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Several Goethals–Seidel Sequences with Special Structures

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Abstract: In this paper, we develop a novel method to construct Goethals–Seidel (GS) sequences with special structures. In the existing methods, utilizing Turyn sequences is an effective and convenient approach; however, this method cannot cover all GS sequences. Motivated by this, we are devoted to designing some sequences that can potentially construct all GS sequences. Firstly, it is proven that a quad of ± 1 polynomials can be considered a linear combination of eight polynomials with coefficients uniquely belonging to $\{0, \pm 1\}$. Based on this fact, we change the construction of a quad of Goethals–Seidel sequences to find eight sequences consisting of 0 and ± 1. One more motivation is to obtain these sequences more efficiently. To this end, we make use of the k-block, of which some properties of (anti) symmetry are discussed. After this, we can then look for the sequences with the help of computers since the symmetry properties facilitate reducing the search range. Moreover, we find that one of the eight blocks, which we utilize to construct GS sequences directly, can also be combined with Williamson sequences to generate GS sequences with more order. Several examples are provided to verify the theoretical results. The main contribution of this work is in building a bridge linking the GS sequences and eight polynomials, and the paper also provides a novel insight through which to consider the existence of GS sequences.

Keywords: Goethals–Seidel sequences; k-block and k-partition; symmetry and antisymmetry

MSC: 05B05; 05B20

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1. Introduction

A square matrix H of order n is called a Hadamard matrix (HM) if its entries are ± 1 and any two different rows (columns) are orthogonal. The order n satisfies $n = 1, 2, 4m$ with m being a positive integer, and a well-known conjecture related to HMs is whether a Hadamard matrix of order $4m$ exists for any m. HMs are widely applied in many fields, including signal processing, coding and cryptography, while the smallest order of an unconstructed HM is 668. More interesting properties and applications of HMs can be found in [1–4] and the references therein.

The construction of HMs is a classic problem in combinatorics, and many works have been devoted to it in past decades, such as Kronecker products [5], orthogonal designs [6], difference families [7] and many other methods [1,8–13]. In the existing methods, many are required to construct circulant matrices and then plug these constructed circulant matrices into some type of arrays such as the Williamson array and GS array [3,14]. In this paper, we will make use of a GS array taking the form of

$$G = \begin{pmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{pmatrix},$$

where A, B, C and D are four circulant matrices of order n satisfying

$$AA^T + BB^T + CC^T + DD^T = 4nI_n, \tag{1}$$

and R is the back-diagonal identity matrix of order n. The fact that A, B, C and D are circulant matrices implies that they are sufficient for the purposes of constructing the first rows of them, which are denoted by four sequences, i.e., $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$, respectively. If matrices A, B, C and D with entries ±1 satisfy condition (1), then $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are called a quad of GS sequences, and they are particularly said to be a quad of Williamson sequences if A, B, C and D are also symmetrical.

In [15], Goethals and Seidel conducted pioneering work on the GS array and obtained the HMs of a GS type with orders of 36 and 52. In [16,17], Whiteman utilized the Parseval relation to theoretically construct GS sequences of order $\frac{q_1+1}{4}$ and Williamson sequences of order $\frac{q_2+1}{2}$ in a finite field $GF(q_1^2)$ and $GF(q_2^2)$, respectively, where $q_1 \equiv (3mod8)$ and $q_2 \equiv (1mod4)$ are both prime powers. With the help of computers by exhaustive search, Doković studied the GS array and GS sequences in numerous works, where many different orders were obtained, as seen in [18–21] et al. Making use of Lagrange identity for polynomials (LIP), Yang—in [22]—proved that a quad of Williamson sequences of order n and a four-symbol δ-code of order m can be used to construct a quad of GS sequences of order mn. Yang also presented some other results [23–26], where the construction of GS sequences was mainly based on using two groups of sequences that were known beforehand.

In addition to the methods mentioned above, utilizing T-sequences directly is an alternative method, where a quad of GS sequences could be considered a linear combination of a quad of T-sequences, as shown in, e.g., [27]. The existing methods, however, have a slight drawback that not each GS sequences can be represented by a linear combination of T-sequences, as seen in Remark 1.

Motivated by this, we firstly defined the k-block and k-partition in this paper, which aid in dividing a quad of sequences into k parts. Next, we proved that a quad of ±1 polynomials $\{F_i(\xi)\}_{i=1}^4$ associated with sequences $\{f_i\}_{i=1}^4$ can uniquely be considered a linear combination of eight polynomials $\{G_i(\xi)\}_{i=1}^8$ that are associated with sequences $\{g_i\}_{i=1}^8$ consisting of 0 and ±1. For now, all of the GS sequences could be taken into consideration compared with the construction method by using T-sequences. In other words, the construction of GS sequences $\{f_i\}_{i=1}^4$ could be transformed into finding a group of k-partition $\{g_i\}_{i=1}^8$. Then, by supposing that $\{f_i\}_{i=1}^4$ are a quad of GS sequences, some relationships between associated polynomials $\{G_i(\xi)\}_{i=1}^8$ were revealed. To reduce the complexity of discussion, it is natural and necessary to impose some constraints on $\{G_i(\xi)\}_{i=1}^8$, e.g., the properties of symmetry or antisymmetry. Finally, by using k-partitions or k-blocks directly, we obtained some types of GS sequences with different symmetrical structures of $G_i(\xi)$. One was established by utilizing an eight partition, where three were based on nine partitions, and two used nine blocks. As an additional application, the eight partition mentioned above of order n, when combined with a quad of Williamson sequences of order m, can also lead to a quad of GS sequences with order mn. The theoretical results proposed in this paper are validated by some examples. This paper represents the first time that a quad of ±1 sequences have been considered a combination of eight blocks, which ensures that all the "existing" GS sequences can be taken into consideration and that consequently more GS sequences can be potentially discovered. Moreover, when comparing with the results in [28] (where a rough discussion of GS sequences and k-partition was presented and there was no rigorous proof to reveal the bijective relation), in this paper, we extended the results that we not only proved the uniqueness of the linear combination, but also investigated some of the necessary conditions for the existence of these sequences.

The rest of the paper is organized as follows. In Section 2, we introduce some of the necessary notations and definitions needed in later analysis. In Section 3, it is proven that a quad of ±1 sequences can be considered a linear combination of an eight block uniquely. Then, based on a k-block with (anti)symmetry properties, we constructed several

GS sequences and presented some examples to verify the theoretical results. In Section 4, by combining a quad of Williamson sequences of order m and an eight partition of order n (which was obtained above), a quad of GS sequences of order mn was constructed. Some conclusions will be made in Section 5.

2. Preliminaries

For a sequence $\boldsymbol{a} = (a_0, a_1, \ldots, a_{n-1})$, its periodic autocorrelation function $R_{\boldsymbol{a}}(\tau)$ is defined as

$$R_{\boldsymbol{a}}(\tau) = \sum_{i=0}^{n-1} a_i \bar{a}_{i+\tau}, \quad \tau = 0, 1, \ldots, n-1,$$

where $\bar{a}_i$ is the conjugate of a_i, and the sum $i + \tau$ is evaluated as modulo-n. A polynomial

$$\Phi_{\boldsymbol{a}}(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_{n-1}\xi^{n-1}$$

is called the associated polynomial of sequence $\boldsymbol{a}$, where ξ is the n-th root of unity $e^{\frac{2\pi}{n}I}$ and $I = \sqrt{-1}$. The finite Parseval relation [17], also named the Wiener–Khinchin theorem [29,30], between $R_{\boldsymbol{a}}(\tau)$ and $\Phi_{\boldsymbol{a}}(\xi)$ is presented in the following identity

$$R_{\boldsymbol{a}}(\tau) = \frac{1}{n} \sum_{j=0}^{n-1} \|\Phi_{\boldsymbol{a}}(\xi^j)\|^2 \xi^{j\tau}, \quad \tau = 0, 1, \ldots, n-1,$$

and its inverse form is

$$\|\Phi_{\boldsymbol{a}}(\xi^j)\|^2 = \sum_{\tau=0}^{n-1} R_{\boldsymbol{a}}(\tau)\xi^{-j\tau}, \quad j = 0, 1, \ldots, n-1.$$

For the HMs of a GS type, their four circulant matrices possess the following property.

Lemma 1 ([16])**.** *Let A, B, C and D denote four circulant matrices of order n whose first rows are four sequences $\boldsymbol{a} = \{a_i\}_{i=0}^{n-1}$, $\boldsymbol{b} = \{b_i\}_{i=0}^{n-1}$, $\boldsymbol{c} = \{c_i\}_{i=0}^{n-1}$ and $\boldsymbol{d} = \{d_i\}_{i=0}^{n-1}$, respectively. Then, $AA^T + BB^T + CC^T + DD^T = 4nI_n$ if and only if*

$$\|\Phi_{\boldsymbol{a}}(\xi^j)\|^2 + \|\Phi_{\boldsymbol{b}}(\xi^j)\|^2 + \|\Phi_{\boldsymbol{c}}(\xi^j)\|^2 + \|\Phi_{\boldsymbol{d}}(\xi^j)\|^2 = 4n,$$

$j = 0, 1, \ldots, n-1$, where ξ is the n-th root of unity.

Hereafter, without special clarification, a capital letter such as $F_i(\xi)$ denotes the associated polynomial, the bold letter $\boldsymbol{f}_i$ represents the sequence and the lower case letter f_{ij} denotes the j-th element in $\boldsymbol{f}_i$, where i and j rely on different cases. Before the discussion, some definitions are necessary to give.

Definition 1 (GS sequences, [22])**.** *Four ± 1 sequences $\boldsymbol{q}_i = (q_{i0}, q_{i1}, \ldots, q_{i,n-1})$, $i = 1, 2, 3, 4$ are said to be a quad of GS sequences of order n if their associated polynomials $Q_i(\xi)$ satisfy*

$$\sum_{i=1}^{4} \|Q_i(\xi^j)\|^2 = 4n,$$

where ξ is the n-th root of unity for $j = 0, \ldots, n-1$.

Motivated by the definition of L-matrices ([3], Definition 4.15), we define a k-block and k-partition as follows.

Definition 2 (k-block and k-partition)**.** *A class of sequences $\boldsymbol{g}_i = (g_{i0}, \ldots, g_{i,n-1}), i = 1, \ldots, k$, is said to be a k-block of order n, if it holds*

$$\begin{array}{ll} \text{(i)} & g_{ij} \in \{0, \pm 1\}, \quad j = 0, 1, \ldots, n-1, \ i = 1, 2, \ldots k, \\ \text{(ii)} & \sum\limits_{i=1}^{k} |g_{ij}| = 1, \qquad j = 0, 1, \ldots, n-1. \end{array} \tag{2}$$

If a k-block $\{\boldsymbol{g}_i\}_{i=1}^{k}$ of order n additionally satisfies

$$\text{(iii)} \quad \sum_{i=1}^{k} R_{\boldsymbol{g}_i}(\tau) = n, \ \ \tau = 0, \ldots, n-1, \tag{3}$$

we call $\{\boldsymbol{g}_i\}_{i=1}^{k}$ a k-partition, where ξ is the n-th root of unity.

Definition 3 (symmetry and antisymmetry, [22])**.** *Let $F_i(\xi)$ be a polynomial associated with sequences $\boldsymbol{f}_i = (f_{i0}, \ldots, f_{i,n-1})$. $F_i(\xi)$ is symmetrical (or antisymmetrical) if it satisfies*

$$\overline{F_i(\xi)} = F_i(\xi) \ \ (or \ \ \overline{F_i(\xi)} = -F_i(\xi)),$$

where ξ is the n-th root of unity. In other words, the coefficients $(f_{i0}, \ldots, f_{i,n-1})$ satisfy $f_{ij} = f_{i,n-j}$ (or $f_{ij} = -f_{i,n-j}$), $j = 1, 2, \ldots, n-1$.

3. Main Results

Inspired by [27], we extended the construction of GS sequences from four sequences to eight sequences. Then, we obtained the main result that a quad of ± 1 sequences can be uniquely considered a linear combination of an eight block, as stated in the following lemma.

Lemma 2. *The associated polynomials of sequences $\{\boldsymbol{f}_i\}_{i=1}^{4}$ and $\{\boldsymbol{g}_i\}_{i=1}^{8}$ are denoted by $\{F_i(\xi)\}_{i=1}^{4}$ and $\{G_i(\xi)\}_{i=1}^{8}$, respectively. Then, given a quad of ± 1 sequences $\{\boldsymbol{f}_i\}_{i=1}^{4}$ of order n, there exists a unique eight block $\{\boldsymbol{g}_i\}_{i=1}^{8}$ of order n such that the associated polynomials $\{F_i(\xi)\}_{i=1}^{4}$ can be uniquely written as a linear combination of the associated polynomials $\{G_i(\xi)\}_{i=1}^{8}$ that*

$$\begin{aligned} F_1(\xi) = {} & G_1(\xi) + G_2(\xi) + G_3(\xi) + G_4(\xi) + G_5(\xi) + G_6(\xi) + G_7(\xi) - G_8(\xi), \\ F_2(\xi) = {} & G_1(\xi) + G_2(\xi) - G_3(\xi) - G_4(\xi) + G_5(\xi) + G_6(\xi) - G_7(\xi) + G_8(\xi), \\ F_3(\xi) = {} & G_1(\xi) - G_2(\xi) + G_3(\xi) - G_4(\xi) + G_5(\xi) - G_6(\xi) + G_7(\xi) + G_8(\xi), \\ F_4(\xi) = {} & G_1(\xi) - G_2(\xi) - G_3(\xi) + G_4(\xi) - G_5(\xi) + G_6(\xi) + G_7(\xi) + G_8(\xi), \end{aligned} \tag{4}$$

where ξ is the n-th root of unity.

Proof. We first prove the existence. In (4), it is evident that the coefficients on the left and right hand sides are equal to each other correspondingly. Thus, we can equivalently rewrite (4) in the form of the matrix multiplication

$$\mathbf{F} = (\widehat{\boldsymbol{H}} \ \ \widetilde{\boldsymbol{H}}) \begin{pmatrix} \widehat{\mathbf{G}} \\ \widetilde{\mathbf{G}} \end{pmatrix},$$

where we denote

$$\mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}, \quad \widehat{\mathbf{G}} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}, \quad \widetilde{\mathbf{G}} = \begin{pmatrix} g_5 \\ g_6 \\ g_7 \\ g_8 \end{pmatrix},$$
$$\widehat{\boldsymbol{H}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \widetilde{\boldsymbol{H}} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Then, we split $\mathbf{F}$ into $\mathbf{F} = \widehat{\mathbf{F}} + \widetilde{\mathbf{F}}$ satisfying

$$\widehat{\mathbf{F}} = \widehat{\boldsymbol{H}}\widehat{\mathbf{G}} \text{ and } \widetilde{\mathbf{F}} = \widetilde{\boldsymbol{H}}\widetilde{\mathbf{G}},$$

which implies

$$\widehat{\mathbf{G}} = \frac{1}{4}\widehat{\boldsymbol{H}}\widehat{\mathbf{F}} \text{ and } \widetilde{\mathbf{G}} = \frac{1}{4}\widetilde{\boldsymbol{H}}\widetilde{\mathbf{F}}$$

since both $\widehat{\boldsymbol{H}}$ and $\widetilde{\boldsymbol{H}}$ are symmetrical Hadamard matrices.

Denote by $\mathbf{F}_j$ the j-th column of a Matrix $\mathbf{F}$, and by $p(j)$ the number of 1 in $\mathbf{F}_j$. Taking the structure of $\widehat{\boldsymbol{H}}$ and the property of k-block (2) into consideration, it follows that $p(j) \in \{0, 2, 4\}$, and similarly we get $p(j) \in \{1, 3\}$ for $\widetilde{\boldsymbol{H}}$. Then, it is natural to define

$$\widehat{\mathbf{F}}_j = \begin{cases} \mathbf{F}_j, & p(j) \in \{0,2,4\}, \\ \mathbf{0}, & p(j) \in \{1,3\}, \end{cases} \text{ and } \widetilde{\mathbf{F}}_j = \begin{cases} \mathbf{F}_j, & p(j) \in \{1,3\}, \\ \mathbf{0}, & p(j) \in \{0,2,4\}, \end{cases} \tag{5}$$

$j = 1, 2, \ldots, n$, which then guarantees the existence of the eight block $\{g_i\}_{i=1}^{8}$.

Further, we proceeded with the proof of uniqueness. Supposing that there exists another

$$\mathbf{G}^* = \begin{pmatrix} \widehat{\mathbf{G}}^* \\ \widetilde{\mathbf{G}}^* \end{pmatrix},$$

then we have

$$\widehat{\mathbf{F}}^* = \widehat{\boldsymbol{H}}\widehat{\mathbf{G}}^* \text{ and } \widetilde{\mathbf{F}}^* = \widetilde{\boldsymbol{H}}\widetilde{\mathbf{G}}^*.$$

Still, we considered it in view of each column. For a given j, either $\widehat{\mathbf{G}}_j^*$ or $\widetilde{\mathbf{G}}_j^*$ is equal to $\mathbf{0}$, because $\mathbf{G}^*$ also consists of an eight block $\{g_i^*\}_{i=1}^{8}$, which means only one of $\widehat{\mathbf{G}}_j^*$ or $\widetilde{\mathbf{G}}_j^*$ contains a non-zero element. This fact yields that either $\widehat{\mathbf{F}}_j^*$ or $\widetilde{\mathbf{F}}_j^*$ is equal to $\mathbf{0}$, and it must correspond to the splitting (5). Otherwise, the converse case $\widehat{\mathbf{F}}_j = \widetilde{\mathbf{F}}_j^*$ and $\widetilde{\mathbf{F}}_j = \widehat{\mathbf{F}}_j^*$ could not guarantee that the entries of $\mathbf{G}^*$ belong to $\{0, \pm 1\}$. As a result, we know $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}^*$ and $\widetilde{\mathbf{F}} = \widetilde{\mathbf{F}}^*$, which eventually ensures the uniqueness of splitting (5). □

Next, we investigated the relationships between $G_1(\xi), G_2(\xi), \ldots, G_8(\xi)$. From (4), we arrive at

$$\sum_{i=1}^{4} \|F_i(\xi)\|^2 = 4\sum_{i=1}^{8} \|G_i(\xi)\|^2 + 2U(\xi) + 2\overline{U(\xi)}$$

with

$$\begin{aligned} U(\xi) = \quad & G_1(\xi)\overline{G_5(\xi)} + G_1(\xi)\overline{G_6(\xi)} + G_1(\xi)\overline{G_7(\xi)} + G_1(\xi)\overline{G_8(\xi)} \\ & + G_2(\xi)\overline{G_5(\xi)} + G_2(\xi)\overline{G_6(\xi)} - G_2(\xi)\overline{G_7(\xi)} - G_2(\xi)\overline{G_8(\xi)} \\ & + G_3(\xi)\overline{G_5(\xi)} - G_3(\xi)\overline{G_6(\xi)} + G_3(\xi)\overline{G_7(\xi)} - G_3(\xi)\overline{G_8(\xi)} \\ & - G_4(\xi)\overline{G_5(\xi)} + G_4(\xi)\overline{G_6(\xi)} + G_4(\xi)\overline{G_7(\xi)} - G_4(\xi)\overline{G_8(\xi)}. \end{aligned} \tag{6}$$

Further, if $f_1, \ldots, f_4$ are a quad of GS sequences of order n, then we obtain

$$4n = \sum_{i=1}^{4} \|F_i(\xi)\|^2 = 4\sum_{i=1}^{8} \|G_i(\xi)\|^2 + 2U(\xi) + 2\overline{U(\xi)}. \tag{7}$$

Remark 1. *Note that the definition of the k-partition is actually the special case of L-matrices ([3], Definition 4.15). The reason why we emphasize it specifically in this paper is due to the important role it plays in the construction of GS sequences. After such construction has taken place, then it will be convenient to describe them. In particular, a quad of T-sequences [14] is a four partition.*

Remark 2. *In the existing works, e.g., [27], the method for constructing GS sequences is based on a quad of a four partition and the structure $\widetilde{\boldsymbol{H}}$. In the proof of Theorem 2, it is seen that this method could not guarantee that all GS sequences can be taken into consideration. The result is extended that we construct the GS sequences from using a four partition into an eight partition.*

3.1. GS Sequences Based on a k-Partition

In this subsection, we begin with the identities (4) and (7) to construct GS sequences. From the definition of an eight partition, it is natural to obtain the following lemma.

Lemma 3. *For an eight partition $\{g_i\}_{i=1}^{8}$, $\{f_i\}_{i=1}^{4}$ are a quad of GS sequences if and only if*

$$U(\xi) + \overline{U(\xi)} = 0,$$

with ξ being the n-th root of unity, where $U(\xi)$ and $\overline{U(\xi)}$ are defined in (6).

Proof. A combination of (3) and (7) leads to the results immediately. □

Thus, we only need to construct an eight partition satisfying $U(\xi) + \overline{U(\xi)} = 0$. However, it is still challenging to find an eight partition directly, and—as a reduction—we imposed some conditions on the polynomials $G_i(\xi), i = 1, 2, \ldots, 8$, such as properties of symmetry or antisymmetry. We first recall an existing result.

Lemma 4 ([28]). *Let $\{g_i\}_{i=1}^{8}$ be an eight partition of order n and their associated polynomials $\{G_i(\xi)\}_{i=1}^{8}$ satisfy the following symmetry properties*

$$\begin{aligned} G_1(\xi) &= \overline{G_1(\xi)}, & G_2(\xi) &= \overline{G_2(\xi)}, & G_3(\xi) &= \overline{G_3(\xi)}, & G_4(\xi) &= \overline{G_4(\xi)}, \\ G_5(\xi) &= -\overline{G_5(\xi)}, & G_6(\xi) &= -\overline{G_6(\xi)}, & G_7(\xi) &= -\overline{G_7(\xi)}, & G_8(\xi) &= -\overline{G_8(\xi)}, \end{aligned} \tag{8}$$

where ξ is the n-th root of unity. Then, there exist a quad of GS sequences $\{f_i\}_{i=1}^{4}$ that are associated with the polynomials $F_1(\xi), \ldots, F_4(\xi)$ generated by (4).

It is evident that there exist a great deal of polynomial groups satisfying $U(\xi) + \overline{U(\xi)} = 0$. Here, we simply provide one more condition with different types of $\{G_i(\xi)\}_{i=1}^{8}$.

Theorem 1. *For an eight partition $\{g_1\}_{i=1}^{8}$, if their associated polynomials $\{G_i(\xi)\}_{i=1}^{8}$ satisfy the following symmetry properties*

$$\begin{aligned} G_1(\xi) &= \overline{G_1(\xi)}, & G_2(\xi) &= \overline{G_2(\xi)}, & G_3(\xi) &= -\overline{G_3(\xi)}, \\ G_4(\xi) &= -\overline{G_4(\xi)}, & G_5(\xi) &= -\overline{G_6(\xi)}, & G_7(\xi) &= -\overline{G_8(\xi)}, \end{aligned} \tag{9}$$

with ξ being the n-th root of unity, then $f_1, \ldots, f_4$ are a quad of GS sequences formed in (4).

Proof. It is easy to verify $U(\xi) + \overline{U(\xi)} = 0$ from (6) and (9). □

Two following groups of sequences are shown to verify Theorem 1. For $n = 8$,

$$\begin{aligned} g_1 &= (+,0,0,0,-,0,0,0), & g_2 &= (0,0,0,0,0,0,0,0), \\ g_3 &= (0,0,0,0,0,0,0,0), & g_4 &= (0,0,+,0,0,0,-,0), \\ g_5 &= (0,0,0,0,0,0,0,0), & g_6 &= (0,0,0,0,0,0,0,0), \\ g_7 &= (0,0,0,-,0,0,0,-), & g_8 &= (0,+,0,0,0,+,0,0), \end{aligned}$$

by (4), the GS sequences of order eight are

$$\begin{aligned} f_1 &= (+,-,+,-,-,-,-,-), & f_2 &= (+,+,-,+,-,+,+,+), \\ f_3 &= (+,+,-,-,-,+,+,-), & f_4 &= (+,+,+,-,-,+,-,-),, \end{aligned}$$

for $n = 9$, they are

$$\begin{aligned} g_1 &= (+,0,0,0,0,0,0,0,0), & g_2 &= (0,0,0,0,0,0,0,0,0), \\ g_3 &= (0,0,0,0,0,0,0,0,0), & g_4 &= (0,0,0,0,0,0,0,0,0), \\ g_5 &= (0,0,0,0,0,+,-,0,0), & g_6 &= (0,0,0,+,-,0,0,0,0), \\ g_7 &= (0,0,0,0,0,0,0,-,-), & g_8 &= (0,+,+,0,0,0,0,0,0), \end{aligned}$$

and the GS sequences are

$$\begin{aligned} f_1 &= (+,-,-,+,-,+,-,-,-), & f_2 &= (+,+,+,+,-,+,-,+,+), \\ f_3 &= (+,+,+,-,+,+,-,-,-), & f_4 &= (+,+,+,+,-,-,+,-,-).. \end{aligned}$$

In the process of creating the constructions above, discovering the relations between $g_1, g_2, \ldots, g_8$ still seemed complex. As such, we next changed the structure of $\{G_i(\xi)\}_{i=1}^{8}$ further. For a quad of Williamson sequences [31] $\boldsymbol{w}_i = (w_{i0}, w_{i1}, \ldots, w_{i,n-1}), i = 1,2,3,4,$ it holds $w_{10} = w_{20} = w_{30} = w_{40} = 1$ and the associated polynomials potentially take the form of

$$\begin{aligned} W_1(\xi) &= 1 - G_1(\xi) + G_2(\xi) + G_3(\xi) + G_4(\xi), \\ W_2(\xi) &= 1 + G_1(\xi) - G_2(\xi) + G_3(\xi) + G_4(\xi), \\ W_3(\xi) &= 1 + G_1(\xi) + G_2(\xi) - G_3(\xi) + G_4(\xi), \\ W_4(\xi) &= 1 + G_1(\xi) + G_2(\xi) + G_3(\xi) - G_4(\xi), \end{aligned}$$

where ξ is the n-th root of unity and the coefficients of $\{G_i(\xi)\}_{i=1}^{4}$ are of a four block. The associated polynomials $W_1(\xi), W_2(\xi), W_3(\xi), W_4(\xi)$ satisfy

$$4n = \sum_{i=1}^{4} \|W_i(\xi)\|^2 = \sum_{i=1}^{4} \|2G_i(\xi) + 1\|^2.$$

Inspired by this, it is reasonable to assume that the constant in (4) is contained in $G_1(\xi)$ and is 1, and following the analogous manner we can separate the constant 1 out. As a result, and slightly different from (4), the associated polynomials $F_1(\xi), \ldots, F_4(\xi)$ can be rewritten as

$$\begin{aligned} F_1(\xi) = &\ 1 + G_1(\xi) + G_2(\xi) + G_3(\xi) + G_4(\xi) + G_5(\xi) + G_6(\xi) + G_7(\xi) - G_8(\xi), \\ F_2(\xi) = &\ 1 + G_1(\xi) + G_2(\xi) - G_3(\xi) - G_4(\xi) + G_5(\xi) + G_6(\xi) - G_7(\xi) + G_8(\xi), \\ F_3(\xi) = &\ 1 + G_1(\xi) - G_2(\xi) + G_3(\xi) - G_4(\xi) + G_5(\xi) - G_6(\xi) + G_7(\xi) + G_8(\xi), \\ F_4(\xi) = &\ 1 + G_1(\xi) - G_2(\xi) - G_3(\xi) + G_4(\xi) - G_5(\xi) + G_6(\xi) + G_7(\xi) + G_8(\xi), \end{aligned} \tag{10}$$

where ξ is the n-th root of unity and the coefficients of $G_1(\xi), \ldots, G_8(\xi)$ are of an eight block. Then, we have

$$\sum_{i=1}^{4} \|F_i(\xi)\|^2 = 4 + 4\sum_{i=1}^{8} \|G_i(\xi)\|^2 + 2U(\xi) + 2\overline{U(\xi)} + 2V(\xi) + 2\overline{V(\xi)}, \tag{11}$$

with

$$\begin{aligned} U(\xi) = \;& G_1(\xi)\overline{G_5(\xi)} + G_1(\xi)\overline{G_6(\xi)} + G_1(\xi)\overline{G_7(\xi)} + G_1(\xi)\overline{G_8(\xi)} \\ &+ G_2(\xi)\overline{G_5(\xi)} + G_2(\xi)\overline{G_6(\xi)} - G_2(\xi)\overline{G_7(\xi)} - G_2(\xi)\overline{G_8(\xi)} \\ &+ G_3(\xi)\overline{G_5(\xi)} - G_3(\xi)\overline{G_6(\xi)} + G_3(\xi)\overline{G_7(\xi)} - G_3(\xi)\overline{G_8(\xi)} \\ &- G_4(\xi)\overline{G_5(\xi)} + G_4(\xi)\overline{G_6(\xi)} + G_4(\xi)\overline{G_7(\xi)} - G_4(\xi)\overline{G_8(\xi)} \end{aligned}$$

and

$$V(\xi) = G_5(\xi) + G_6(\xi) + G_7(\xi) + G_8(\xi) + 2G_1(\xi).$$

Consequently, we only need to construct the eight block $\{\boldsymbol{g}_i\}_{i=1}^{8}$ of order n, which together with $\boldsymbol{e} := (1, 0, \ldots, 0)$ of order n actually makes up a nine partition.

Analogous to Lemma 3, $\{f_i\}_{i=1}^{4}$ are a quad of GS sequences if and only if $U(\xi) + \overline{U(\xi)} + V(\xi) + \overline{V(\xi)} = 0$. In this case, an observation of the structure of $V(\xi)$ led to some more concrete relationships between $G_1(\xi)$ and $G_5(\xi)$-$G_8(\xi)$. We still added some symmetry properties, as shown in the theorems below, and omitted the proof for compactness.

Theorem 2. *For a nine partition $\boldsymbol{e}, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_8$ of order n, if the associated polynomials $G_i(\xi)$ of sequences $\boldsymbol{g}_i, i = 1, \ldots, 8$, satisfy*

$$\begin{aligned} &G_1(\xi) = -\overline{G_1}(\xi), \quad G_2(\xi) = \overline{G_2(\xi)}, \quad G_3(\xi) = \overline{G_4(\xi)}, \\ &G_5(\xi) = -\overline{G_5}(\xi), \quad G_6(\xi) = -\overline{G_6(\xi)}, \quad G_7(\xi) = -\overline{G_7}(\xi), \quad G_8(\xi) = -\overline{G_8(\xi)}, \end{aligned}$$

then we obtain a quad of GS sequences generated by (10).

In this case, note that all polynomials $G_i(\xi), i = 5, 6, 7, 8$, are antisymmetrical. The following two examples are shown to verify Theorem 2. For $n = 10$, we have

$$\begin{aligned} &\boldsymbol{g}_1 = (0,0,0,0,0,0,0,0,0,0), && \boldsymbol{g}_2 = (0,0,0,0,+,-,+,0,0,0), \\ &\boldsymbol{g}_3 = (0,0,0,0,0,0,0,+,+,0), && \boldsymbol{g}_4 = (0,0,+,+,0,0,0,0,0,0), \\ &\boldsymbol{g}_5 = (0,0,0,0,0,0,0,0,0,0), && \boldsymbol{g}_6 = (0,0,0,0,0,0,0,0,0,0), \\ &\boldsymbol{g}_7 = (0,0,0,0,0,0,0,0,0,0), && \boldsymbol{g}_8 = (0,+,0,0,0,0,0,0,0,-), \end{aligned}$$

which together with (10) lead to a quad of GS sequences of order 10 as follows

$$\begin{aligned} &f_1 = (+,-,+,+,+,-,+,+,+,+), && f_2 = (+,+,-,-,+,-,+,-,-,-,), \\ &f_3 = (+,+,-,-,-,+,-,+,+,-), && f_4 = (+,+,+,+,-,+,-,-,-,-). \end{aligned}$$

For $n = 12$,

$$\begin{aligned} &\boldsymbol{g}_1 = (0,0,0,0,0,0,0,0,0,0,0,0), && \boldsymbol{g}_2 = (0,0,+,0,0,0,+,0,0,0,+,0), \\ &\boldsymbol{g}_3 = (0,+,0,0,0,-,0,0,0,+,0,0), && \boldsymbol{g}_4 = (0,0,0,+,0,0,0,-,0,0,0,+), \\ &\boldsymbol{g}_5 = (0,0,0,0,0,0,0,0,0,0,0,0), && \boldsymbol{g}_6 = (0,0,0,0,0,0,0,0,0,0,0,0), \\ &\boldsymbol{g}_7 = (0,0,0,0,0,0,0,0,0,0,0,0), && \boldsymbol{g}_8 = (0,0,0,0,+,0,0,0,-,0,0,0), \end{aligned}$$

we can obtain a quad of GS sequences of order 12

$$\begin{aligned} &f_1 = (+,+,+,+,-,-,+,-,+,+,+,+), && f_2 = (+,-,+,-,+,+,+,+,-,-,+,-), \\ &f_3 = (+,+,-,-,+,-,-,+,-,+,-,-), && f_4 = (+,-,-,+,+,+,-,-,-,-,-,+). \end{aligned}$$

Similarly, we can construct two more types of GS sequences in view of $V(\xi)$.

Corollary 1. *For $\boldsymbol{e}, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_8$ of order n being a nine partition, the associated polynomials satisfy*

$$\begin{aligned} G_1(\xi) &= -\overline{G_1}(\xi), \quad G_2(\xi) = \overline{G_2(\xi)}, \quad G_3(\xi) = -\overline{G_4(\xi)}, \\ G_5(\xi) &= -\overline{G_5}(\xi), \quad G_6(\xi) = -\overline{G_6(\xi)}, \quad G_7(\xi) = -\overline{G_8(\xi)}, \end{aligned}$$

with the n-th root of unity ξ. Then, we obtain a quad of GS sequences $\{\boldsymbol{f}_i\}_{i=1}^4$ defined by (10).

Here, in $G_i(\xi)$, $i = 5, 6, 7, 8$, we obviously choose two of them as they were antisymmetrical and another two as they were antisymmetrical with each other. Again two examples are illustrated to verify Corollary 1. For $n = 6$, we have

$$\begin{aligned} \boldsymbol{g}_1 &= (0,0,0,0,0,0), & \boldsymbol{g}_2 &= (0,0,0,+,0,0), \\ \boldsymbol{g}_3 &= (0,0,0,0,-,0), & \boldsymbol{g}_4 &= (0,0,+,0,0,0), \\ \boldsymbol{g}_5 &= (0,0,0,0,0,0), & \boldsymbol{g}_6 &= (0,0,0,0,0,0), \\ \boldsymbol{g}_7 &= (0,0,0,0,0,-), & \boldsymbol{g}_8 &= (0,+,0,0,0,0), \end{aligned}$$

which together with (10) leads to a quad of GS sequences of order 6

$$\begin{aligned} \boldsymbol{f}_1 &= (+,-,+,+,-,-), & \boldsymbol{f}_2 &= (+,+,-,+,+,+), \\ \boldsymbol{f}_3 &= (+,+,-,-,-,-,), & \boldsymbol{f}_4 &= (+,+,+,-,+,-). \end{aligned}$$

For $n = 9$, we have

$$\begin{aligned} \boldsymbol{g}_1 &= (0,0,0,0,0,0,0,0,0), & \boldsymbol{g}_2 &= (0,0,0,0,0,0,0,0,0), \\ \boldsymbol{g}_3 &= (0,0,0,0,0,+,-,0,0), & \boldsymbol{g}_4 &= (0,0,0,+,-,0,0,0,0), \\ \boldsymbol{g}_5 &= (0,0,0,0,0,0,0,0,0), & \boldsymbol{g}_6 &= (0,0,0,0,0,0,0,0,0), \\ \boldsymbol{g}_7 &= (0,0,0,0,0,0,0,-,-), & \boldsymbol{g}_8 &= (0,+,+,0,0,0,0,0,0), \end{aligned}$$

which we can use to obtain a quad of GS sequences of order 9

$$\begin{aligned} \boldsymbol{f}_1 &= (+,-,-,+,-,+,-,-,-), & \boldsymbol{f}_2 &= (+,+,+,-,+,-,+,+,+), \\ \boldsymbol{f}_3 &= (+,+,+,-,+,+,-,-,-), & \boldsymbol{f}_4 &= (+,+,+,+,-,-,+,-,-). \end{aligned}$$

Corollary 2. *For a nine partition $\boldsymbol{e}, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_8$ of order n, the associated polynomials $G_i(\xi)$ satisfy*

$$\begin{aligned} G_1(\xi) &= -\overline{G_1(\xi)}, \quad G_2(\xi) = \overline{G_2(\xi)}, \quad G_3(\xi) = \overline{G_4(\xi)}, \\ G_5(\xi) &= -\overline{G_6(\xi)}, \quad G_7(\xi) = -\overline{G_8(\xi)}, \end{aligned}$$

where ξ is the n-th root of unity. Then, the $\{\boldsymbol{f}_i\}_{i=1}^4$ defined in (10) is a quad of GS sequences.

The last case is that in these four polynomials, two pairs are antisymmetrical with each other. We also provide two examples to verify Corollary 2. For $n = 6$, we have

$$\begin{aligned} \boldsymbol{g}_1 &= (0,0,0,0,0,0), & \boldsymbol{g}_2 &= (0,0,0,+,0,0), \\ \boldsymbol{g}_3 &= (0,0,0,0,+,0), & \boldsymbol{g}_4 &= (0,+,0,0,0,0), \\ \boldsymbol{g}_5 &= (0,0,0,0,0,-), & \boldsymbol{g}_6 &= (0,+,0,0,0,0), \\ \boldsymbol{g}_7 &= (0,0,0,0,0,0), & \boldsymbol{g}_8 &= (0,0,0,0,0,0). \end{aligned}$$

(10) yields a quad of GS sequences of order 6

$$f_1 = (+,+,+,+,+,-), \quad f_2 = (+,+,-,+,-,-),$$
$$f_3 = (+,-,-,-,+,-), \quad f_4 = (+,+,+,-,-,+).$$

For $n = 9$, we have

$$g_1 = (0,0,0,0,0,0,0,0,0), \quad g_2 = (0,0,0,0,0,0,0,0,0),$$
$$g_3 = (0,0,0,1,0,0,0,1,0), \quad g_4 = (0,0,1,0,0,0,1,0,0),$$
$$g_5 = (0,0,0,0,1,0,0,0,-1), \quad g_6 = (0,1,0,0,0,-1,0,0,0),$$
$$g_7 = (0,0,0,0,0,0,0,0,0), \quad g_8 = (0,0,0,0,0,0,0,0,0),$$

which together with (10) results in a quad of GS sequences of order 9

$$f_1 = (+,+,+,+,+,-,+,+,-), \quad f_2 = (+,+,-,-,+,-,-,-,-),$$
$$f_3 = (+,-,-,+,+,+,-,+,-), \quad f_4 = (+,+,+,-,-,-,+,-,+).$$

3.2. GS Sequences Based on a Nine Block

In addition, if we only discuss the term $U(\xi) + \overline{U(\xi)}$ in (11), then we can obtain some results related to GS sequences.

Corollary 3. *For a nine block $\mathbf{e}, g_1, \ldots, g_8$, the associated polynomials mentioned in (10) satisfy*

$$G_1(\xi) = -\overline{G_1(\xi)}, \quad G_2(\xi) = -\overline{G_2(\xi)}, \quad G_3(\xi) = \overline{G_3(\xi)},$$
$$G_4(\xi) = \overline{G_4(\xi)}, \quad G_5(\xi) = \overline{G_6(\xi)}, \quad G_7(\xi) = \overline{G_8(\xi)},$$

and

$$\sum_{i=1}^{4} \|F_i(\xi)\|^2 = 4\sum_{i=1}^{4} \|G_i(\xi)\|^2 + \sum_{i=5}^{8} \|2G_i(\xi) + 1\|^2 = 4n,$$

where ξ is the n-th root of unity. Then, we have a quad of GS sequences by (10).

There is an example through which to verify Corollary 3. For $n = 5$, we have

$$g_1 = (0,0,0,0,0), \quad g_2 = (0,0,-,+,0),$$
$$g_3 = (0,0,0,0,0), \quad g_4 = (0,0,0,0,0),$$
$$g_5 = (0,0,0,0,0), \quad g_6 = (0,0,0,0,0),$$
$$g_7 = (0,+,0,0,0), \quad g_8 = (0,0,0,0,+),$$

which together with (10) leads to a quad of GS sequences

$$f_1 = (+,+,-,+,-), \quad f_2 = (+,-,-,+,+),$$
$$f_3 = (+,+,+,-,+), \quad f_4 = (+,+,+,-,+).$$

Corollary 4. *For a nine block $\mathbf{e}, g_1, \ldots, g_8$, the associated polynomials satisfy*

$$G_1(\xi) = \overline{G_2(\xi)}, G_3(\xi) = \overline{G_4(\xi)}, G_5(\xi) = -\overline{G_6(\xi)}, G_7(\xi) = \overline{G_8(\xi)}$$

and

$$\sum_{i=1}^{4} \|F_i(\xi)\|^2 = 4\sum_{i=3}^{6} \|G_i(\xi)\|^2 + \sum_{i\in\{1,2,7,8\}} \|2G_i(\xi) + 1\|^2 = 4n,$$

where ξ is the n-th root of unity. Then, a quad of GS sequences are generated by (10).

One example is presented to verify the results of Corollary 4. For $n = 9$, we have

$$\begin{aligned} \boldsymbol{g}_1 &= (0,0,0,0,0,0,0,0,0), & \boldsymbol{g}_2 &= (0,0,0,0,0,0,0,0,0), \\ \boldsymbol{g}_3 &= (0,0,0,+,-,0,0,0,0), & \boldsymbol{g}_4 &= (0,0,0,0,0,-,+,0,0), \\ \boldsymbol{g}_5 &= (0,+,+,0,0,0,0,0,0), & \boldsymbol{g}_6 &= (0,0,0,0,0,0,0,-,-), \\ \boldsymbol{g}_7 &= (0,0,0,0,0,0,0,0,0), & \boldsymbol{g}_8 &= (0,0,0,0,0,0,0,0,0). \end{aligned}$$

which also yields a quad of GS sequences

$$\begin{aligned} \boldsymbol{f}_1 &= (+,+,+,+,-,-,+,-,-), & \boldsymbol{f}_2 &= (+,+,+,-,+,+,-,-,-), \\ \boldsymbol{f}_3 &= (+,+,+,+,-,+,-,+,+), & \boldsymbol{f}_4 &= (+,-,-,-,+,-,+,-,-). \end{aligned}$$

Remark 3. *In order to construct the GS sequences, we transformed it into the construction of eight polynomials $G_1(\xi), \ldots, G_8(\xi)$. For some special cases, we were able to obtain $G_i(\xi)$ via a four partition such as through T-sequences and, in actuality, we also searched them directly with computers in some more general cases, where utilizing known symmetry and antisymmetry properties may significantly reduce the search range.*

4. GS Structures of Two Groups of Polynomials

We analyzed a quad of GS sequences with different structures in Section 3, and we now intend to utilize two groups of polynomials $\{E_i(\xi)\}_{i=1}^8$ and $\{G_i(\xi)\}_{i=1}^8$, which are associated with sequences $\{\boldsymbol{e}_i\}_{i=1}^8$ and $\{\boldsymbol{g}_i\}_{i=1}^8$ to construct several different GS sequences.

We changed the conditions from an eight partition to a four partition, which produced the following result.

Theorem 3. *Let $\{E_i(\xi)\}_{i=1}^4$ be the associated polynomials of Williamson sequences $\{\boldsymbol{e}_i\}_{i=1}^4$ of order m, and let $G_1(\xi), G_2(\xi), G_7(\xi), G_8(\xi)$ of order n be chosen in Theorem 1, i.e., satisfying*

$$G_1(\xi) = \overline{G_1(\xi)}, G_2(\xi) = \overline{G_2(\xi)}, G_7(\xi) = -\overline{G_8(\xi)}.$$

Then, the four new polynomials, which are defined by

$$\begin{aligned} F_1(\xi) &= E_1(\xi)G_1(\xi) + E_2(\xi)G_2(\xi) + E_3(\xi)G_7(\xi) + E_4(\xi)G_8(\xi), \\ F_2(\xi) &= E_1(\xi)G_2(\xi) - E_2(\xi)G_1(\xi) + E_3(\xi)G_8(\xi) - E_4(\xi)G_7(\xi), \\ F_3(\xi) &= E_1(\xi)G_7(\xi) - E_2(\xi)G_8(\xi) - E_3(\xi)G_1(\xi) + E_4(\xi)G_2(\xi), \\ F_4(\xi) &= E_1(\xi)G_8(\xi) + E_2(\xi)G_7(\xi) - E_3(\xi)G_2(\xi) - E_4(\xi)G_1(\xi), \end{aligned} \tag{12}$$

satisfy

$$\sum_{i=1}^{4} \|F_i(\xi)\|^2 = \sum_{i=1}^{4} \|E_i(\xi)\|^2 \sum_{i=1}^{4} \|G_i(\xi)\|^2 = 4mn.$$

Moreover, if $(m,n) = 1$, then the sequences $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_4$ made up of the coefficients of $F_1(\xi), \ldots, F_4(\xi)$ are a quad of GS sequences.

Proof. Since Williamson sequences $\{\boldsymbol{e}_i\}_{i=1}^4$ are symmetrical, it is easy to verify the results $\sum_{i=1}^{4} \|F_i(\xi)\|^2 = 4mn$. Further, $(m,n) = 1$ guarantees that $\{\boldsymbol{f}_i\}_{i=1}^4$ consists of ± 1. $\square$

We now give an example for the sequences $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_7, \boldsymbol{g}_8$ of the associated polynomials $G_1(\xi), G_2(\xi), G_7(\xi), G_8(\xi)$ in Theorem 3 of order $n = 8$,

$$\begin{aligned} \boldsymbol{g}_1 &= (+,0,0,0,+,0,0,0), & \boldsymbol{g}_2 &= (0,0,+,0,0,0,+,0), \\ \boldsymbol{g}_7 &= (0,0,0,+,0,0,0,-), & \boldsymbol{g}_8 &= (0,+,0,0,0,-,0,0) \end{aligned}$$

and a quad of Williamson sequences $\boldsymbol{e}_i$ of order $m = 7$

$$\begin{aligned} \boldsymbol{e}_1 &= (+,+,-,-,-,-,+), \quad & \boldsymbol{e}_2 &= (+,-,+,+,+,+,-), \\ \boldsymbol{e}_3 &= (+,+,-,+,+,-,+), \quad & \boldsymbol{e}_4 &= (+,+,-,+,+,-,+). \end{aligned}$$

As the application of (12), we can obtain a quad of GS sequences of order $mn = 56$ as follows

$$\begin{aligned} \boldsymbol{f}_1 = (&+,-,+,-,-,-,-,+,+,+,+,-,-,+,+,+,-,-,+,+,+,+,-,-,-,-,+,-, \\ &+,+,+,+,-,+,-,-,+,-,+,+,-,-,+,-,-,+,+,-,+,-,-,+,-,+,+,+), \\ \boldsymbol{f}_2 = (&+,-,+,+,+,-,-,-,-,+,+,+,+,+,-,-,+,-,+,-,-,+,-,+,+,-,+,+, \\ &+,+,+,-,+,+,-,+,-,-,+,-,+,-,-,+,+,+,+,+,-,-,-,-,+,+,+,-), \\ \boldsymbol{f}_3 = (&+,+,+,-,+,+,-,+,+,-,-,-,-,-,-,-,-,-,-,-,+,-,-,+,+,-,+,+, \\ &+,-,+,+,+,-,-,-,+,+,-,+,-,+,-,+,-,+,-,+,+,+,-,-,+,+,+,-), \\ \boldsymbol{f}_4 = (&+,-,-,-,+,-,+,-,+,+,+,-,-,+,+,-,-,+,+,-,+,-,+,+,+,+,-,+, \\ &+,+,-,+,+,+,+,+,+,-,+,+,-,-,+,+,-,-,+,+,+,+,+,-,+,-,-,-). \end{aligned}$$

5. Conclusions

In this paper, we studied several special structures of a quad of GS sequences by using k-partitions or k-blocks with different symmetry properties. It has been rigorously proven that a quad of ± 1 sequences can be determined uniquely by an eight block. Then, we can write a quad of GS sequences into two forms (4) or (10), and we can then let $U(\xi) + \overline{U(\xi)} = 0$ in (7) or $U(\xi) + \overline{U(\xi)} + V(\xi) + \overline{V(\xi)} = 0$ in (11), respectively. This, consequently, reveals some of the relationships between these k-partitions or k-blocks, which are based on whether we can add some symmetry properties to obtain GS sequences with different structures. Moreover, through making use of some of the special structures of $\{G_i(\xi)\}_{i=1}^{8}$ of order n and Williamson sequences of order m, we managed to construct a quad of GS sequences of order $4mn$.

For now, to obtain the k-partitions and k-blocks, we completely made use of the computer by using an exhaustive search based on the symmetry and antisymmetry properties, which reduced the degree of computational consumption significantly. In the future, we will be devoted to discussing more sufficient or necessary conditions for the existence of a k-block in order to obtain more of the relationships between a k-block serving for the purposes of improving searching efficiency, and we will also try to determine the k-partition theoretically.

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