* $\mathbf{t}_0 \in \Lambda'$ *there exists* $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ *such that there exists* $l(\tau) > 0$ *such that, for every* $\mathbf{t} \in \Lambda$, $x \in B$, $l \geq l(\tau)$ *and* $\mathbf{s} \in \mathbf{t} + l\Omega$, *there exists an element* $y_{\mathbf{s};x} \in \rho(F(\mathbf{s}; x))$ *such that* (12) *holds.*

We omit the term "ν*" from the notation if* $\nu \equiv 1$ *and the term "*Λ'*" if* $\Lambda' = \Lambda$*; furthermore, we omit the term "*$\mathcal{B}$*" from the notation if* $X = \{0\}$ *and the term "*ρ*" from the notation if* $\rho = \mathrm{I}$.

Before proceeding any further, we would like to observe the following fact:

Remark 1. *In place of* (12), *we can also consider a slightly stronger condition:*

$$\sup_{l_2 \geq l; x \in B} \Big\| F(\cdot + \tau; x) - y_{\cdot;x} \Big\|_{P_{\epsilon,l_2,\mathbb{F},\nu}} \leq \epsilon. \tag{13}$$

If $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n}$, *then Theorem 1 shows that this condition is equivalent to* (12).

Any Bohr $(\mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic function has to be equi-Weyl $(1, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic, provided that $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$. It is also clear that any equi-Weyl $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic function is Weyl $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic.

For simplicity, we will consider here the case in which $\phi(x) \equiv x$. The notion of (equi)-Weyl-(Doss)-m-almost periodicity for a Lebesgue measurable function $F : \mathbb{R}^n \to Y$ is obtained by plugging $\mathbb{F}(\mathbf{t}, l) \equiv l^{-n/p}$, $\Lambda' = \mathbb{R}^n$, $\rho = \mathrm{I}$, $\Omega = [0, 1]^n$ and $\nu(\cdot) \equiv 1$.

We continue by stating the following result. We can also consider general measures here (cf. Proposition 3.4 of Ref. [19]):

Proposition 1. *Suppose that* (WM1-1) *holds. Then, we have the following:*

(i) *Suppose that* $F \in (e-)W^{p(\mathbf{u}),x,\mathbb{F},\nu}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *or* $F \in (e-)W^{p,x,\mathbb{F},\nu}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ *for some* $p \in (0, 1)$. *Then* $F(\cdot;\cdot)$ *is (equi-)Weyl* $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m, \nu)$*-almost periodic.*

(ii) *Suppose that* $\rho = T \in L(Y)$, $F(\cdot;\cdot)$ *is (equi-)Weyl* $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m, \nu)$*-almost periodic, there exists* $M > 0$ *such that* $\nu(\mathbf{t}) \leq M$ *for all* $\mathbf{t} \in \Lambda$, *there exists* $l_0 > 0$ *such that* $\sup_{l \geq l_0, \mathbf{t} \in \Lambda} [\mathbb{F}(l, \mathbf{t}) l^{n/p}] < +\infty$ *and for each set* $B \in \mathcal{B}$ *we have* $\sup_{x \in B; \mathbf{t} \in \Lambda} \|F(\mathbf{t}; x)\|_Y < +\infty$. *Then, for every* $p \geq 1$, *we have* $F \in (e-)W^{p,x,\mathbb{F},\nu}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$.

Proof. The proof of (i) is almost trivial and therefore is omitted. To deduce (ii), define

$$A_{\epsilon,l,\mathbf{t},x} := \big\{ \mathbf{s} \in \mathbf{t} + l\Omega : \|F(\mathbf{s} + \tau; x) - F(\mathbf{s}; x)\|_Y \cdot \nu(\mathbf{s}) \geq \epsilon \big\},\ \epsilon > 0,\ l > 0,\ \mathbf{t} \in \Lambda,\ x \in X.$$

Then, the final conclusion simply follows from the prescribed assumptions and the estimate ($B \in \mathcal{B}$; $\tau \in \Lambda'$):

$$\sup_{x\in B;\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t}) \left(\int_{\mathbf{t}+l\Omega} \|F(\mathbf{s}+\tau;x) - TF(\mathbf{s};x)\|_Y^p \cdot \nu^p(\mathbf{s})\, d\mathbf{s} \right)^{1/p}$$

$$\leq \sup_{x\in B;\mathbf{t}\in\Lambda} \mathbb{F}(l,\mathbf{t}) \left(m\big(A_{\epsilon,l,\mathbf{t},x}\big) \left[(1+\|T\|) \cdot \sup_{x\in B;\mathbf{t}\in\Lambda} \|F(\mathbf{t};x)\|_Y \right]^p + \epsilon^p l^n M^p \right)^{1/p}.$$

□

We will not reconsider the statement of Proposition 3.8 of Ref. [19] here. Concerning the statement of Theorem 3.10 of Ref. [19], we have the following result:

Theorem 2. *Suppose that $M > 0$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a given function, $\rho = T \in L(Y)$, $\mathbb{F} : (0,\infty) \to (0,\infty)$, $\nu : \Lambda \to [0,\infty)$ and $\Lambda + \Lambda' \subseteq \Lambda$. Suppose further that, for every $k \in \mathbb{N}$, the function $F_k : \Lambda \times X \to Y$ is (equi-)Weyl-$(\mathbb{F},\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-almost periodic and, for every $\epsilon > 0$, $l > 0$ and $B \in \mathcal{B}$, we have*

$$\lim_{k\to+\infty} \sup_{x\in B} \|F_k(\cdot;x) - F(\cdot;x)\|_{P_{\epsilon,l,\mathbb{F},\nu}} = 0.$$

Then, $F(\cdot;\cdot)$ is (equi-)Weyl-$(\mathbb{F},\mathcal{B},\Lambda',\rho,\Omega,m',\nu)$-almost periodic, provided that the assumptions A, B, $C \subseteq \mathbb{R}^n$ and $A \subseteq B \cup C$ imply $m'(A) \leq m'(B) + m'(C)$, and the assumption $\mathbf{v} \in \Lambda + l\Omega + \tau$ for some $\tau \in \Lambda'$ and $l > 0$ implies $\nu(\mathbf{v} - \tau) \leq M\nu(\mathbf{v})$.

Proof. The proof is very similar to the proof of the above-mentioned theorem. If $k \in \mathbb{N}$, $x \in X$ and $\tau \in \Lambda'$, then the estimates (8)–(10) imply:

$$\begin{aligned}
&\|F(\cdot+\tau;x) - TF(\cdot;x)\|_{P_{\epsilon,l,\mathbb{F},\nu}} \\
&\leq \|F(\cdot+\tau;x) - F_k(\cdot+\tau;x)\|_{P_{\epsilon/2,l,\mathbb{F},\nu}} + \|F_k(\cdot+\tau;x) - TF(\cdot;x)\|_{P_{\epsilon/2,l,\mathbb{F},\nu}} \\
&\leq \|F(\cdot+\tau;x) - F_k(\cdot+\tau;x)\|_{P_{\epsilon/2,l,\mathbb{F},\nu}} + \|F_k(\cdot+\tau;x) - TF_k(\cdot;x)\|_{P_{\epsilon/4,l,\mathbb{F},\nu}} \\
&+ \|TF_k(\cdot;x) - TF(\cdot;x)\|_{P_{\epsilon/4,l,\mathbb{F},\nu}} \\
&\leq \|F(\cdot;x) - F_k(\cdot;x)\|_{P_{\epsilon/2M,l,\mathbb{F},\nu}} + \|F_k(\cdot+\tau;x) - TF_k(\cdot;x)\|_{P_{\epsilon/4,l,\mathbb{F},\nu}} \\
&+ \|F_k(\cdot;x) - F(\cdot;x)\|_{P_{\epsilon/4\|T\|,l,\mathbb{F},\nu}}.
\end{aligned}$$

This simply completes the proof. □

4. Multidimensional Doss ρ-Almost-Periodic-Type Functions in General Measure

In this section, we extend the class of Doss ρ-almost periodic functions from Definition 2, provided that the function $\mathbb{F}(\cdot;\cdot)$ does not depend on the second argument, by considering the class of Doss $(\mathbb{F},\mathcal{B},\Lambda',\rho,m',\nu)$-almost periodic functions. The precise definition goes as follows:

Definition 5. *Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a given function, ρ is a binary relation on Y, $R(F) \subseteq D(\rho)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$, $\nu : \Lambda \to [0,\infty)$, and $\Lambda + \Lambda' \subseteq \Lambda$. Then, we say that $F(\cdot;\cdot)$ is Doss $(\mathrm{F},\mathcal{B},\Lambda',\rho,m',\nu)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0,L) \cap \Lambda'$ such that, for every $\mathbf{s} \in \Lambda$ and $x \in B$, there exists an element $y_{\mathbf{s};x} \in \rho(F(\mathbf{s};x))$ such that*

$$\limsup_{l\to+\infty} \left[\mathrm{F}(l) \sup_{x\in B} m'\Big(\big\{ \mathbf{s} \in \Lambda_l : \|F(\mathbf{s}+\tau;x) - y_{\mathbf{s};x}\|_Y \cdot \nu(\mathbf{s}) \geq \epsilon \big\} \Big) \right] \leq \epsilon. \tag{14}$$

We omit the term "ν" from the notation if $\nu \equiv 1$ and the term "Λ'" if $\Lambda' = \Lambda$. Furthermore, we omit the term "$\mathcal{B}$" from the notation if $X = \{0\}$ and the term "ρ" from the notation if $\rho = \mathrm{I}$.

The proof of the following result is simple and therefore omitted:

Proposition 2. *Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ is a given function, ρ is a binary relation on Y, $R(F) \subseteq D(\rho)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$, $\nu : \Lambda \to [0,\infty)$ and $\Lambda + \Lambda' \subseteq \Lambda$.*

(i) If $F(\cdot;\cdot)$ is Doss-$(p, \mathrm{F}, \mathcal{B}, \Lambda', \rho, \nu)$-almost periodic and $p \in D_+(\Lambda)$, then $F(\cdot;\cdot)$ is Doss $(\mathrm{F}, \mathcal{B}, \Lambda', \rho, m, \nu)$-almost periodic.

(ii) If $\rho = T \in L(Y)$, there exists $M > 0$ such that $\nu(\mathbf{s}) \leq M$ for a.e. $\mathbf{s} \in \Lambda$,

$$\limsup_{l \to +\infty} [\mathrm{F}(l) l^{n/p}] < +\infty,$$

$F(\cdot;\cdot)$ is Doss $(\mathrm{F}, \mathcal{B}, \Lambda', \rho, m, \nu)$-almost periodic and $\sup_{x \in B; \mathbf{t} \in \Lambda} \|F(\mathbf{t};x)\|_Y < +\infty$ for each set $B \in \mathcal{B}$, then $F(\cdot;\cdot)$ is Doss-$(p, \mathrm{F}, \mathcal{B}, \Lambda', \rho, \nu)$-almost periodic for each $p \geq 1$.

(iii) If $0 \in \Lambda$, $l\Omega = \Lambda_l$ for all $l > 0$ and $F(\cdot;\cdot)$ is Weyl $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic with $\mathbb{F}(l, \mathbf{t}) \equiv \mathbb{F}(l)$, then $F(\cdot;\cdot)$ is Doss $(\mathrm{F}, \mathcal{B}, \Lambda', \rho, m', \nu)$-almost periodic with $\mathrm{F} = \mathbb{F}$.

The class of Doss $(\mathrm{F}, \mathcal{B}, \Lambda', \rho, m, \nu)$-uniformly recurrent functions can be introduced in the same way as above. Keeping this observation in mind, Proposition 2(ii) and the estimate established on p. 113, line 1 of Ref. [7] can be used to prove that there does not exist a nonempty subset Λ' of $\mathbb{R}$ such that the function $f(\cdot)$ analyzed in Example 3.2.3 of Ref. [7] is Doss $(l^{-1}, \Lambda', \rho, m, 1)$-uniformly recurrent with $\rho(t) = 1$ for all $t \in \mathbb{R}$. The interested reader may try to reformulate Theorem 2 and the conclusions established in Example 3.2.4 of Ref. [7] for multidimensional Doss almost periodic type functions in general measure. The statement of the Theorem 3.7 of Ref. [19] cannot be properly formulated for these classes of functions, which follows from the analysis carried out in Example 3.2.7 of Ref. [7].

5. Some Applications

In this section, we present some applications of Weyl and Doss almost-periodic-type functions to the abstract Volterra integro-differential inclusions in Banach spaces.

5.1. Invariance of Generalized Almost Periodicity in Measure under the Actions of Convolution Products

In this subsection, we examine the invariance of Weyl and Doss almost periodicity in measure under the actions of convolution products. For simplicity, we consider the one-dimensional setting only. We start by stating the following result:

Theorem 3. *Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}$, $f : \mathbb{R} \to Y$ is a bounded function, $\rho = T \in L(Y)$, $\mathbb{F} : (0,\infty) \times \mathbb{R} \to (0,\infty)$, $\mathbb{F}_1 : (0,\infty) \times \mathbb{R} \to (0,\infty)$ and $\Omega = [0,1]$. Suppose, further, that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family such that $\int_{(0,\infty)} \|R(t)\|\,dt < \infty$. If $f(\cdot)$ is (equi-)Weyl $(\mathbb{F}, \Lambda', \rho, \Omega, m)$-almost periodic, then the function $F : \mathbb{R} \to Y$, given by*

$$F(t) := \int_{-\infty}^{t} R(t-s) f(s)\,ds, \quad t \in \mathbb{R}, \tag{15}$$

is bounded, continuous, and (equi-)Weyl $(\mathbb{F}_1, \Lambda', \rho, \Omega, m)$-almost periodic, provided that the following condition holds true:

(Q1) For every $\epsilon > 0$, there exists $\epsilon' \in (0, \epsilon/[2(1 + \int_{(0,\infty)} \|R(t)\|\,dt)])$ such that, for every $l > 0$ and $t \in \mathbb{R}$, we have

$$m\left(\left\{ s \in [t, t+l] : (1 + \|T\|)\|f\|_\infty \sum_{k=0}^{+\infty} \frac{\|R(\cdot)\|_{L^\infty[kl,(k+1)l]}}{\mathbb{F}(l, s-(k+1)l)} < \frac{\epsilon}{2\epsilon'} \right\}\right) \geq l - \frac{\epsilon}{\mathbb{F}_1(l, \mathbf{t})}.$$

Proof. We consider the class of equi-Weyl $(\mathbb{F}, \Lambda', \rho, [0,1], m)$-almost periodic functions only. We can simply prove that $F(\cdot)$ is well-defined, bounded, and continuous. Let $\epsilon > 0$ be given, and let $\epsilon' > 0$ be determined from condition (Q1). Then, we know that there exist $l > 0$ and $L > 0$ such that for each $t_0 \in \Lambda'$ there exists $\tau \in B(t_0, L) \cap \Lambda'$ such that

$$\left\| f(\cdot + \tau) - Tf(\cdot) \right\|_{P_{\epsilon', l, \mathbb{F}}} \leq \epsilon'.$$

Furthermore, we have

$$\begin{aligned}
&\|F(s+\tau) - TF(s)\|_Y \\
&\leq \int_0^{+\infty} \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|_Y \, dr \\
&= \sum_{k=0}^{+\infty} \int_{kl}^{(k+1)l} \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|_Y \, dr \\
&\leq (1+\|T\|)\|f\|_\infty \cdot \epsilon' \cdot \sum_{k=0}^{+\infty} \frac{\|R(\cdot)\|_{L^\infty[kl,(k+1)l]}}{F(l, s-(k+1)l)} + \epsilon' \cdot \sum_{k=0}^{+\infty} \int_{kl}^{(k+1)l} \|R(r)\| \, dr.
\end{aligned}$$

Since $\epsilon / < 1/[2(1 + \int_0^\infty \|R(r)\| \, dr)]$, the above calculation implies

$$\begin{aligned}
&\left\{ s \in [t, t+l] : \sum_{k=0}^{+\infty} \frac{\|R(\cdot)\|_{L^\infty[kl,(k+1)l]}}{F(l, s-(k+1)l)} < \frac{\epsilon}{2\epsilon' \|f\|_\infty (1+\|T\|)} \right\} \\
&\qquad \subseteq \left\{ s \in [t, t+l] : \|F(s+\tau) - TF(s)\|_Y < \epsilon \right\},
\end{aligned}$$

the final conclusion simply follows from condition (Q1). □

We continue with the following illustrative example:

Example 1. *Suppose that the function $f \in C^2(\mathbb{R}^3)$ has a compact support. Then, we know that the function*

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(x-y)}{|y|} \, dy, \quad x \in \mathbb{R}^3,$$

is a unique classical solution of the partial differential equations $\Delta u = -f$. Since for each $T \in L(\mathbb{R}^3)$, $l > 0$ and $\epsilon' > 0$ we have

$$\begin{aligned}
|u(x+\tau) - Tu(x)| &\leq \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|f(x-y+\tau) - Tf(x-y)|}{|y|} \, dy, \\
&\leq \frac{\epsilon'}{4\pi} \int_{\mathbb{R}^3} \frac{dy}{|y| \cdot \mathbb{F}(l, x-y)}, \quad x \in \mathbb{R}^3,
\end{aligned}$$

it readily follows that the (equi-)Weyl $(\mathbb{F}, \Lambda', T, \Omega, m)$-almost periodicity of $f(\cdot)$ implies the (equi-)Weyl $(\mathbb{F}_1, \Lambda', T, \Omega, m)$-almost periodicity of $u(\cdot)$, provided that the following condition holds:

(D) For every $\epsilon > 0$, there exists $\epsilon' > 0$ such that, for every $l > 0$ and $\mathbf{t} \in \mathbb{R}^3$, we have

$$m\left(\left\{ x \in \mathbf{t} + l\Omega : \int_{\mathbb{R}^3} \frac{dy}{|y| \cdot \mathbb{F}(l, x-y)} < \frac{\epsilon}{4\pi\epsilon'} \right\} \right) \geq m(l\Omega) - \frac{\epsilon}{\mathbb{F}_1(l, \mathbf{t})}.$$

Remark 2. *The invariance of (equi-)Weyl-p-almost periodicity under the actions of the infinite convolution products has been investigated in Proposition 2.11.1, Theorem 2.11.4 of Ref. [6]. These results can be also formulated with a general operator $\rho = T \in L(Y)$ and a general set Λ' of the corresponding Weyl-ϵ-almost periods. Because of that, the case in which $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$ will not occupy our attention here.*

For the sake of completeness, we provide the main details of the proof of the following result:

Theorem 4. *Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}$, $f : \mathbb{R} \to Y$ is a bounded function, $\rho = T \in L(Y)$, $\mathrm{F} : (0,\infty) \to (0,\infty)$, $\mathrm{F}_1 : (0,\infty) \to (0,\infty)$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family such that $\int_{(0,\infty)} \|R(t)\|\,dt < \infty$. If $f(\cdot)$ is Doss $(\mathrm{F},\Lambda',T,m)$-almost periodic, then the function $F : \mathbb{R} \to Y$, given by (15), is bounded, continuous and Doss $(\mathrm{F}_1,\Lambda',T,m)$-almost periodic, provided that the following condition holds true:*

(Q2) *For every $\epsilon > 0$, there exist $\epsilon' \in (0,\epsilon/[2(1+\int_{(0,\infty)} \|R(t)\|\,dt)])$ and $l_0 > 0$ such that, for every $l \geq l_0$, we have*

$$m\left(\left\{s \in [-l,l] : (1+\|T\|)\|f\|_\infty \cdot \epsilon' \cdot \limsup_{v\to+\infty} \frac{\|R(\cdot)\|_{L^\infty[0,v]}}{\mathrm{F}(v+|s|)} < \frac{\epsilon}{2}\right\}\right) \geq 2l - \frac{\epsilon}{\mathrm{F}_1(l)}.$$

Proof. Let $\epsilon > 0$ be given, and let $\epsilon' > 0$ and $l_0 > 0$ be determined from condition (Q1). Then, we know that there exists $L > 0$ such that for each $t_0 \in \Lambda'$ there exists $\tau \in B(t_0,L) \cap \Lambda'$ such that

$$\mathrm{F}(l)m\Big(\big\{s \in [-l,l] : \|f(s+\tau) - Tf(\cdot)\|_Y \geq \epsilon'\big\}\Big) \leq \epsilon'.$$

Furthermore, we have:

$$\begin{aligned}
&\|F(s+\tau) - TF(s)\|_Y \\
&\leq \int_0^{+\infty} \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|_Y\,dr \\
&= \lim_{v\to+\infty} \int_0^l \|R(r)\| \cdot \|f(s+\tau-r) - Tf(s-r)\|_Y\,dr \\
&\leq (1+\|T\|)\|f\|_\infty \cdot \epsilon' \cdot \limsup_{v\to+\infty} \frac{\|R(\cdot)\|_{L^\infty[0,v]}}{\mathrm{F}(v+|s|)} + \epsilon' \cdot \sum_{k=0}^{+\infty} \int_0^{+\infty} \|R(r)\|\,dr,
\end{aligned}$$

so that

$$\begin{aligned}
&\left\{s \in [-l,l] : (1+\|T\|)\|f\|_\infty \cdot \epsilon' \cdot \limsup_{v\to+\infty} \frac{\|R(\cdot)\|_{L^\infty[0,v]}}{\mathrm{F}(v+|s|)} < \frac{\epsilon}{2}\right\} \\
&\qquad \subseteq \left\{s \in [-l,l] : \|F(s+\tau) - TF(s)\|_Y < \epsilon\right\}.
\end{aligned}$$

The final conclusion now follows from condition (Q2). □

It is clear that Theorems 3 and 4 can be applied in the analysis of Weyl [Doss] almost periodic solutions in the Lebesgue measure for a large class of the abstract Volterra integro-differential inclusions without boundary conditions (cf. [6] for more details about this problematic).

In this context, we end this section with the following illustrative application to the Gaussian semigroup. The Gaussian semigroup is related to the solution of certain partial differential equations (PDEs), particularly those involving heat conduction or diffusion processes.

Example 2. *Let $p \in [1,\infty)$ and $(\mathcal{G}(t))$ be the Gaussian semigroup*

$$(\mathcal{G}f)(x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x-y)e^{-\frac{|y|^2}{4t}}\,dy, \quad t>0, \quad f \in L^p(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

This semigroup can be extended to a bounded analytic C_0-semigroup of angle $\frac{\pi}{2}$, generated by the Laplacian Δ_{L^p}, with maximal distributional domain in $L^p(\mathbb{R}^n)$. Suppose that, $\emptyset \neq \Lambda' \subseteq \Lambda = \mathbb{R}^n$, $\Omega = [0,1]^n$, $\rho = T \in L(L^p(\mathbb{R}^n))$, $f \in L(\mathbb{R}^n : \mathbb{C})$ is (equi-)Weyl $(\mathbb{F}_1, \Lambda', \rho, \Omega, m)$-almost periodic function and $\sup_{t\in\mathbb{R}^n} \|f(t)\| < \infty$. Furthermore, suppose that the functions $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ and $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ does not depend on t and (Q1) holds. By Theorem 3, the function $u(x, t_0) \equiv (\mathcal{G}(t_0)f)(x)$ is (equi-)Weyl $(\mathbb{F}_1, \Lambda', \rho, \Omega, m)$-almost periodic.

5.2. Abstract Semilinear Cauchy Inclusions

In this subsection, we analyze the existence and uniqueness of (equi-)Weyl almost periodic solutions for the following abstract semilinear Cauchy inclusion

$$D^t_{\gamma,+}u(t) \in \mathcal{A}u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{16}$$

where $D^t_{\gamma,+}u(t)$ denotes the Weyl–Liouville fractional derivative of order $\gamma \in (0,1)$, $\mathcal{A}$ is a multivalued linear operator and $f(\cdot,\cdot)$ has some extra features. It is well-known that there exists a large class of multivalued linear operators $\mathcal{A}$ for which the solution operator family $(R(t))_{t>0}$ for (16) has the growth

$$\|R(t)\| \leq M\frac{t^{\beta-1}}{1+t^{\gamma}}, \quad t > 0, \tag{17}$$

for some finite real constants $M \geq 1$, $\beta \in (0,1]$ and $\gamma > \beta$ (cf. [6] for more details). Moreover, a unique mild solution of (16) is given by

$$u(t) = \int_{-\infty}^{t} R(t-s)f(s, u(s))\, ds, \quad t \in \mathbb{R}. \tag{18}$$

Here, we consider the space $e - W^{\infty}(\mathbb{R} : X) := C_b(\mathbb{R} : X) \bigcap_{p\geq 1} e - W^p AP(\mathbb{R} : X)$. It is clear that $e - W^{\infty}(\mathbb{R} : X) := C_b(\mathbb{R} : X) \bigcap_{p>1/\beta} e - W^p AP(\mathbb{R} : X)$; equipped with the sup-norm, $e - W^{\infty}(\mathbb{R} : X)$ is a Banach space. Observe also that Proposition 1 yields that $e - W^{\infty}(\mathbb{R}^n : X)$ is the Banach space of all bounded continuous functions $f : \mathbb{R} \to X$ which are equi-Weyl almost periodic in the Lebesgue measure. By $e - W^{\infty,rc}(\mathbb{R} : X)$ we denote the Banach subspace of $e - W^{\infty}(\mathbb{R} : X)$ which contains functions with a relatively compact range.

Keeping in mind the Banach contraction principle, Proposition 2.11.1, Theorem 2.11.4 of the Ref. [6], and the composition principles established in Theorem 2.2, Theorem 2.3 of Ref. [23], it readily follows that there exists a unique mild solution of (16) which belongs to the space $e - W^{\infty,rc}(\mathbb{R} : X)$, provided that the following conditions hold:

(i) There exists $L > 0$ such that $L \cdot \int_0^{\infty} \|R(r)\|\, dr < 1$ and $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$ for all $t \in \mathbb{R}$ and $x,\ y \in X$.
(ii) For every relatively compact set $K \in X$, the set $\{f(t,x) : t \in \mathbb{R},\ x \in K\}$ is relatively compact in X.
(iii) The conditions (i) and (iii) given in the formulation of the Theorem 2.2 of Ref. [23] hold for any exponent $p > 1/\beta$.

We can similarly analyze the existence and uniqueness of (equi-)Weyl almost periodic solutions for the following semilinear integral equation:

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{-\infty}^{t_1}\int_{-\infty}^{t_2}\cdots\int_{-\infty}^{t_n} a(\mathbf{t}-\mathbf{s})F(\mathbf{s}; u(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n, \tag{19}$$

where $X = Y$ is a finite-dimensional complex Banach space, $a \in L^1((0,\infty)^n)$ and $f \in W^{\infty}(\mathbb{R}^n : X)$ (cf. [19] for more details).

6. Conclusions and Final Remarks

In this paper, we analyzed various classes of multidimensional Weyl and Doss ρ-almost-periodic-type functions in general measure. This establishes a foundation by explaining the notation and presenting basic definitions. The introduced classes of (equi-)Weyl $(\mathbb{F}, \mathcal{B}, \Lambda', \rho, \Omega, m', \nu)$-almost periodic functions and Doss $(\mathrm{F}, \mathcal{B}, \Lambda', \rho, m', \nu)$-almost periodic functions are analyzed, revealing essential structural results. The obtained structural results contribute to a deeper comprehension of these mathematical entities. The applications to abstract Volterra integro-differential inclusions in Banach spaces highlight the practical implications of the theoretical framework. This paper concludes with valuable remarks and observations, contributing to the ongoing exploration of generalized almost periodicity and almost automorphy in measure. Overall, this paper makes a contribution to advancing the understanding of these mathematical concepts and their practical implications.

Let us finally mention some important topics which are not considered in this paper:

1. In Definition 2.13.2 of Ref. [6], we introduced the class of one-dimensional Besicovitch-Doss-p-almost periodic functions. The conditions (ii), (iii) (already performed for Doss-p-almost periodic functions in measure), and (iv) in this definition can be further extended by considering the same condition in view of the general measure. In such a way, we can extend the class of Besicovitch–Doss-p-almost periodic functions ($p \geq 1$). We can also consider the case in which $0 < p < 1$ here and some multidimensional analogs.
2. We can analyze various classes of Stepanov quasi-asymptotically ρ-almost periodic type functions in general measure.
3. We can also consider Weyl and Besicovitch almost-automorphic-type functions in general measure.

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