mathematics

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A New Methodology for the Development of Efficient Multistep Methods for First-Order IVPs with Oscillating Solutions

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Abstract: In this research, we provide a novel approach to the development of effective numerical algorithms for the solution of first-order IVPs. In particular, we detail the fundamental theory behind the development of the aforementioned approaches and show how it can be applied to the Adams–Bashforth approach in three steps. The stability of the new scheme is also analyzed. We compared the performance of our novel algorithm to that of established approaches and found it to be superior. Numerical experiments confirmed that, in comparison to standard approaches to the numerical solution of Initial Value Problems (IVPs), including oscillating solutions, our approach is significantly more effective.

Keywords: numerical solution; initial value problems (IVPs); Adams–Bashforth methods; trigonometric fitting; multistep methods

MSC: 65L05; 65L06

Citation: Simos, T.E. A New Methodology for the Development of Efficient Multistep Methods for First-Order IVPs with Oscillating Solutions. *Mathematics* **2024**, *12*, 504. https://doi.org/10.3390/math12040504

Academic Editor: Ravi P. Agarwal

Received: 5 January 2024
Revised: 31 January 2024
Accepted: 4 February 2024
Published: 6 February 2024

1. Introduction

Equations or systems of equations of the form

$$\mathbf{y}'(\mathbf{t}) = \mathbf{f}(\mathbf{t}, \mathbf{y}),\ \mathbf{y}(\mathbf{t_0}) = \mathbf{y_0}, \tag{1}$$

are used to solve problems in a wide variety of fields, including astrophysics, chemistry, physics, chemistry, electronics, nanotechnology, materials science, and more. The category of equations with an oscillatory/periodic solution deserves extra consideration (see [1,2]).

Significant effort has been put into studying the numerical solution to the above equation or system of equations during the past two decades (for examples, see [3–17], and the references therein). For a more in-depth look at the techniques used to solve (1) with solutions presenting oscillating behavior, refer to [3,8,18] and the references therein; Quinlan and Tremaine [10] as well as [6,7,19]; and so on. All existing numerical methods for solving (1) in the literature have several commonalities, the most prominent one being that they are multistep or hybrid approaches. In addition, the vast majority of these techniques were developed for the numerical solution of second-order differential equations. We mention the following basic categories of methods and the bibliography for them:

- Exponentially fitted, trigonometrically fitted, phase-fitted and amplification-fitted Runge–Kutta and Runge–Kutta–Nyström methods and Runge–Kutta and Runge–Kutta–Nyström methods with minimal phase lag (see [20–57]).
- Exponentially fitted and trigonometrically fitted phase-fitted and amplification-fitted multistep methods and multistep methods with minimal phase lag (see [58–121]).

From the above bibliography, it is easy to see that there has not been, until now, a contribution to the development of multistep methods with minimal phase lag or phase-fitted multistep methods for first-order IVPs.

In this paper, we introduce the theory for the calculation of the phase lag and amplification error (or amplification factor) of the multistep methods for first-order IVPs.

The paper is developed as follows:

- In Section 2, we develop the general theory for the calculation of the phase lag and amplification error (or amplification factor) of the multistep methods for first-order IVPs.
- In Section 3, we present methodologies for the achievement of the phase lag, amplification factor, phase-fitted, and amplification-fitted multistep methods. More specifically, we present methodologies for the achievement of minimal phase lag of the multistep method, a methodology for the achievement of the amplification-fitted multistep method, and a methodology for the achievement of the phase-fitted and amplification-fitted multistep method.
- In Section 4, we present the stability analysis for the new proposed methods.
- In Section 5, we present the numerical results.

The numerical results show that the methodology for the development of phase-fitted and amplification-fitted multistep methods produced the most efficient ones for problems with solutions of oscillating behavior.

2. The Theory

In order to study the phase lag of multistep methods for the problems (1), the following scalar test equation is used:

$$y'(t) = I\,\omega\,y(t) \tag{2}$$

The solution of the above equation is given by:

$$y(t) = \exp(I\,\omega\,t) \tag{3}$$

Consider the multistep methods for the numerical solution of the above-mentioned problem (1):

$$y_{n+k} - y_{n+k-1} = h \sum_{j=1}^{k} \Big[A_{n+k-j}(\omega\,h)\, f_{n+k-j} \Big] \tag{4}$$

where $A_{n+k-j}(\omega\,h)$, $j = 1, 2, \ldots, k$ are polynomials of $\omega\,h$, and h is the step length of the integration.

Applying (4) to (2), we achieve:

$$y_{n+k} - y_{n+k-1} = I\,\omega\,h \sum_{j=1}^{k} \Big[A_{n+k-j}(\omega\,h)\, y_{n+k-j} \Big] \tag{5}$$

Taking into account:

$$v = \omega\,h \tag{6}$$

(5) gives:

$$y_{n+k} - y_{n+k-1} = I\,v \sum_{j=1}^{k} \Big[A_{n+k-j}(v)\, y_{n+k-j} \Big] \tag{7}$$

and

$$y_{n+k} - (1 + I\,v\,A_{n+k-1}(v))\,y_{n+k-1} - I\,v\,\sum_{j=2}^{k}\Big[A_{n+k-j}(v)\,y_{n+k-j}\Big] = 0 \tag{8}$$

The characteristic equation of the above difference equation is given by:

$$\lambda^k - [1 + I\,v\,A_{n+k-1}(v)]\,\lambda^{k-1} - I\,v\,\sum_{j=2}^{k}\Big[A_{n+k-j}(v)\,\lambda^{k-j}\Big] = 0 \tag{9}$$

Definition 1. *Taking into account that the theoretical solution of the scalar test Equation (1) for $t = h$ is equal to $\exp(I\,\omega\,h)$, i.e., $\exp(I\,v)$ (see (6)), and the numerical solution of the scalar test Equation (1) for $t = h$ is equal to $\exp(I\,\theta(v))$, we have the following definition for the phase lag:*

$$\Phi = v - \theta(v) \tag{10}$$

The order of the phase lag is equal to q if and only if the quantity $\Phi = O\big(v^{q+1}\big)$ as $v \longrightarrow 0$.

Taking into account the following relation:

$$\lambda^n = \exp^{n\,I\,\theta(v)} = \cos[n\,\theta(v)] + I\,\sin[n\,\theta(v)]\; n = 1, 2, \ldots \tag{11}$$

we obtain:

$$\begin{aligned}\cos[k\,\theta(v)] + I\,\sin[k\,\theta(v)] - (1 + I\,v\,A_{n+k-1})\,\Big\{\cos[(k-1)\,\theta(v)] + I\,\sin[(k-1)\,\theta(v)]\Big\}\\ - \sum_{j=2}^{k-1} I\,v\,A_{n+k-j}(v)\,\Big\{\cos[(k-j)\,\theta(v)] + I\,\sin[(k-j)\,\theta(v)]\Big\} - I\,v\,A_n(v) = 0\end{aligned} \tag{12}$$

The following lemmas must be used to analyze the above relation (12).

Lemma 1. *The following relation is valid:*

$$\cos[\theta(v)] = cos(v) + c\,v^{q+2} + O\Big(v^{q+4}\Big) \tag{13}$$

For the proof, see Appendix A.

Lemma 2. *The following relation is valid:*

$$\sin[\theta(v)] = sin(v) - c\,v^{q+1} + O\Big(v^{q+3}\Big) \tag{14}$$

For the proof, see Appendix B.

Lemma 3. *The following relation is valid:*

$$\cos[j\,\theta(v)] = cos(j\,v) + c\,j^2\,v^{q+2} + O\Big(v^{q+4}\Big) \tag{15}$$

$$\sin[j\,\theta(v)] = sin(j\,v) - c\,j\,v^{q+1} + O\Big(v^{q+3}\Big) \tag{16}$$

For the proof, see Appendix C.

Taking into account relations (15) and (16), relation (12) becomes:

$$\begin{aligned}
&\cos[k\,v] + c\,k^2\,v^{q+2} + I\left[\sin[k\,v] - c\,k\,v^{q+1}\right] \\
&-(1 + I\,v\,A_{n+k-1})\left\{\left[\cos[(k-1)\,v] + c\,(k-1)^2\,v^{q+2}\right]\right. \\
&\left. +I\left[\sin[(k-1)\,v] - c\,(k-1)\,v^{q+1}\right]\right\} \\
&-\sum_{j=2}^{k-1} I\,v\,A_{n+k-j}(v)\left\{\left[\cos[(k-j)\,v] + c\,(k-j)^2\,v^{q+2}\right]\right. \\
&\left. +I\left[\sin[(k-j)\,v] - c\,(k-j)\,v^{q+1}\right]\right\} - I\,v\,A_n(v) = 0
\end{aligned} \tag{17}$$

The above relation (17) can be divided into two parts: the real part and the imaginary part.

- **The Real Part**

  The real part gives:

$$\begin{aligned}
&\cos[k\,v] + c\,k^2\,v^{q+2} - \cos[(k-1)\,v] - c\,(k-1)^2\,v^{q+2} \\
&+v\,A_{n+k-1}\left[\sin[(k-1)\,v] - c\,(k-1)\,v^{q+1}\right] \\
&+\sum_{j=2}^{k-1} v\,A_{n+k-j}(v)\left[\sin[(k-j)\,v] - c\,(k-j)\,v^{q+1}\right] = 0
\end{aligned} \tag{18}$$

  The relation (18) gives:

$$\begin{aligned}
&\cos[k\,v] - \cos[(k-1)\,v] + \sum_{j=1}^{k-1} v\,A_{n+k-j}(v)\,\sin[(k-j)\,v] \\
&= -c\,v^{q+2}\left[k^2 - (k-1)^2 - \sum_{j=1}^{k-1} v\,A_{n+k-j}(v)\right] \Longrightarrow \\
-c\,v^{q+2} &= \frac{\cos[k\,v] - \cos[(k-1)\,v] + \sum_{j=1}^{k-1} v\,A_{n+k-j}(v)\,\sin[(k-j)\,v]}{2\,k - 1 - \sum_{j=1}^{k-1} A_{n+k-j}(v)\,(k-j)}
\end{aligned} \tag{19}$$

  This is the direct formula for the computation of the phase lag of the multistep method (4). Below, we will describe the procedure for the computation of the phase lag of method (4).

- **The Imaginary Part**

  The imaginary part gives:

$$\begin{aligned}
&\sin[k\,v] - c\,k\,v^{q+1} - \sin[(k-1)\,v] + c\,(k-1)\,v^{q+1} \\
&-v\,A_{n+k-1}\left[cos[(k-1)\,v] + c\,(k-1)^2\,v^{q+2}\right] \\
&-\sum_{j=2}^{k-1} v\,A_{n+k-j}(v)\left[\cos[(k-j)\,v] + c\,(k-j)^2\,v^{q+2}\right] - v\,A_n = 0
\end{aligned} \tag{20}$$

  Relation (20) gives:

$$\sin[k\,v] - \sin[(k-1)\,v] - \sum_{j=1}^{k-1} v\,A_{n+k-j}(v)\,\cos[(k-j)\,v] - v\,A_n$$
$$= -c\,v^{q+1}\Big[-1 - \sum_{j=1}^{k-1} v^2\,A_{n+k-j}(v)(k-j)^2\Big] \Longrightarrow$$
$$-c\,v^{q+1} = \frac{\sin[k\,v] - \sin[(k-1)\,v] - \sum_{j=1}^{k-1} v\,A_{n+k-j}(v)\,\cos[(k-j)\,v] - v\,A_n}{-1 - \sum_{j=1}^{k-1} v^2\,A_{n+k-j}(v)(k-j)^2} \tag{21}$$

This is the direct formula for the computation of the amplification factor of the multistep method (4).

3. Methodologies for Achievement of the Phase Lag, Amplification Factor, Phase-Fitted and Amplification-Fitted Methods

3.1. Classical Methods—Methods with Constant Coefficients Known in the Literature

We will focus our examples on the well-known method of Adams–Bashforth of the third algebraic order, i.e., on the method:

$$\psi_{n+1} - \psi_n = \frac{h}{12}\Big(23\,\psi'_n - 16\,\psi'_{n-1} + 5\,\psi'_{n-2}\Big) \tag{22}$$

with the local truncation error (LTE) given by:

$$LTE = \frac{3}{8}\,h^4\,y^{\{(4)\}}(t) + O\Big(h^5\Big) \tag{23}$$

In order to determine the phase lag and amplification error of this method, we apply the theory developed in Section 2.

Application of the method (22) to the test Equation (2) leads to the difference Equation (7) with $k = 3$ and

$$A_2(v) = \frac{23}{12},\ A_1(v) = -\frac{4}{3},\ A_0(v) = \frac{5}{12} \tag{24}$$

Based on the above formula (19) and using Taylor series expansion for $\cos(m\,v)$, $\sin(m\,v)$, $m = 1,\,2$, we obtain:

$$\frac{\cos(3\,v) - \cos(2\,v) + v\,A_2(v)\,\sin(2\,v) + v\,A_1(v)\,\sin(v)}{5 - 2\,A_2(v) - A_1(v)} = \frac{3}{20}\,v^4 - \frac{61}{360}\,v^6 + \ldots \tag{25}$$

Consequently, $q = 2$ and $c = -\frac{3}{20}$. The third algebraic order Adams–Bashforth method is of the second-order phase lag.

Based now on the above formula (21) and using Taylor series expansion for $\cos(m\,v)$, $\sin(m\,v)$, $m = 1,\,2$, we obtain:

$$\frac{\sin(3\,v) - \sin(2\,v) - v\,A_2(v)\,\cos(2\,v) - v\,A_1(v)\,\cos(v) - v\,A_0(v)}{-1 - 4\,v^2\,A_2(v) - v^2\,A_1(v)} = -\frac{193}{360}\,v^5 + \frac{54967}{15120}\,v^7 + \ldots \tag{26}$$

Consequently, $q = 4$ and $c = \frac{193}{360}$. The third algebraic order Adams–Bashforth method is of the fourth order amplification error. For our computational purposes, we will call the third algebraic order Adams–Bashforth method **Method I**.

3.2. Minimal Phase Lag

In order to investigate the minimization of the phase lag, we study the following general four-step method:

$$\psi_{n+1} - \psi_n = h\left(A_2(v), \psi'_n + A_1(v)\,\psi'_{n-1} + A_0(v)\,\psi'_{n-2}\right) \tag{27}$$

3.2.1. Algorithm for the Minimization of the Phase Lag

The algorithm for the minimization of the phase lag is the following:

- Elimination of the amplification factor.
- Computation of the phase lag based on the coefficient obtained by the previous step.
- Taylor series expansion of the phase lag computed above.
- Determination of the system of equations in order to achieve minimal phase lag.
- Computation of the new coefficients.

Based on the above algorithm, we obtain the following two methods with minimal phase lag.

3.2.2. First Method with Minimal Phase Lag (Method with Minimal Phase Lag and Eliminated Amplification Factor with Third Algebraic Order)

We consider the method (27) with $A_0(v) = \frac{5}{12}$

Using the direct formula for the computation of the amplification factor (21), we obtain:

$$AF = \frac{\sin(3\,v) - \sin(2\,v) - A_2(v)\,v\,\cos(2\,v) - A_1(v)\,v\,\cos(v) - A_0(v)\,v}{-4\,v^2\,A_2(v) - v^2\,A_1(v) - 1} \tag{28}$$

where AF declares the amplification factor.

Requiring the elimination of the amplification factor, i.e., requiring $AF = 0$, we obtain:

$$A_2(v) = \frac{-A_1(v)\,v\,\cos(v) - A_0(v)\,v + \sin(3\,v) - \sin(2\,v)}{v\cos(2\,v)} \tag{29}$$

Substituting the values of $A_0(v)$ and $A_2(v)$ given above into the direct formula for the computation of the phase lag (19), we obtain:

$$PhErr = \frac{v\left(\frac{5}{6}\,\sin(v)\,\cos(v)\,v + A_1(v)\,v\,\sin(v) - \cos(v) + 1\right)}{A_3(v)} \tag{30}$$

where

$$\begin{aligned} A_3(v) &= 2\,[\cos(v)]^2\,v\,A_1(v) + 8\,\sin(v)\,[\cos(v)]^2 - 10\,[\cos(v)]^2\,v \\ &-2\,A_1(v)\,v\,\cos(v) - 4\,\sin(v)\,\cos(v) - A_1(v)\,v + \frac{25\,v}{6} - 2\,\sin(v) \end{aligned} \tag{31}$$

and $PhErr$ declares the phase lag.

Taking the Taylor series expansion of the formula (30), we obtain:

$$PhErr = \frac{\left(\frac{4}{3} + A_1(v)\right)v^2}{-A_1(v) - \frac{23}{6}} + \frac{v^4\,A_4(v)}{-A_1(v) - \frac{23}{6}} + \dots \tag{32}$$

where

$$A_4(v) = -\frac{43}{72} - \frac{1}{6}\,A_1(v) + 2\,\frac{(4 + 3\,A_1(v))\left(-A_1(v) + \frac{11}{3}\right)}{6\,A_1(v) + 23} \tag{33}$$

Requesting the minimization of the phase lag, we obtain the following equation:

$$\frac{\left(\frac{4}{3}+A_1(v)\right)v^2}{-A_1(v)-\frac{23}{6}}=0 \Longrightarrow A_1(v)=-\frac{4}{3}. \tag{34}$$

The characteristics of this new method are:

$$\begin{aligned} A_0(v) &= \frac{5}{12} \\ A_1(v) &= -\frac{4}{3} \\ A_2(v) &= \frac{1}{12}\,\frac{16\,v\,\cos(v)+48\,\sin(v)\,\cos(v)^2-12\,\sin(v)-24\,\sin(v)\,\cos(v)-5\,v}{v\,\cos(2\,v)} \\ LTE &= \frac{3}{8}\,h^4\,y^{\{(4)\}}(t)+O\left(h^5\right) \\ PhErr &= \frac{3}{20}\,v^4+\frac{467}{1800}\,v^6+\ldots \end{aligned} \tag{35}$$

The Taylor series expansion of the coefficient $A_2(v)$ is given by:

$$A_2(v)=\frac{23}{12}+\frac{193}{360}\,v^4+\frac{12583}{15120}\,v^6+\ldots \tag{36}$$

We will call the above new method **Method II** for our computational purposes.

3.2.3. Second Method with Minimal Phase Lag (Method with Minimal Phase Lag and Eliminated Amplification Factor with Second Algebraic Order)

We consider again the general four-step method presented in (27). Our strategy is the following.

Elimination of the Amplification Factor

In order to achieve this target, we have the following procedure:

- Using the direct formula for the computation of the amplification factor (21) for $k=3$, we obtain the relation (28).
- Requiring the elimination of the amplification factor for $k=3$, i.e., requiring $AF=0$, we obtain the relation (29).

Minimization of the Phase Lag

- Using the direct formula for the computation of the phase lag (19) with the value of $A_2(v)$ calculated by (29), we obtain:

$$PhErr=\frac{v\left(2\,\cos(v)\,\sin(v)\,v\,A_0(v)+A_1(v)\,v\,\sin(v)-\cos(v)+1\right)}{A_5(v)} \tag{37}$$

where

$$\begin{aligned} A_5(v) &= 2\,(\cos(v))^2\,v\,A_1(v)+8\,\sin(v)\,[\cos(v)]^2 \\ &-10\,[\cos(v)]^2\,v-2\,A_1(v)\,v\,\cos(v)-4\,\cos(v)\,\sin(v) \\ &-A_1(v)\,v-2\,A_0(v)\,v-2\,\sin(v)+5\,v \end{aligned} \tag{38}$$

and $PhErr$ declares the phase lag.

- Taking the Taylor series expansion of the formula (37), we obtain:

$$PhErr = -\frac{1}{2}\,\frac{\left(4\,A_0(v)+2\,A_1(v)+1\right)v^2}{A_1(v)+3+2\,A_0(v)}+\frac{1}{24}\,\frac{A_6(v)\,v^4}{\left(A_1(v)+3+2\,A_0(v)\right)^2}$$
$$-\frac{A_7(v)\,v^6}{720\left(A_1(v)+3+2\,A_0(v)\right)^3}+\frac{A_8(v)\,v^8}{120960\left(A_1(v)+3+2\,A_0(v)\right)^4}$$
$$-\frac{A_9(v)\,v^{10}}{725760\left(A_1(v)+3+2\,A_0(v)\right)^5}+\ldots \tag{39}$$

where

$$\begin{aligned}
A_6(v) &= 28\,A_1(v)^2+88\,A_1(v)\,A_0(v)+64\,A_0(v)^2 \\
&\quad -63\,A_1(v)-78\,A_0(v)-41 \\
A_7(v) &= 1266\,A_1(v)^3+4536\,A_1(v)^2\,A_0(v)+4392\,A_1(v)\,A_0(v)^2 \\
&\quad +768\,A_0(v)^3-3331\,A_1(v)^2-7324\,A_1(v)\,A_0(v)-4204\,A_0(v)^2 \\
&\quad +6852\,A_1(v)+11244\,A_0(v)+4717 \\
A_8(v) &= 305448\,A_1(v)^4+1251120\,A_1(v)^3\,A_0+1587168\,A_1(v)^2\,A_0(v)^2 \\
&\quad +625728\,A_1(v)\,A_0(v)^3+24576\,A_0(v)^4-1138857\,A_1(v)^3-3145374\,A_1(v)^2\,A_0(v) \\
&\quad -2160972\,A_1(v)\,A_0(v)^2+116376\,A_0(v)^3+2964871\,A_1(v)^2+7033180\,A_1(v)\,A_0(v) \\
&\quad +3960796\,A_0(v)^2-3414051\,A_1(v)-6053454\,A_0(v)-2816419 \\
A_9(v) &= 2654582\,A_1(v)^5+12221120\,A_1(v)^4\,A_0(v)+18932720\,A_1(v)^3\,A_0(v)^2 \\
&\quad +11125120\,A_1(v)^2\,A_0(v)^3+1823200\,A_1(v)\,A_0(v)^4+16384\,A_0(v)^5 \\
&\quad -12358419\,A_1(v)^4-40966656\,A_1(v)^3\,A_0(v)-38619576\,A_1(v)^2\,A_0(v)^2 \\
&\quad -4641600\,A_1(v)\,A_0(v)^3+3584400\,A_0(v)^4+42485492\,A_1(v)^3 \\
&\quad +121231992\,A_1(v)^2 A_0(v)+91483728\,A_1(v)\,A_0(v)^2+5748064\,A_0(v)^3 \\
&\quad -73555030\,A_1(v)^2-167703472\,A_1(v)\,A_0(v)-84236488\,A_0(v)^2 \\
&\quad +63314574\,A_1(v)+132127224\,A_0(v)+61142641
\end{aligned} \tag{40}$$

- Requesting the minimization of the phase lag, we obtain the following system of equations:

$$\begin{aligned}
\frac{\left(4\,A_0(v)+2\,A_1(v)+1\right)}{A_1(v)+3+2\,A_0(v)} &= 0 \\
\frac{A_6(v)}{\left(A_1(v)+3+2\,A_0(v)\right)^2} &= 0
\end{aligned} \tag{41}$$

- Solving the above system of Equation (41), we obtain:

$$A_1(v) = -\frac{7}{12},\; A_0(v) = \frac{1}{24}. \tag{42}$$

The characteristics of this new method are:

$$A_0(v) = \frac{1}{24}$$
$$A_1(v) = -\frac{7}{12}$$
$$A_2(v) = \frac{1}{24}\,\frac{14\,v\,\cos(v) + 24\,\sin(3\,v) - 24\,\sin(2\,v) - v}{v\,\cos(2\,v)}$$
$$LTE = \frac{3}{8}\,h^3\left(y^{\{(3)\}}(t) + \omega^2\,y'(t)\right) + O\left(h^5\right)$$
$$PhErr = -\frac{11}{3600}\,v^6 - \frac{2423}{504000}\,v^8 + \ldots \tag{43}$$

The Taylor series expansion of the coefficient $A_2(v)$ is given by:

$$A_2(v) = \frac{37}{24} - \frac{3}{8}\,v^2 + \frac{7}{1440}\,v^4 - \frac{761}{60480}\,v^6 - \ldots \tag{44}$$

We will call the above new method **Method III** for our computational purposes.

3.3. Amplification Fitted Method

We consider the method (27) with $A_1(v) = -\frac{4}{3}$ and $A_0(v) = \frac{5}{12}$.
Using the direct formula for the computation of the amplification factor (21), we obtain:

$$AF = -\frac{1}{4}\,\frac{A_{10}(v)}{12\,A_2(v)\,v^2 - 4\,v^2 + 3} \tag{45}$$

where AF declares the amplification factor and

$$A_{10}(v) = -24\,A_2(v)\,v[\cos(v)]^2 + 48\,\sin(v)\,[\cos(v)]^2$$
$$-24\,\sin(v)\,\cos(v) + 16\,v\cos(v) + 12\,A_2(v)\,v - 12\,\sin(v) - 5\,v \tag{46}$$

Requiring the elimination of the amplification factor, i.e., requiring $AF = 0$, we obtain:

$$A_2(v) = \frac{1}{12}\,\frac{16\,v\,\cos(v) + 12\,\sin(3\,v) - 12\,\sin(2\,v) - 5\,v}{v\,\cos(2\,v)} \tag{47}$$

Substituting the value of $A_2(v)$ given above into the direct formula for the computation of the phase lag (19), we obtain:

$$PhErr = \frac{v\left(5\,v\sin(v)\,\cos(v) - 8\,v\,\sin(v) - 6\,\cos(v) + 6\right)}{A_{11}(v)} \tag{48}$$

where

$$A_{11}(v) = 48\,\sin(v)\,[\cos(v)]^2 - 76\,[\cos(v)]^2\,v$$
$$-24\,\sin(v)\,\cos(v) + 16\,v\cos(v) - 12\,\sin(v) + 33\,v \tag{49}$$

Taking the Taylor series expansion of the formula (48), we obtain:

$$PhErr = \frac{3}{20}\,v^4 + \frac{467}{1800}\,v^6 + \frac{245629}{504000}\,v^8 + \ldots \tag{50}$$

Remark 1. *Consequently, $q = 2$ and $c = -\frac{3}{20}$. The amplification-fitted Adams–Bashforth method developed in this section is of the second-order phase lag, i.e., has the same phase lag order of the the third algebraic order Adams–Bashforth method.*

The Taylor series expansion of the coefficient $A_2(v)$ is given by:

$$A_2(v) = \frac{23}{12} + \frac{193}{360}\, v^4 + \frac{12583}{15120}\, v^6 + \frac{13973}{10368}\, v^8 + \dots \tag{51}$$

The characteristics of this new method are:

$$\begin{aligned} A_0(v) &= \frac{5}{12} \\ A_1(v) &= -\frac{4}{3} \\ A_2(v) &= \frac{1}{12}\, \frac{16\, v\, \cos(v) + 12\, \sin(3\, v) - 12\, \sin(2\, v) - 5\, v}{v\, \cos(2\, v)} \\ LTE &= \frac{3}{8}\, h^4\, y^{\{(4)\}}(t) + O\Big(h^5\Big) \\ PhErr &= \frac{3}{20}\, v^4 + \frac{467}{1800}\, v^6 + \dots \end{aligned} \tag{52}$$

We will call the above new method **Method IV** for our computational purposes.

3.4. Phase Fitted and Amplification Fitted Method

We consider the method (27).

Using the direct formula for the computation of the phase lag and (19) the amplification factor (21), we obtain:

$$PhErr = \frac{\cos(3\, v) - \cos(2\, v) + A_2(v)\, v\, \sin(2\, v) + A_1(v)\, v\, \sin(v)}{5 - 2\, A_2(v) - A_1(v)} \tag{53}$$

$$AF = \frac{\sin(3\, v) - \sin(2\, v) - A_2(v)\, v\, \cos(2\, v) - A_1(v)\, v\, \cos(v) - A_0(v)\, v}{-4\, A_2(v)\, v^2 - v^2\, A_1(v) - 1} \tag{54}$$

where $PhErr$ is the phase lag, and AF declares the amplification factor.

Requiring the elimination of the phase lag and the amplification factor, i.e., requiring $PhErr = 0$ and $AF = 0$, and considering that $A_0 = \frac{5}{12}$, we obtain:

$$A_1(v) = \frac{1}{6}\, \frac{5\, v\, \Big[\sin(v)\Big]^3 - 5\, v\, \sin(v) + 6\, \Big[\cos(v)\Big]^2 - 6\, \cos(v)}{\sin(v)\, \cos(v)\, v} \tag{55}$$

$$A_2(v) = \frac{1}{12}\, \frac{5\, v\, \sin(v) + 12 - 24\, \Big[\cos(v)\Big]^2 + 12\, \cos(v)}{v\, \sin(v)} \tag{56}$$

Taking the Taylor series expansion of the above formulae, we obtain:

$$A_1(v) = -\frac{4}{3} + 3/8\, v^2 - \frac{7\, v^4}{180} + \frac{89\, v^6}{120960} - \frac{23\, v^8}{362880} - \frac{1963\, v^{10}}{479001600} \tag{57}$$

$$A_2(v) = \frac{23}{12} - 3/8\, v^2 + \frac{v^4}{80} - \frac{11\, v^6}{13440} - \frac{v^8}{26880} - \frac{233\, v^{10}}{53222400} + \dots \tag{58}$$

The characteristics of this new method are:

$$\begin{aligned} A_0(v) &= \frac{5}{12} \\ A_1(v) &\text{ see } (55) \\ A_2(v) &\text{ see } (56) \\ LTE &= \frac{3}{8}\, h^4\, \Big[y^{\{(4)\}}(t) + \omega^2\, y^{\{(2)\}}(t)\Big] + O\Big(h^5\Big) \\ PhErr &= 0 \\ AF &= 0 \end{aligned} \tag{59}$$

We will call the above new method **Method V** for our computational purposes.

4. Stability Analysis

Let us consider the general form of Adams–Bashforth four-step methods:

$$\psi_{n+1} - \psi_n = h \left(C_2 \, f_n + C_1 \, f_{n-1} + C_0 \, f_{n-2} \right) \tag{60}$$

where $f_{n+j} = y'_{n+j}$, $j = -2(1)0$

The obtained methods in the previous section, i.e., the methods (24), (35), (36), (43), (44), (51), (52), (58), and (59), belong to the general method (60).

Applying the scheme (60) to the scalar test equation

$$\psi' = \lambda \, y \quad \text{where } \lambda \in \mathcal{C} \tag{61}$$

we obtain the following difference equation:

$$\psi_{n+1} - A(H) \, \psi_n - B(H) \, \psi_{n-1} - C(H) \, \psi_{n-2} = 0 \tag{62}$$

where $H = \lambda \, h$ and

$$A(H) = 1 + C_2 \, H, \;\; B(H) = C_1 \, H, \;\; C(H) = C_3 \, H \tag{63}$$

The characteristic equation of (62) is given by

$$r^3 - A(H) \, r^2 - B(H) \, r - C(H) = 0 \tag{64}$$

Solving the above equation in H and substituting $r = \exp(i \, \theta)$, where $i = \sqrt{-1}$, we can plot the stability regions for $\theta \in [0, 2\,\pi]$. In Figures 1–5, we present the stability region for the obtained **Methods I–V**. For the cases of Methods II–V, we present the stability regions for $v = 1$, $v = 10$, and $v = 100$.

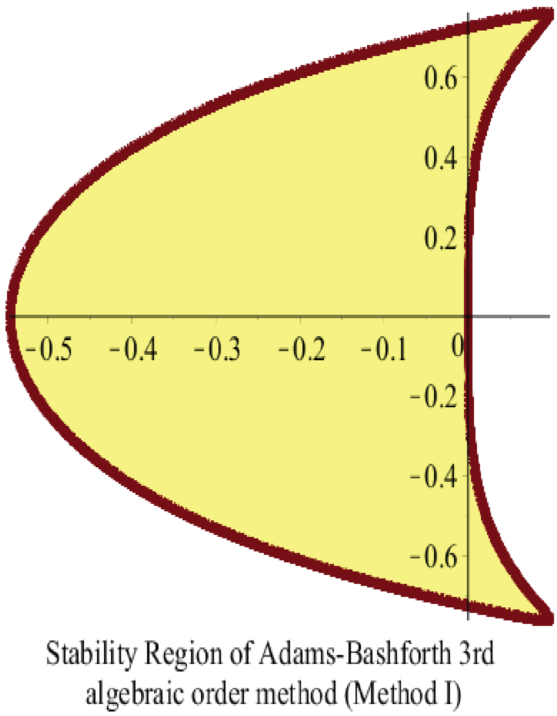

Figure 1. Stability region for the classical third-order Adams–Bashforth method (Method I).

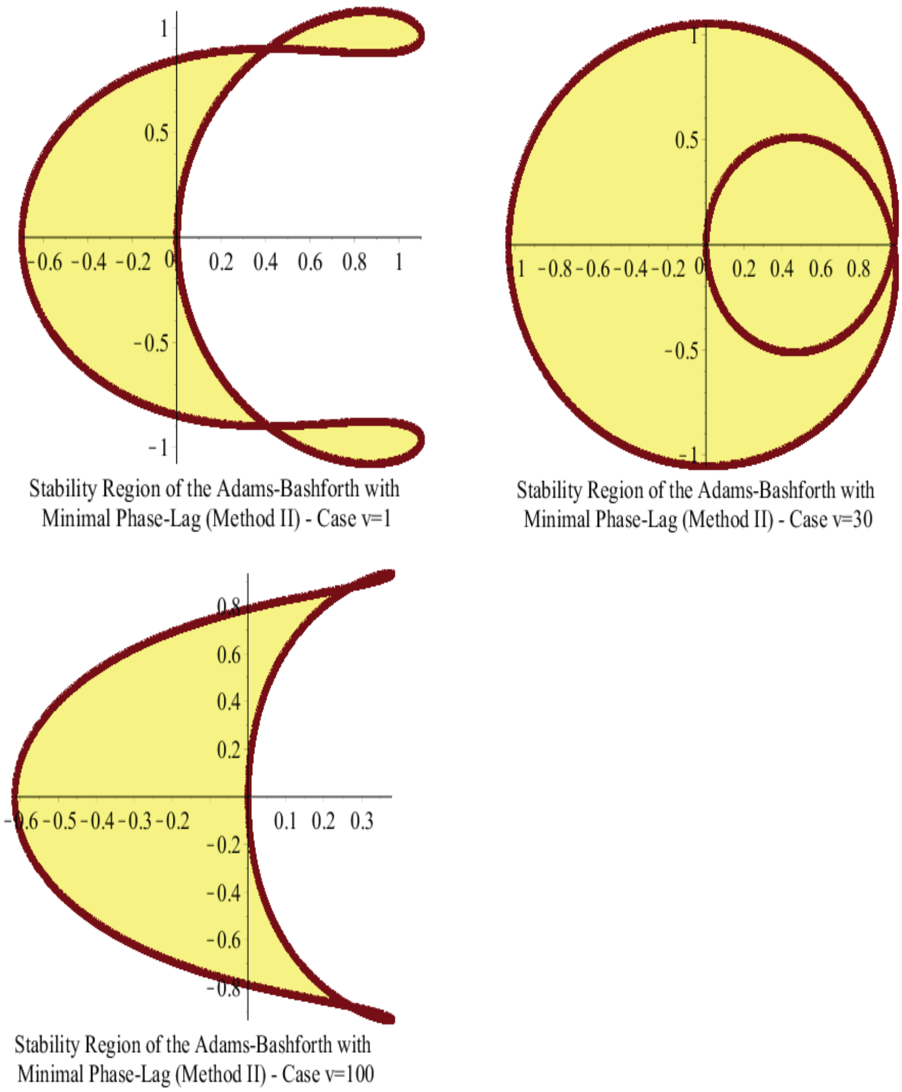

Figure 2. Stability region for the amplification-fitted Adams–Bashforth method (Method II).

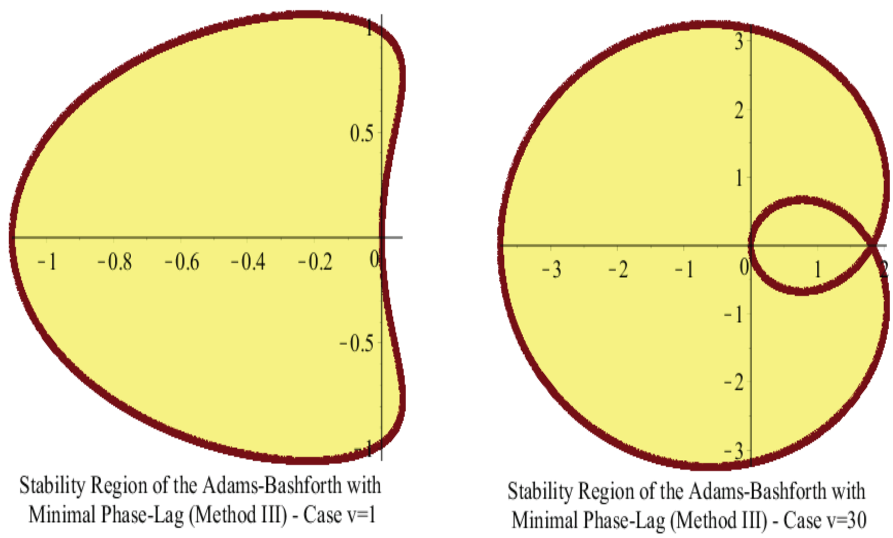

Figure 3. *Cont*.

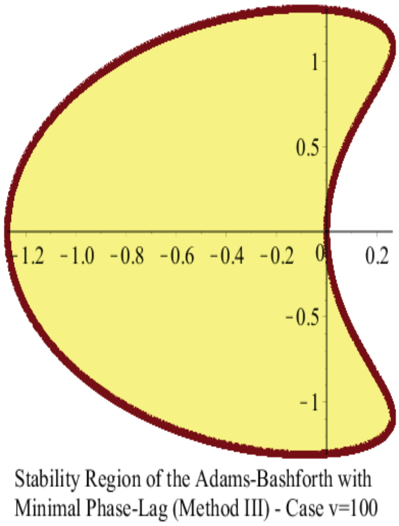

Figure 3. Stability region for the amplification-fitted Adams–Bashforth method (Method III).

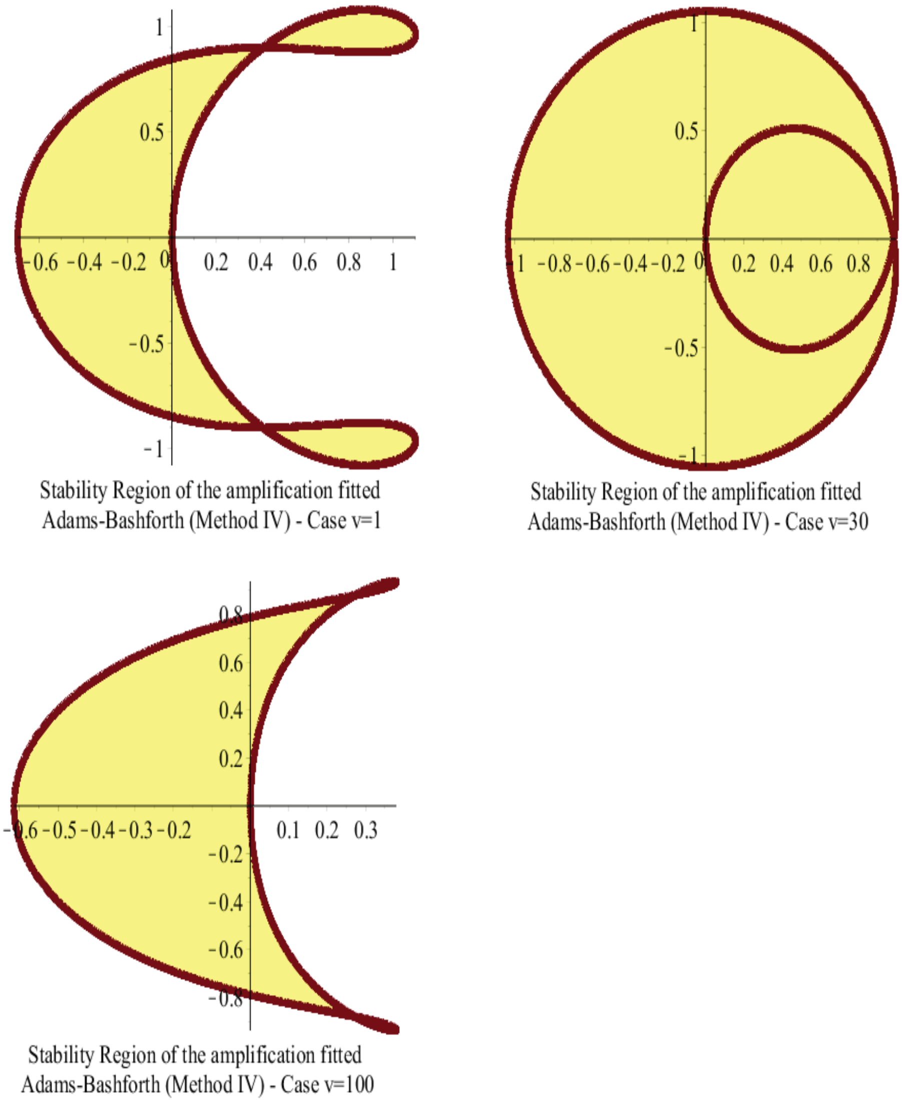

Figure 4. Stability region for the amplification-fitted Adams–Bashforth method (Method IV).

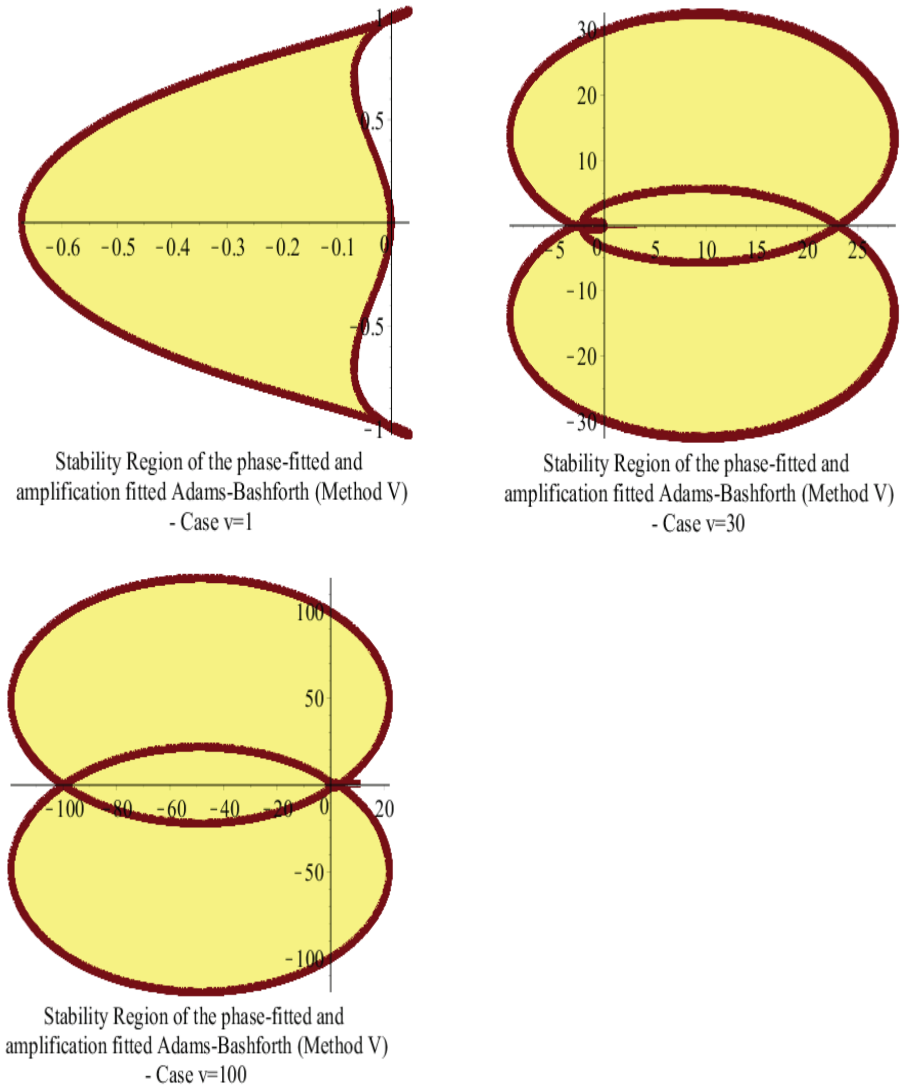

Figure 5. Stability region for the phase-fitted and amplification-fitted Adams–Bashforth method (Method V).

5. Numerical Results

5.1. Problem of Stiefel and Bettis

We consider the following almost periodic orbit problem studied by Stiefel and Bettis [122]:

$$\begin{aligned} y_1''(x) &= -y_1(x) + 0.001\cos(x), \quad y_1(0) = 1, \quad y_1'(0) = 0 \\ y_2''(x) &= -y_2(x) + 0.001\sin(x), \quad y_2(0) = 0, \quad y_2'(0) = 0.9995 \end{aligned} \tag{65}$$

The exact solution is

$$\begin{aligned} y_1(x) &= \cos(x) + 0.0005x\sin(x), \\ y_2(x) &= \sin(x) - 0.0005x\cos(x). \end{aligned} \tag{66}$$

For this problem, we use $\omega = 1$.

Equation (65) is solved numerically for $0 \leq x \leq 100{,}000$ using the following methods:

- The classical Adams–Bashforth method of the third order, which is mentioned as **Comp. Meth. I.**
- The classical Adams–Bashforth method of the fifth order, which is mentioned as **Comp. Meth. II.**
- The Runge–Kutta Dormand and Prince fourth-order method [54], which is mentioned as **Comp. Meth. III.**
- The Runge–Kutta Dormand and Prince fifth-order method [54], which is mentioned as **Comp. Meth. IV**.
- The Runge–Kutta Fehlberg fourth-order method [123], which is mentioned as **Comp. Meth. V**.
- The Runge–Kutta Fehlberg fifth-order method [123], which is mentioned as **Comp. Meth. VI**.
- The Runge–Kutta Cash and Karp fifth-order method [124], which is mentioned as **Comp. Meth. VII**.
- The Adams–Bashforth method with minimal phase lag (1st Case) which is developed in Section 3.2.2, which is mentioned as **Comp. Meth. VIII**.
- The Adams–Bashforth method with minimal phase lag (2nd Case) which is developed in Section 3.2.3, which is mentioned as **Comp. Meth. IX**.
- The Adams–Bashforth amplification fitted method which is developed in Section 3.3, which is mentioned as **Comp. Meth. X**.
- The Adams–Bashforth phase-fitted and amplification-fitted method, which is developed in Section 3.4, which is mentioned as **Comp. Meth. XI**.

In Figure 6, we present the maximum absolute error of the solution achieved by each of the above-mentioned numerical methods for the problem of Stiefel and Bettis [122].

From Figure 6, we can observe the following:

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X give approximately the same results
- Comp. Meth. IV gives more accurate results than the Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods.
- Comp. Meth. VII gives more accurate results than the Comp. Meth. IV method.
- Comp. Meth. II gives more accurate results than the Comp. Meth. VII method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. VI gives more accurate results than the Comp. Meth. V method.
- Comp. Meth. III gives better results than the Comp. Meth. VI method for the most step sizes, but for small step sizes, it gives approximately the same results as Comp. Meth. VI.
- Comp. Meth. IX gives mixed results. For big step sizes, it gives better results than Comp. Meth. III. For middle step sizes, it gives better results than Comp. Meth. VII but worse results than Comp. Meth. III, Comp. Meth. V, and Comp. Meth. VI. For small step sizes, it gives better results than Comp. Meth. IV but worse results than Comp. Meth. VII.
- Finally, Comp. Meth. XI, gives the most accurate results.

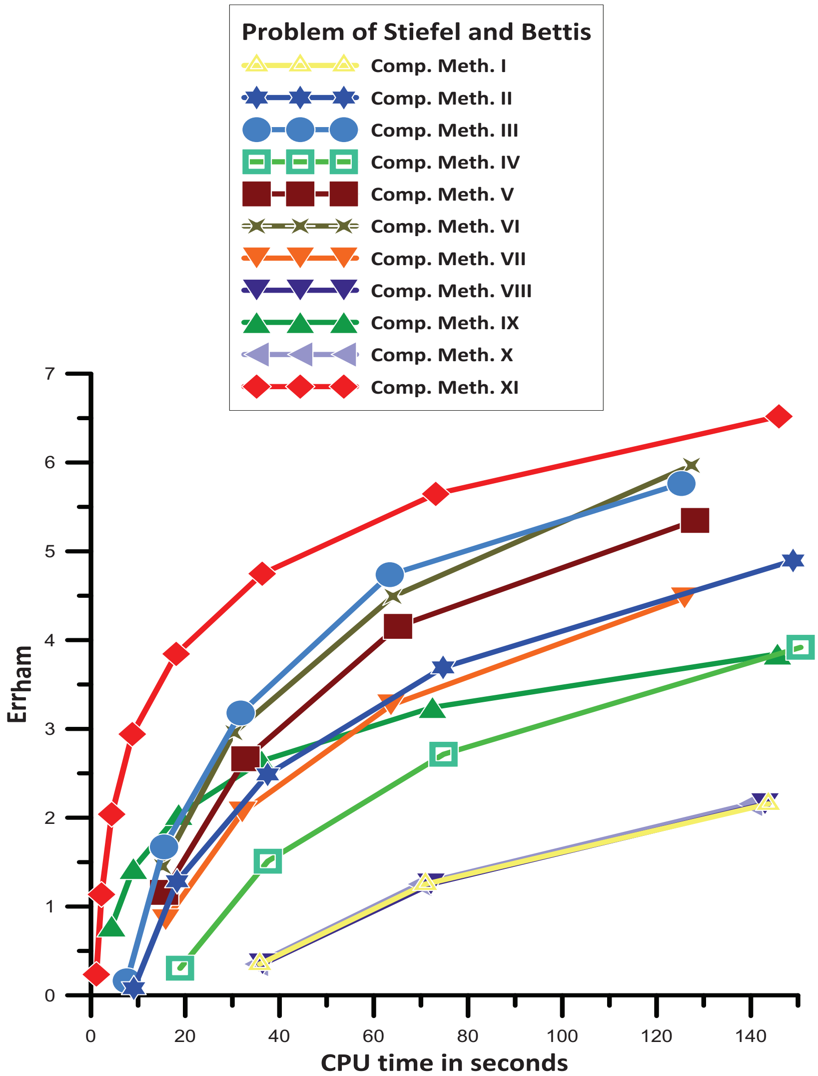

Figure 6. Numerical results for the problem of Stiefel and Bettis [122].

5.2. Problem of Franco et al. [125]

We consider the following inhomogeneous linear problem studied by Franco et al. [125]:

$$\begin{aligned} y_1''(x) &= -\frac{1}{2}\left(\mu^2+1\right)y_1(x)-\frac{1}{2}\left(\mu^2-1\right)y_2(x), \quad y_1(0)=1, \quad y_1'(0)=1 \\ y_2''(x) &= -\frac{1}{2}\left(\mu^2-1\right)y_1(x)-\frac{1}{2}\left(\mu^2+1\right)y_2(x), \quad y_2(0)=-1, \quad y_2'(0)=-1 \end{aligned} \tag{67}$$

The exact solution is

$$\begin{aligned} y_1(x) &= \cos(x)+\sin(x), \\ y_2(x) &= -\cos(x)-\sin(x). \end{aligned} \tag{68}$$

where $\mu = 10^4$. For this problem, we use $\omega = 1$.

The system of Equation (67) is solved numerically for $0 \le x \le 100{,}000$ using the methods presented in Section 5.1.

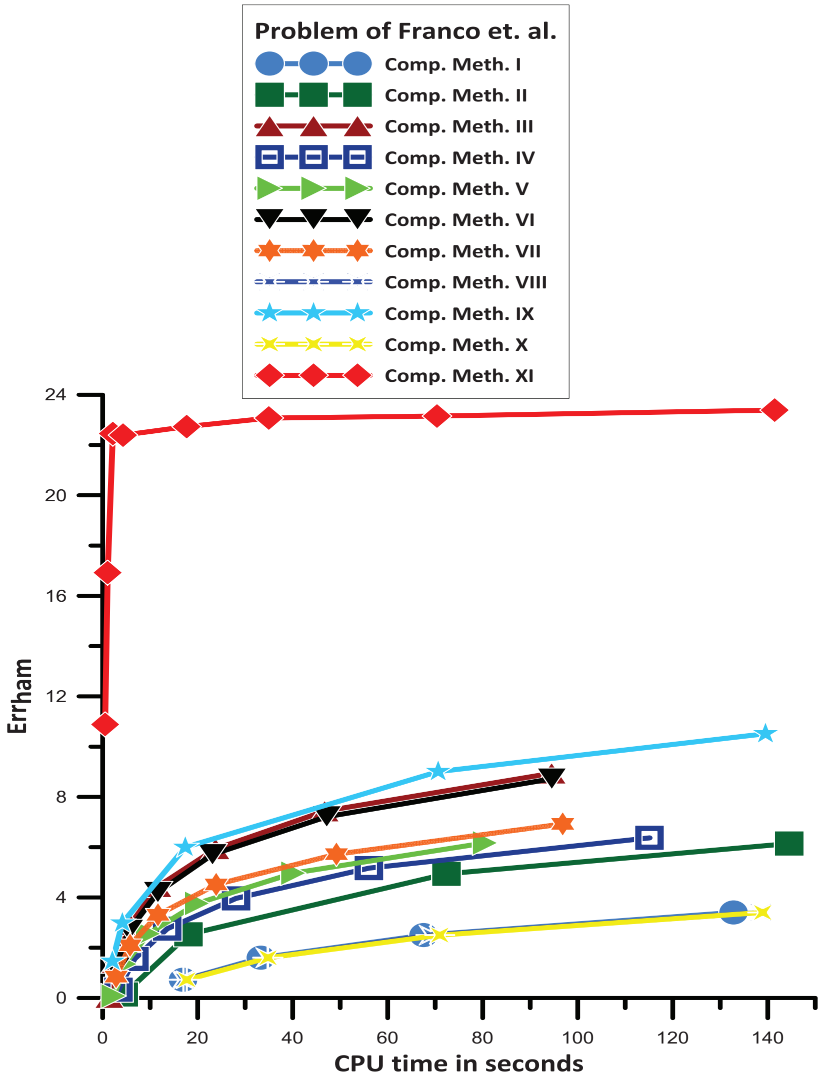

Figure 7. Numerical results for the problem of Franco et al. [125].

From the Figure 7, we can observe the following:

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X give approximately the same results
- Comp. Meth. II gives more accurate results than the Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods.
- Comp. Meth. IV gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. IV method.
- Comp. Meth. VII gives more accurate results than the Comp. Meth. V method.
- Comp. Meth. III gives more accurate results than the Comp. Meth. VII method.
- Comp. Meth. VI gives results with the same accuracy as the Comp. Meth. III method.
- Comp. Meth. IX gives better results than the Comp. Meth. VI and Comp. Meth. III methods.
- Finally, Comp. Meth. XI gives the most accurate results.

5.3. Problem of Franco and Palacios [126]

We consider the following problem studied by Franco and Palacios [126]:

$$\begin{aligned} y_1''(x) &= -y_1(x) + \varepsilon \cos(\vartheta x), \quad y_1(0) = 1, \quad y_1'(0) = 0 \\ y_2''(x) &= -y_2(x) + \varepsilon \sin(\vartheta x), \quad y_2(0) = 0, \quad y_2'(0) = 1 \end{aligned} \tag{69}$$

The exact solution is

$$y_1(x) = \frac{1-\varepsilon-\vartheta^2}{1-\vartheta^2}\cos(x) + \frac{\varepsilon}{1-\vartheta^2}\cos(\vartheta\, x),$$
$$y_2(x) = \frac{1-\varepsilon\,\vartheta-\vartheta^2}{1-\vartheta^2}\sin(x) + \frac{\varepsilon}{1-\vartheta^2}\sin(\vartheta\, x). \tag{70}$$

where $\varepsilon = 0.001$ and $\vartheta = 0.01$. For this problem, we use $\omega = \max\Big(1, |\vartheta|\Big)$.

The system of Equation (69) is solved numerically for $0 \leq x \leq 100{,}000$ using the methods presented in Section 5.1.

From Figure 8, we can observe the following:

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X give approximately the same results.
- Comp. Meth. IV gives more accurate results than the Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods.
- Comp. Meth. II gives more accurate results than the Comp. Meth. IV method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. VII gives more accurate results than the Comp. Meth. V method.
- Comp. Meth. III gives more accurate results than the Comp. Meth. VII method.
- Comp. Meth. VI gives results with the same accuracy as the Comp. Meth. III method.
- Comp. Meth. IX gives better results than the Comp. Meth. VI and Comp. Meth. III methods.
- Finally, Comp. Meth. XI gives the most accurate results.

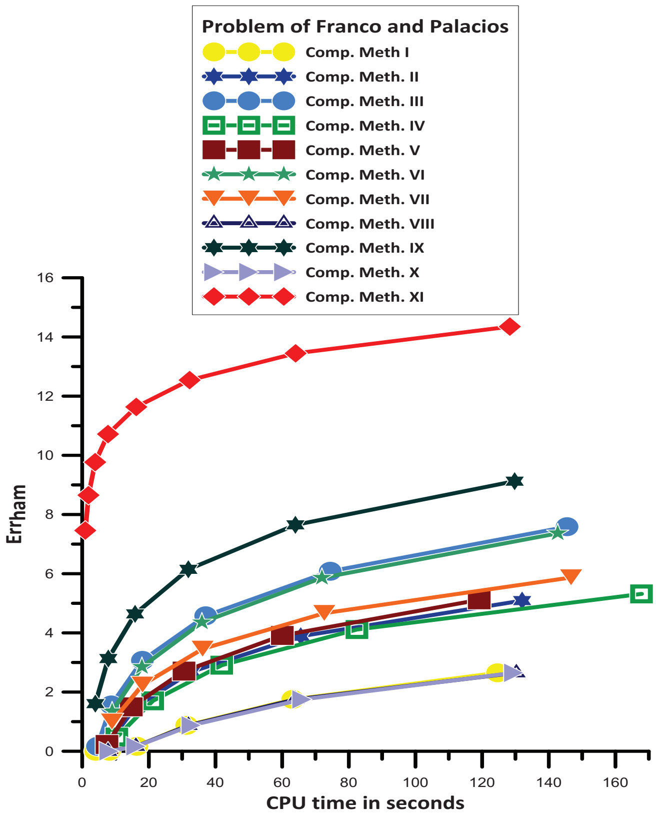

Figure 8. Numerical results for the problem of Franco and Palacios [126].

5.4. A Nonlinear Orbital Problem [127]

We consider the following nonlinear orbital problem studied by Simos in [127]:

$$\begin{aligned} y_1''(x) &= -\varphi^2\, y_1(x) + \frac{2\, y_1(x)\, y_2(x) - \sin(2\,\varphi\, x)}{\left(y_1(x)^2 + y_2(x)^2\right)^{\frac{3}{2}}}, \quad y_1(0) = 1, \quad y_1'(0) = 0 \\ y_2''(x) &= -\varphi^2\, y_2(x) + \frac{y_1(x)^2 - y_2(x)^2 - \cos(2\,\varphi\, x)}{\left(y_1(x)^2 + y_2(x)^2\right)^{\frac{3}{2}}}, \quad y_2(0) = 0, \quad y_2'(0) = \varphi \end{aligned} \quad (71)$$

The exact solution is

$$y_1(x) = \cos(\varphi\, x), \quad y_2(x) = \sin(\varphi\, x). \quad (72)$$

where $\varphi = 10$. For this problem, we use $\omega = 10$.

The system of Equation (71) is solved numerically for $0 \le x \le 100{,}000$ using the methods presented in Section 5.1.

From Figure 9, we can observe the following:

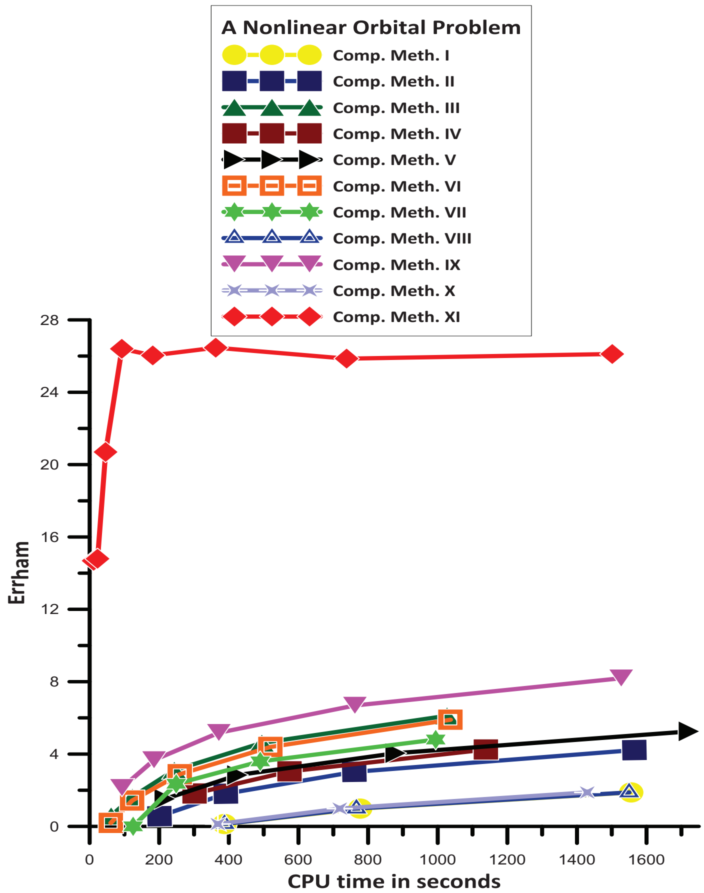

Figure 9. Numerical results for the nonlinear orbital problem of [127].

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X give approximately the same results.
- Comp. Meth. II gives more accurate results than the Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods.
- Comp. Meth. IV gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. IV method.
- Comp. Meth. VII gives more accurate results than the Comp. Meth. V method.

- Comp. Meth. VI gives results with the same accuracy as the Comp. Meth. VII method.
- Comp. Meth. III gives more accurate results than the Comp. Meth. VI method.
- Comp. Meth. IX gives better results than the Comp. Meth. VI and Comp. Meth. III method.
- Finally, Comp. Meth. XI gives the most accurate results.

5.5. Nonlinear Problem of Petzold [128]

We consider the following nonlinear problem studied by Petzold [128]:

$$\begin{aligned} y_1'(x) &= \lambda\, y_2(x), \quad y_1(0) = 1 \\ y_2'(x) &= -\lambda\, y_1(x) + \frac{\alpha}{\lambda}\, \sin(\lambda\, x), \quad y_2(0) = -\frac{\alpha}{2\,\lambda^2} \end{aligned} \tag{73}$$

The exact solution is

$$\begin{aligned} y_1(x) &= \left(1 - \frac{\alpha}{2\,\lambda}\, x\right)\, \cos(\lambda\, x), \\ y_2(x) &= -\left(1 - \frac{\alpha}{2\,\lambda}\, x\right)\, \sin(\lambda\, x) - \frac{\alpha}{2\,\lambda^2}\, \cos(\lambda\, x) \end{aligned} \tag{74}$$

where $\lambda = 1000$, $\alpha = 100$. For this problem, we use $\omega = 1000$.

The system of Equation (73) is solved numerically for $0 \leq x \leq 1000$ using the methods presented in Section 5.1.

From Figure 10, we can observe the following.

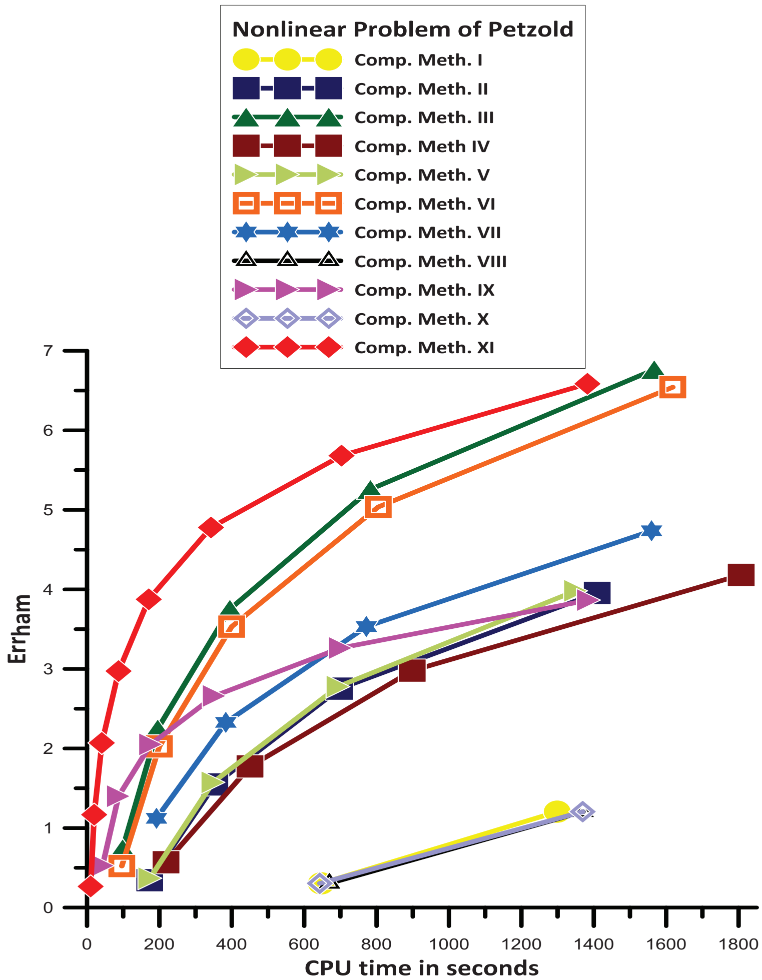

Figure 10. Numerical results for the nonlinear problem of [128].

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X give approximately the same results.
- Comp. Meth. IV gives more accurate results than the Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods.
- Comp. Meth. II gives more accurate results than the Comp. Meth. IV method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. VII gives more accurate results than the Comp. Meth. V method.
- Comp. Meth. VI gives results with the same accuracy as the Comp. Meth. VII method.
- Comp. Meth. III gives more accurate results than the Comp. Meth. VI method.
- Comp. Meth. IX gives mixed results. For big step sizes, it gives better results than Comp. Meth. III. For middle step sizes, it gives better results than Comp. Meth. VII but worse results than Comp. Meth. III and Comp. Meth. VI. For small step sizes, it gives better results than Comp. Meth. IV and Comp. Meth. V but worse results than Comp. Meth. VII.
- Finally, Comp. Meth. XI gives the most accurate results.

5.6. Two-Body Gravitational Problem

We consider the two-body gravitational problem

$$\begin{aligned} y_1''(x) &= -\frac{y_1(x)}{\left(y_1(x)^2+y_2(x)^2\right)^{\frac{3}{2}}}, \qquad y_1(0)=1, \quad y_1'(0)=0 \\ y_2''(x) &= -\frac{y_2(x)}{\left(y_1(x)^2+y_2(x)^2\right)^{\frac{3}{2}}}, \qquad y_2(0)=0, \quad y_2'(0)=1 \end{aligned} \tag{75}$$

The exact solution is

$$\begin{aligned} y_1(x) &= \cos(x), \\ y_2(x) &= \sin(x). \end{aligned} \tag{76}$$

For this problem, we use $\omega = \frac{1}{\left(y_1(x)^2+y_2(x)^2\right)^{\frac{3}{4}}}$.

The system of Equations (75) is solved numerically for $0 \leq x \leq 100{,}000$ using the methods presented in Section 5.1.

From Figure 11, we can observe the following:

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X are not convergent to the solution.
- Comp. Meth. II gives more accurate results than the Comp. Meth. VII method.
- Comp. Meth. VI gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. VI method.
- Comp. Meth. III gives more accurate results than the Comp. Meth. V method.
- Comp. Meth. IX gives results with the same accuracy as the Comp. Meth. III method.
- Comp. Meth. IV gives results with approximately the same accuracy as the results given by the Comp. Meth. IX method.
- Finally, Comp. Meth. XI gives the most accurate results.

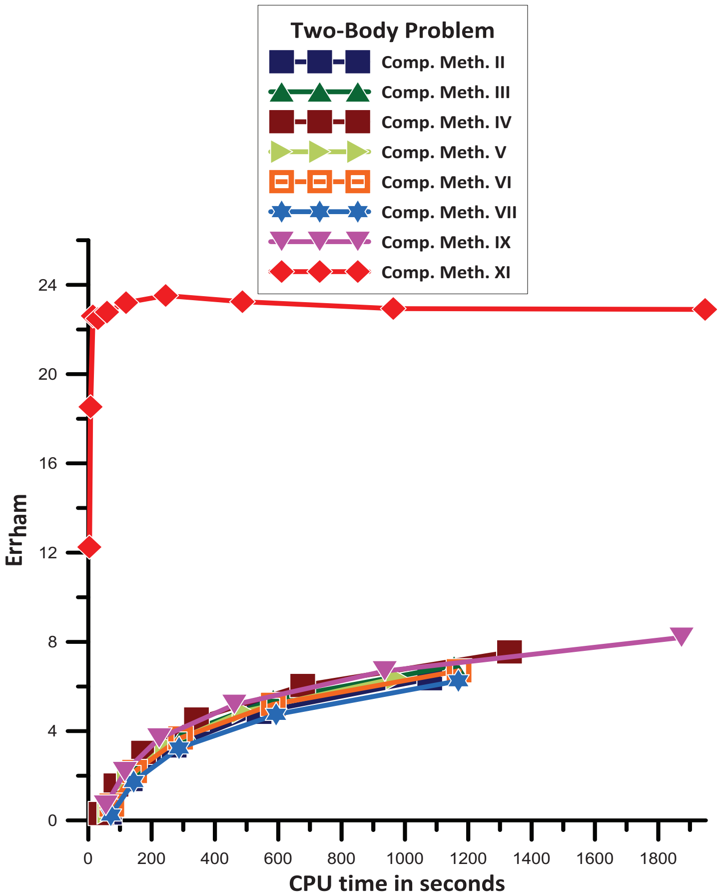

Figure 11. Numerical results for two-body gravitational problem (Kepler's plane problem).

5.7. Perturbed Two-Body Gravitational Problem

5.7.1. Case $\mu = 0.1$

We consider the perturbed two-body Kepler's problem

$$
\begin{aligned}
y_1''(x) &= -\frac{y_1(x)}{\left(y_1(x)^2 + y_2(x)^2\right)^{\frac{3}{2}}} - \mu\left(\mu + 2\right)\frac{y_1(x)}{\left(y_1(x)^2 + y_2(x)^2\right)^{\frac{5}{2}}}, \\
& \qquad\qquad\qquad\qquad y_1(0) = 1, \quad y_1'(0) = 0 \\
y_2''(x) &= -\frac{y_2(x)}{\left(y_1(x)^2 + y_2(x)^2\right)^{\frac{3}{2}}} - \mu\left(\mu + 2\right)\frac{y_2(x)}{\left(y_1(x)^2 + y_2(x)^2\right)^{\frac{5}{2}}}, \\
& \qquad\qquad\qquad\qquad y_2(0) = 0, \quad y_2'(0) = 1 + \mu
\end{aligned}
\tag{77}
$$

The exact solution is

$$
\begin{aligned}
y_1(x) &= \cos(x + \mu\, x), \\
y_2(x) &= \sin(x + \mu\, x).
\end{aligned}
\tag{78}
$$

For this problem, we use $\omega = \dfrac{\sqrt{1+\mu\left(\mu+2\right)}}{\left(y_1(x)^2+y_2(x)^2\right)^{\frac{3}{4}}}$.

The system of Equation (77) is solved numerically for $0 \leq x \leq 100{,}000$ with $\mu = 0.1$ and using the methods presented in Section 5.1. From Figure 11, it is obvious that the

Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods are not convergent to the solution.

From Figure 12, we can observe the following.

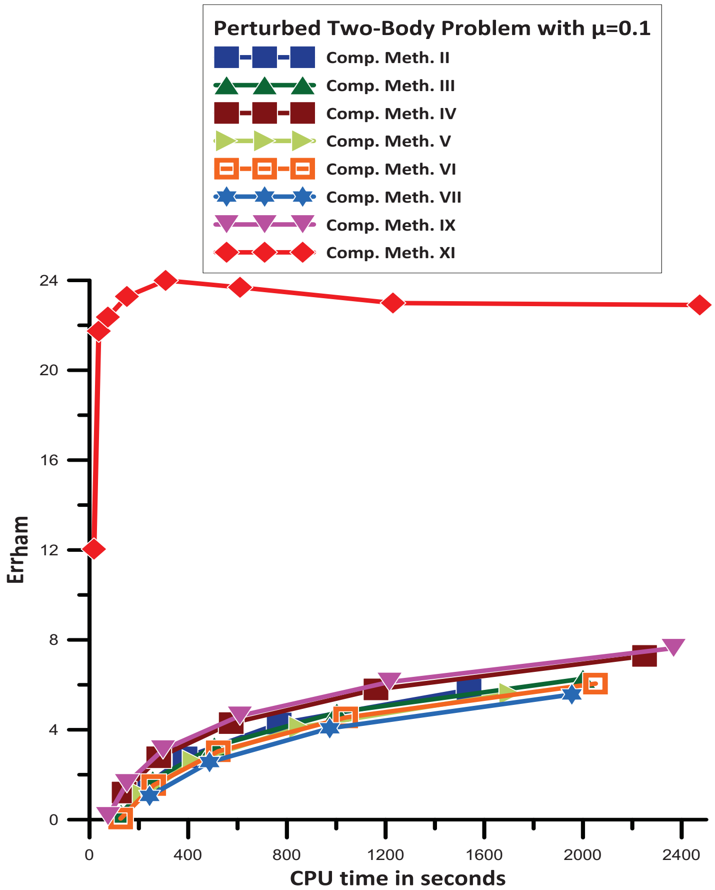

Figure 12. Numerical results for perturbed two-body gravitational problem (perturbed Kepler's problem) with $\mu = 0.1$.

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X are not convergent to the solution.
- Comp. Meth. VI gives more accurate results than the Comp. Meth. VII method.
- Comp. Meth. V gives more accurate results than the Comp. Meth. VI method.
- Comp. Meth. III gives more accurate results than the Comp. Meth. V method.
- Comp. Meth. II gives more accurate results than the Comp. Meth. III method.
- Comp. Meth. IV gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. IX gives more accurate results than the Comp. Meth. IV method.
- Finally, Comp. Meth. XI gives the most accurate results.

5.7.2. Case $\mu = 0.4$

The system of Equation (77) is solved numerically for $0 \leq x \leq 100{,}000$ with $\mu = 0.4$ and using the methods presented in Section 5.1. From Figure 12, it is obvious that the Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X methods are not convergent to the solution.

From Figure 13, we can observe the following:

- Comp. Meth. I, Comp. Meth. VIII, and Comp. Meth. X are not convergent to the solution.
- Comp. Meth. V gives more accurate results than the Comp. Meth. VII method.
- Comp. Meth. VI gives more accurate results than the Comp. Meth. V method.

- Comp. Meth. III gives more accurate results than the Comp. Meth. VI method.
- Comp. Meth. II gives more accurate results than the Comp. Meth. III method.
- Comp. Meth. IV gives more accurate results than the Comp. Meth. II method.
- Comp. Meth. IX gives more accurate results than the Comp. Meth. IV method.
- Finally, Comp. Meth. XI gives the most accurate results.

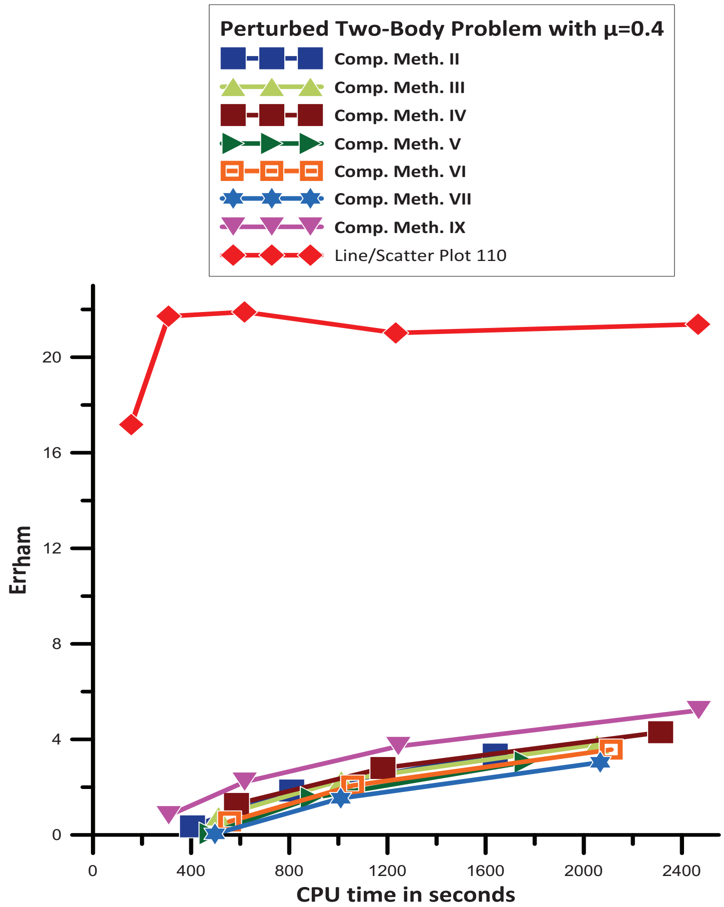

Figure 13. Numerical results for perturbed two-body gravitational problem (perturbed Kepler's problem) with $\mu = 0.4$.

For the above numerical illustrations, we have the following conclusions:

- The phase-fitted and amplification-fitted methods (Comp. Meth. XI) give the most efficient results for all the problems.
- The Adams–Bashforth method with minimal phase lag (1st Case) (Comp. Meth. VIII) and the Adams–Bashforth amplification-fitted method (Comp. Meth. X) give approximately the same results as the classical Adams–Bashforth method of the third order (Comp. Meth. I).
- The Adams–Bashforth method with minimal phase lag (2nd Case) (Comp. Meth. IX) gives the second most efficient results for most of the problems.

From the above conclusions, we can see that the methodologies presented in this paper which prepare the most efficient methods are as follows:

- The methodology which emphasizes the minimization of the phase lag (ignoring the algebraic order of the method), which is developed in Section 3.2.3;
- The methodology which emphasizes on the vanishing of the phase lag and the amplification error of the method, which is developed in Section 3.4.

It is easy to see that the efficiency of the frequency-dependent methods (like the newly introduced) is dependent on the choice of the parameter v. In many problems, this choice is easy to be defined from the model of the specific problem. For the cases that this is not easy, methodologies for the determination of the parameter v are introduced in the literature (see [129,130]).

Remark 2. *For the solution of systems of high-order ordinary differential equations using the above-mentioned newly introduced techniques, we note that there are well-known methods for reducing a system of high-order ordinary differential equations into a system of first-order differential equations, for example: variable substitution, the introduction of new variables, rewriting the system of higher-order ordinary differential equations as a system of first-order differential equations by introducing new variables for each derivative, etc. (see [131]).*

For the solution of systems of partial differential equations using the above-mentioned newly introduced techniques, we note that there are well-known methods for reducing a system of partial differential equations into a system of first-order differential equations, for example, the method of characteristics (see [132]).

6. Conclusions

In the present paper, we developed the theory of the phase lag and amplification error analysis for the multistep methods of the first-order initial-value problems. Based on the above developed theory, we presented several methodologies for the development of efficient methods for the multistep methods. More specifically, we developed methodologies for the following:

- Methodology for the minimization of the phase lag.
- Methodology for the development of an amplification-fitted method.
- Methodology for the development of a phase-fitted method.

Using the above-mentioned methodologies, we developed several multistep methods. We used as a basic method the Adams–Bashforth method of the third algebraic order.

The above produced methods were applied to several problems with oscillating solutions, in order to test their efficiency.

All calculations adhered to are carried out using a quadruple precision arithmetic data type.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Proof of Lemma 1. The phase lag order q and the phase lag constant c are given by its definition (10) as follows:

$$\Phi = v - \cos[\theta(v)] = c\, v^{q+1} + O\Big(v^{q+2}\Big) \tag{A1}$$

then, and since (see (10))

$$\theta(v) = v - \Phi \tag{A2}$$

using expansions of trigonometric functions, we achieve:

$$\begin{aligned} \cos[\theta(v)] &= \cos(v - \Phi) = \\ &= \cos(v)\ \cos(\Phi) + \sin(v)\ \sin(\Phi) \end{aligned} \tag{A3}$$

Using (A1), the relation (A3) gives:

$$\cos[\theta(v)] = \cos(v)\ \cos\Big[c\, v^{q+1} + O\Big(v^{q+2}\Big)\Big] + \sin(v)\ \sin\Big[c\, v^{q+1} + O\Big(v^{q+2}\Big)\Big] \tag{A4}$$

With the help of the Taylor series expansions of the functions $\cos\left[c\,v^{q+1} + O\left(v^{q+2}\right)\right]$, $\sin\left[c\,v^{q+1} + O\left(v^{q+2}\right)\right]$ and $\sin(v)$, relation (A4) gives:

$$\cos[\theta(v)] = \cos(v) + c\,v^{q+2} - \frac{1}{6}\,c\,v^{q+4} + \ldots = \cos(v) + c\,v^{q+2} + O\left(v^{q+4}\right) \tag{A5}$$

and the lemma is proved. □

Appendix B

Proof of Lemma 2. Using expansions of trigonometric functions, we achieve:

$$\begin{aligned} \sin[\theta(v)] &= \sin(v - \Phi) = \\ &= \sin(v)\,\cos(\Phi) - \cos(v)\,\sin(\Phi) \end{aligned} \tag{A6}$$

Using (A1), the relation (A6) gives:

$$\sin[\theta(v)] = \sin(v)\,\cos\left[c\,v^{q+1} + O\left(v^{q+2}\right)\right] - \cos(v)\,\sin\left[c\,v^{q+1} + O\left(v^{q+2}\right)\right] \tag{A7}$$

With the help of the Taylor series expansions of the functions $\cos\left[c\,v^{q+1} + O\left(v^{q+2}\right)\right]$, $\sin\left[c\,v^{q+1} + O\left(v^{q+2}\right)\right]$ and $\cos(v)$, relation (A7) gives:

$$\sin[\theta(v)] = \sin(v) - c\,v^{q+1} + \frac{1}{2}\,c\,v^{q+3} + \ldots = \sin(v) - c\,v^{q+1} + O\left(v^{q+3}\right) \tag{A8}$$

and the lemma is proved. □

Appendix C

Proof of Lemma 3 (Proof for the relation (15)).

We will prove first the relation (15). We can write (15) as follows:

- Let us examine the case $j = 1$ for the relation (15):

$$\begin{aligned} \cos[1\,\theta(v)] = \cos(1\,v) + c\,1^2\,v^{q+2} + O\left(v^{q+4}\right) \Longrightarrow \\ \cos[\theta(v)] = \cos(v) + c\,v^{q+2} + O\left(v^{q+4}\right) \end{aligned} \tag{A9}$$

  which is valid (see Lemma 1).
- Let us consider the relations (15) to be valid for $j = k$, i.e., let us consider that the relations:

$$\cos[k\,\theta(v)] = \cos(k\,v) + c\,k^2\,v^{q+2} + O\left(v^{q+4}\right) \tag{A10}$$

  are valid.
- We will prove that the relation (15) is valid for $j = k+1$. For $j = k+1$, we have:

$$\begin{aligned} \cos[(k+1)\,\theta(v)] = \cos\left[(k)\,\theta(v) + \theta(v)\right] = \\ \cos\left[(k)\,\theta(v)\right]\,\cos\left[\theta(v)\right] - \sin\left[(k)\,\theta(v)\right]\,\sin\left[\theta(v)\right] \end{aligned} \tag{A11}$$

  Taking into account the following:
  - $\cos[k\,\theta(v)] = \cos(k\,v) + c\,k^2\,v^{q+2} + O\left(v^{q+4}\right)$ (see (A10))
  - $\cos[\theta(v)] = \cos(v) + c\,v^{q+2} + O\left(v^{q+4}\right)$ (see (13))
  - $\sin[k\,\theta(v)] = \sin(k\,v) - c\,k\,v^{q+1} + O\left(v^{q+3}\right)$ (see (A14))
  - $\sin[\theta(v)] = \sin(v) - c\,v^{q+1} + O\left(v^{q+3}\right)$ (see (14))

  the relation (A11) becomes:

$$\begin{aligned}
\cos[(k+1)\,\theta(v)] &= \cos[(k+1)\,v] + c\,v^{q+2} + c\,k^2\,v^{q+2} \\
&-\frac{1}{2}\,c\,k^2\,v^{q+4} + \frac{1}{24}\,c\,k^4\,v^{q+6} - \frac{1}{2}\,c\,k^2\,v^{q+4} + \frac{1}{24}\,c\,k^2\,v^{q+6} \\
&+c^2\,k^2\,v^{2\,q+4} + \ldots + c\,k\,v^{q+2} - \frac{1}{6}\,c\,k^3\,v^{q+4} + \frac{1}{120}c\,k^5\,v^{q+6} \\
&+c\,k\,v^{q+2} - \frac{1}{6}\,c\,k\,v^{q+4} + \frac{1}{120}c\,k\,v^{q+6} + c\,k\,v^{2\,q+2} + \cdots \Longrightarrow \\
\cos[(k+1)\,\theta(v)] &= \cos[(k+1)\,v] + c\,v^{q+2} + 2\,c\,k\,v^{q+2} + c\,k^2\,v^{q+2} \\
&-\frac{1}{6}\,c\,k\,v^{q+4}\Big(k^2 + 6\,k + 1\Big) + \frac{1}{120}c\;k\,\Big(k^4 + 5\,k^3 + 5\,k + 1\Big)\,v^{q+6} + \ldots \Longrightarrow \\
\cos[(k+1)\,\theta(v)] &= \cos[(k+1)\,v] + c\,(k+1)^2\,v^{q+2} - \frac{1}{6}\,c\,k\,v^{q+4}\Big(k^2 + 6\,k + 1\Big) + \ldots
\end{aligned} \tag{A12}$$

and the relation (15) is proved for $j = k+1$. □

Proof of Lemma 3 (Proof for the relation (16)).

We will prove now the relation (16). We can write (16) as follows:

- Let us examine the case $j = 1$ for the relation (16):

$$\begin{aligned}
\sin[1\,\theta(v)] &= \sin(1\,v) - c\,1\,v^{q+1} + O\Big(v^{q+3}\Big) \Longrightarrow \\
\sin[\theta(v)] &= \sin(v) - c\,v^{q+1} + O\Big(v^{q+3}\Big)
\end{aligned} \tag{A13}$$

  which is valid (see Lemma 2).

- Let us consider that relation (16) is valid for $j = k$, i.e., let us consider that the relation:

$$\sin[k\,\theta(v)] = \sin(k\,v) - c\,k\,v^{q+1} + O\Big(v^{q+3}\Big) \tag{A14}$$

  is valid

- we will prove that relation (16) is valid for $j = k+1$. For $j = k+1$, we have:

$$\begin{aligned}
\sin[(k+1)\,\theta(v)] &= \sin\Big[(k)\,\theta(v) + \theta(v)\Big] = \\
\sin\Big[(k)\,\theta(v)\Big]\;\cos\Big[\theta(v)\Big] &+ \cos\Big[(k)\,\theta(v)\Big]\;\sin\Big[\theta(v)\Big]
\end{aligned} \tag{A15}$$

  We take into account the following:

  - $\sin[k\,\theta(v)] = \sin(k\,v) - c\,k\,v^{q+1} + O\big(v^{q+3}\big)$ (see (A14))
  - $\cos[\theta(v)] = cos(v) + c\,v^{q+2} + O\big(v^{q+4}\big)$ (see (13)
  - $\cos[k\,\theta(v)] = \cos(k\,v) + c\,k^2\,v^{q+2} + O\big(v^{q+4}\big)$ (see (A10))
  - $\sin[\theta(v)] = sin(v) - c\,v^{q+1} + O\big(v^{q+3}\big)$ (see (14))

  and relation (A15) becomes:

$$\begin{aligned}
\sin[(k+1)\,\theta(v)] &= \sin[(k+1)\,v] - c\,k\,v^{q+3} - \frac{1}{6}c\,k^3\,v^{q+5} \\
&+\frac{1}{120}\,c\,k^5\,v^{q+7} - \ldots - c\,v^{q+1} + \frac{1}{2}\,c\,k^2\,v^{q+3} - \frac{1}{24}\,c\,k^4\,v^{q+5} + \ldots \\
&+c\,k^2\,v^{q+3} - \frac{1}{6}\,c\,k^2\,v^{q+5} + \frac{1}{120}\,c\,k^2\,v^{q+7} - \ldots \\
&-c\,k\,v^{q+1} + \frac{1}{2}\,c\,k\,v^{q+3} - \frac{1}{24}c\,k\,v^{q+5} - c\,(k+1)\,v^{2\,q+3} + \cdots \Longrightarrow \\
\sin[(k+1)\,\theta(v)] &= \sin[(k+1)\,v] - c\,(k+1)\,v^{q+1} \\
&+\frac{1}{2}\,c\,k\,v^{q+3}\Big(3\,k^2 + 1\Big) - \frac{1}{24}\,c\,k\,v^{q+5}\,\Big(k^3 + 4\,k^2 + 4\,k + 1\Big) + \ldots
\end{aligned} \tag{A16}$$

and relation (16) is proved for $j = k + 1$. □

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