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Article

Manin Triples and Bialgebras of Left-Alia Algebras Associated with Invariant Theory

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Abstract: A left-Alia algebra is a vector space together with a bilinear map satisfying the symmetric Jacobi identity. Motivated by invariant theory, we first construct a class of left-Alia algebras induced by twisted derivations. Then, we introduce the notions of Manin triples and bialgebras of left-Alia algebras. Via specific matched pairs of left-Alia algebras, we figure out the equivalence between Manin triples and bialgebras of left-Alia algebras.

Keywords: left-Alia algebra; bialgebra; invariant theory; Manin triple; matched pair; representation

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1. Introduction and Main Statements

1.1. Introduction

Let G be a finite group and $\mathbb{K}$ an algebraic closed field of characteristic zero. Suppose that V is an n-dimensional faithful representation of G and $S = \mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]$ is the coordinate ring of V.

The goal of invariant theory is to study the structures of the ring of invariants

$$S^G = \{f \in S : a \cdot f = f, \ \forall a \in G\},$$

in which the group action is extended from the representation of G (see Section 2.1 for more details). In particular, Hilbert proved that S^G is always a finite generated $\mathbb{K}$-algebra [1] and Chevalley [2], Shephard and Todd [3] proved that S^G is a polynomial algebra if and only if G is generated by pseudo-reflections (see Section 2.2 for precise definition).

Twisted derivations [4] (also named σ-derivations) play an important role in the study of deformations of Lie algebras. Motivated by the above Chevalley's Theorem, we apply pseudo-reflections to induce a class of twisted derivations on S (see Theorem 3 for more details). Based on twisted derivations on commutative associative algebras, we obtain a class of left-Alia (left anti-Lie-admissible) algebras [5], which appears in the study of a special class of algebras with a skew-symmetric identity of degree three. Furthermore, we construct Manin triples and bialgebras of left-Alia algebras. Via specific matched pairs of left-Alia algebras, we figure out the equivalence between Manin triples and bialgebras.

Throughout this paper, unless otherwise specified, all vector spaces are finite-dimensional over an algebraically closed field K of characteristic zero and all K-algebras are commutative and associative with the finite Krull dimension, although many results and notions remain valid in the infinite-dimensional case.

1.2. Left-Alia Algebras Associated with Invariant Theory

The notion of a left-Alia algebra was defined for the first time in the table in the Introduction of [5].

Definition 1 ([5])**.** *A **left-Alia algebra** (also named a 0-Alia algebra) is a vector space A together with a bilinear map $[\cdot,\cdot] : A \otimes A \to A$ satisfying the **symmetric Jacobi identity**:*

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = [[y,x],z] + [[z,y],x] + [[x,z],y],\ \forall x,y,z \in A. \tag{1}$$

There are some typical examples of left-Alia algebras. Firstly, when the bilinear map $[\cdot,\cdot]$ is skew-symmetric, $(A,[\cdot,\cdot])$ is a Lie algebra. By contrast, any commutative algebra is a left-Alia algebra and, in particular, a mock-Lie algebra [6] (also known as a Jacobi–Jordan algebra in [7]) with a symmetric bilinear map that satisfies the Jacobi identity is a left-Alia algebra. Secondly, the notion of an anti-pre-Lie algebra [8] was recently studied as a left-Alia algebra with an additional condition. Anti-pre-Lie algebras are the underlying algebra structures of nondegenerate commutative 2-cocycles [9] on Lie algebras and are characterized as Lie-admissible algebras whose negative multiplication operators compose representations of commutator Lie algebras. Condition (1) of the identities of an anti-pre-Lie algebra is just to guarantee $(A,[\cdot,\cdot])$ is a Lie-admissible algebra. Additionally, we also studied left-Alia algebras in terms of their relationships with Leibniz algebras [10] and Lie triple systems [11].

Let $(A,\cdot)$ be a commutative associative algebra and $R : A \to A$ a linear map on A. For brevity, the operation $\cdot$ will be omitted. A linear map $D : A \to A$ is called **a twisted derivation** with respect to an R (also named a σ-derivation in [4]) if D satisfies the twisted Leibniz rule:

$$D(fg) = D(f)g + R(f)D(g),\ f,g \in A. \tag{2}$$

Non-trivial examples of twisted derivations can be constructed in invariant theory. In particular, each pseudo-reflection R on a vector space V induces a twisted derivation D_R on the polynomial ring $\mathbb{K}[V]$ (see Section 2.2 for details).

Define

$$[f,g]_R = D(f)g - R(f)D(g).$$

We then obtain a class of left-Alia algebras in Theorem A.

Theorem A (Theorems 3 and 4)

(a) *For each twisted derivation D on A, $(A,[\cdot,\cdot]_R)$ is a left-Alia algebra.*
(b) *Each pseudo-reflection R on V induces a left-Alia algebra $(\mathbb{K}[V],[\cdot,\cdot]_R)$.*

This applies when $R = I$, $[f,g]_R$ is skew-symmetric and $(A,[\cdot,\cdot]_R)$ is a Lie algebra of the Witt type [12]. Moreover, Theorem A also provides a class of left-Alia algebras on polynomial rings from invariant theory. As a corollary of Theorem A, we see that when S^G is a polynomial algebra, each generator $g \in G$ corresponds to a left-Alia algebra $(S,[\cdot,\cdot]_{R_g})$. The collection of left-Alia algebras is also an interesting research object for further study.

In addition, if we define that $[f,h] = D_R(f)h - fD_R(h)$ on S, $(S,[\cdot,\cdot])$, then it is not a left-Alia algebra in general. However, when $[\cdot,\cdot]$ is restricted to V^*, we obtain a finite-dimensional Lie algebra, which induces a linear Poisson structure on V, and figure out the entrance to the study of twisted relative Poisson structures on graded algebras. See [13] for reference.

1.3. Manin Triples and Bialgebras of Left-Alia Algebras

A bialgebra structure is a vector space equipped with both an algebra structure and a coalgebra structure satisfying certain compatible conditions. Some well-known examples of such structures include Lie bialgebras [14,15], which are closely related to Poisson–Lie groups and play an important role in the infinitesimalization of quantum groups, and antisymmetric infinitesimal bialgebras [16–20] as equivalent structures of double constructions of Frobenius algebras which are widely applied in the 2d topological field and string theory [21,22]. Recently, the notion of anti-pre-Lie bialgebras was studied in [23], which serves as a preliminary to supply a reasonable bialgebra theory for transposed Poisson algebras [24]. The notions of mock-Lie bialgebras [25] and Leibniz bialgebras [26,27] were also introduced with different motivations. These bialgebras have a common property in that they can be equivalently characterized by Manin triples which correspond to nondegenerate invariant bilinear forms on the algebra structures. In this paper, we follow such a procedure to study left-Alia bialgebras.

To develop the bialgebra theory of left-Alia algebras, we first define a representation of a left-Alia algebra to be a triple (l, r, V), where V is a vector space and $l, r : A \to \mathrm{End}(V)$ are linear maps such that the following equation holds:

$$l([x, y])v - l([y, x])v = r(x)r(y)v - r(y)r(x)v + r(y)l(x)v - r(x)l(y)v, \ \forall x, y \in A, v \in V.$$

A representation (ρ, V) of a Lie algebra $(A, [\cdot, \cdot])$ renders representations $(\rho, -\rho, V)$ and $(\rho, 2\rho, V)$ of $(A, [\cdot, \cdot])$ as left-Alia algebras.

Furthermore, we introduce the notion of a quadratic left-Alia algebra, defined as a left-Alia algebra $(A, [\cdot, \cdot])$ equipped with a nondegenerate symmetric bilinear form $\mathcal{B}$ which is invariant in the sense that

$$\mathcal{B}([x, y], z) = \mathcal{B}(x, [z, y] - [y, z]), \ \forall x, y, z \in A.$$

A quadratic left-Alia algebra gives rise to the equivalence between the adjoint representation and the coadjoint representation.

Last, we introduce the notions of a matched pair (Definition 8) of left-Alia algebras, a Manin triple of left-Alia algebras (Definition 11) and a left-Alia bialgebra (Definition 13). Via specific matched pairs of left-Alia algebras, we figure out the equivalence between Manin triples and bialgebras in Theorem B.

Theorem B (Theorems 5 and 6) *Let $(A, [\cdot, \cdot]_A)$ be a left-Alia algebra. Suppose that there is a left-Alia algebra structure $(A^*, [\cdot, \cdot]_{A^*})$ on the dual space A^*, and $\delta : A \to A \otimes A$ is the linear dual of $[\cdot, \cdot]_{A^*}$. Then, the following conditions are equivalent:*

(a) *There is a Manin triple of left-Alia algebras $\big((A \oplus A^*, [\cdot, \cdot]_d, \mathcal{B}_d), A, A^*\big)$, where*

$$\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \ \forall x, y \in A, a^*, b^* \in A^*.$$

(b) *$(A, [\cdot, \cdot]_A, \delta)$ is a left-Alia bialgebra.*

Theorem B naturally leads to the study of Yang–Baxter equations and relative Rota–Baxter operators for left-Alia algebras [28].

2. Pseudo-Reflections and Twisted Deviations in Invariant Theory

2.1. Preliminary on Invariant Theory

Let G be a finite group and $\mathbb{K}$ an algebraic closed field of characteristic zero. Suppose that (ρ, V) is an n-dimensional faithful representation of G and its dual representation is denoted by (ρ, V^*). Let $S = \mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]$ be the coordinate ring of V. Define a G-action on S as

$$g \cdot \sum_{i_1, \ldots, i_n} k_{i_1, \ldots, i_n} x_1^{i_1} \ldots x_n^{i_n} := \sum_{i_1, \ldots, i_n} k_{i_1, \ldots, i_n} (\rho(g)x_1)^{i_1} \ldots (\rho(g)x_n)^{i_n}, \ \forall g \in G. \tag{3}$$

Define the ring of invariants as

$$S^G = \{f \in S : a \cdot f = f, \ \forall a \in G\}.$$

Theorem 1 ([1,29])**.** *(a)* $\;$ *S^G is a finitely generated $\mathbb{K}$-algebra.*
(b) $\;$ *S is a finitely generated S^G-module.*

Definition 2 ([29])**.** *A linear automorphism $R \in Aut(V)$ is called a **pseudo-reflection** if $R^m = I$ for some $m \in \mathbb{N}^*$ and $Im(I - R)$ is one-dimensional.*

In invariant theory, the following theorem gives the equivalent condition that S^G is a polynomial algebra:

Theorem 2 ([2,3]). *S^G is a polynomial algebra if and only if $G \cong \rho(G)$ is generated by pseudo-reflections.*

Then, we figure out the relation between a pseudo-reflection on V and a twisted deviation on $S = \mathbb{K}[V]$.

Lemma 1. *Let R be a pseudo-reflection on V. Then, R induces a pseudo-reflection on V^* (also denoted by R).*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of V such that $W = \mathrm{Span}\{e_1, \dots, e_{n-1}\}$ is fixed by R. By $R^m = \begin{pmatrix} 1 & & & & a_1 \\ & 1 & & & a_2 \\ & & \ddots & & \vdots \\ & & & 1 & a_{n-1} \\ & & & & a_n \end{pmatrix}^m = I$, we see that R is given by the diagonal matrix $diag(1, \dots, 1, \omega)$, where $\omega \neq 1$ is an m-th primitive root over $\mathbb{K}$. Denote $\{x_1, \dots, x_n\}$, the dual basis of V^*, such that $x_i(e_j) = \delta_{ij}$. Thus, the induced automorphism on V^*, defined by $R(x_i)(e_j) := x_i(R^{-1}(e_j))$, satisfies that $R(x_i) = x_i$, $1 \leq i \leq n-1$ and $R(x_n) = (1/\omega)x_n$. Therefore, R is a pseudo-reflection on V^*. $\square$

2.2. Pseudo-Reflections Induced by Twisted Deviations

Let $(A, \cdot)$ be a commutative associative algebra and $R : A \to A$ a linear map on A. Recall from [4] the definition of a twisted derivation (also named a σ-derivation).

Definition 3. *A linear map $D : A \to A$ is called a **twisted derivation** with respect to R if D satisfies the twisted Leibniz rule:*

$$D(fg) = D(f)g + R(f)D(g), \ f, g \in A. \tag{4}$$

Remark 1. *When $R = I$, D is a derivation on A.*

Recall from Section 2.1 that for a fixed non-zero $v_R \in Im(I - R) \subset V$, there exists a $\Delta_R \in V^*$ such that

$$(I - R)v = \Delta_R(v)v_R, \ \ \forall v \in V. \tag{5}$$

By Lemma 1, for a fixed non-zero $l_R \in Im(I - R) \subset V^*$, there also exists a $\Delta_R \in V$ such that

$$(I - R)x = \Delta_R(x)l_R, \ \ \forall x \in V^*. \tag{6}$$

Also, denote $R : S \to S$ as an extension of $R \in \mathrm{Aut}(V)$ satisfying

$$R(k_1 f + k_2 h) = k_1 R(f) + k_2 R(h) \text{ and } R(fh) = R(f)R(h).$$

Theorem 3. *For each $f \in S$, there exists a twisted derivation $D_R : S \to S$ with respect to R such that*

$$R(f) = f - D_R(f)l_R. \tag{7}$$

Proof. First, we prove that $R(f)$ can be uniquely written as $R(f) = f - D_R(f)l_R$ for some $D_R : S \to S$. It follows from (6) that, for $1 \leq a_i \leq n$,

$$\begin{aligned} R(x_{a_1} \dots x_{a_k}) &= (Rx_{a_1}) \dots (Rx_{a_k}) \\ &= (x_{a_1} - \Delta_R(x_{a_1})l_R) \dots (x_{a_k} - \Delta_R(x_{a_k})l_R) \end{aligned}$$

which can be expressed as

$$x_{a_1} \dots x_{a_k} - D_R(x_{a_1} \dots x_{a_k})l_R,$$

where D_R maps the monomial to a polynomial in S. As a consequence, $R(f)$ can be written as

$$R(f) = f - D_R(f)l_R, \tag{8}$$

where $D_R : S \to S$ is a linear map. Then, we prove that D_R is a twisted derivation on S with respect to R. On the one hand,

$$R(fh) = fh - D_R(fh)l_R.$$

On the other hand,

$$\begin{aligned} R(f)R(h) = R(f)(h - D_R(h)l_R) &= (f - D_R(f)l_R)h - R(f)D_R(h)l_R \\ &= fh - (D_R(f)h + R(f)D_R(h))l_R. \end{aligned}$$

Therefore, $R(fh) = D_R(f)h + R(f)D_R(h)$. □

Remark 2. *When restricting D_R to V^*, $D_R = \Delta_R$ on V^*. When restricting $(I - D_R)$ to V^*, $(I - D_R)$ is a pseudo-reflection on V^*.*

3. Left-Alia Algebras and Their Representations

3.1. Left-Alia Algebras and Twisted Derivations

Definition 4 ([5])**.** *A **left-Alia algebra** is a vector space A together with a bilinear map $[\cdot,\cdot] : A \times A \to A$ satisfying the symmetric Jacobi property:*

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = [[y,x],z] + [[z,y],x] + [[x,z],y], \ \forall x,y,z \in A.$$

Remark 3. *A left-Alia algebra $(A,[\cdot,\cdot])$ is a Lie algebra if and only if the bilinear map $[\cdot,\cdot]$ is skew-symmetric. On the other hand, any commutative algebra $(A,[\cdot,\cdot])$ in the sense that $[\cdot,\cdot]$ is symmetric is a left-Alia algebra.*

We can obtain a class of left-Alia algebras from twisted derivations.

Lemma 2. *Let $D : A \to A$ be a twisted derivation of the commutative associative algebra $(A,\cdot)$. Then, D satisfies*

$$xD(y) - D(x)y = R(x)D(y) - D(x)R(y), \quad \forall\, x,y \in A. \tag{9}$$

Proof. By the commutative property of $(A,\cdot)$ and (4), we have

$$D(xy) - D(yx) = D(x)y + R(x)D(y) - D(y)x - R(y)D(x) = 0.$$

Therefore, (9) holds. □

Theorem 4. *Let $(A,\cdot)$ be a commutative associative algebra and D be a twisted derivation. For all $x,y \in A$, define the bilinear map $[\cdot,\cdot]_R : A \times A \to A$ by*

$$[x,y]_R := [x,y]_D = xD(y) - R(y)D(x). \tag{10}$$

Then, $(A,[\cdot,\cdot]_R)$ is a left-Alia algebra.

Proof. Let $x,y,z \in A$. By (10), we have

$$\begin{aligned} [x,y]_R - [y,x]_R &= xD(y) - R(y)D(x) - yD(x) + R(x)D(y) \\ &\overset{(9)}{=} 2(xD(y) - yD(x)), \end{aligned}$$

and

$$D(xD(y) - yD(x)) \overset{(4)}{=} D(x)D(y) - R(y)D^2(x) - D(y)D(x) + R(x)D^2(y)$$
$$= R(x)D^2(y) - R(y)\, D^2(x).$$

Furthermore,

$$\begin{aligned} &\circlearrowleft_{x,y,z} [[x,y]_R - [y,x]_R, z]_R \\ =& \circlearrowleft_{x,y,z} 2[xD(y) - yD(x), z]_R \\ \overset{(10)}{=}& \circlearrowleft_{x,y,z} 2\big(xD(y)D(z) - yD(x)D(z) - R(x)D^2(y)R(z) + R(y)D^2(x)R(z)\big) \\ =& 0. \end{aligned}$$

Therefore, the conclusion holds. □

Remark 4. *Theorem 4 can also be verified in the following way. Let $(A, \cdot)$ be a commutative associative algebra with linear maps $f, g : A \to A$. By [5], there is a left-Alia algebra $(A, [\cdot,\cdot])$ given by*

$$[x, y] = x \cdot f(y) + g(x \cdot y),\ \forall x, y \in A, \tag{11}$$

*which is called a **special left-Alia algebra** with respect to $(A, \cdot, f, g)$. If D is a twisted derivation of $(A, \cdot)$ with respect to R, then we see that $(A, [\cdot,\cdot]_R)$ satisfies (11) for*

$$f = 2D, g = -D.$$

Hence, $(A, [\cdot,\cdot]_R)$ is left-Alia.

3.2. Examples of Left-Alia Algebras

Example 1. *Let R be a reflection defined by $R(x_1) = x_2, R(x_2) = x_1, R(x_3) = x_3$ on three-dimensional vector space V^* with a basis $\{x_1, x_2, x_3\}$. On the coordinate ring $S = \mathbb{K}[x_1, x_2, x_3]$ of V, R can be also denoted an extension of R satisfying $R(fg) = R(f)R(g)$ and $R(k_1 f + k_2 h) = k_1 R(f) + k_2 R(h)$. Let D be the twisted derivation on S induced by the reflection R. It follows from Theorem 3 that $R(f) = f - D(f)(x_1 - x_2)$. Take two polynomials, $f = \sum_i k_i f_i, g = \sum_j h_j g_j, k_i, h_j \in \mathbb{K}$, in S, where f_i, g_j are monomials, $f_i = x_1^{n_{i,1}} x_2^{n_{i,2}} x_3^{n_{i,3}}$, $g_j = x_1^{m_{j,1}} x_2^{m_{j,2}} x_3^{m_{j,3}}$. We have*

$$\begin{aligned} D(x_1) =& 1, D(x_2) = -1, D(x_3) = 0. \\ D(x_1^{n_1}) =& x_1^{n_1-1} + x_1^{n_1-2}x_2 + \ldots + x_2^{n_1-1}. \\ D(x_2^{n_2}) =& -x_1^{n_2-1} - x_1^{n_2-1}x_2 - \ldots - x_2^{n_2-1}. \\ D(x_3^{n_3}) =& 0. \\ D(\sum_i k_i x_1^{n_{i,1}} x_2^{n_{i,2}} x_3^{n_{i,3}}) =& \sum_i k_i (D(x_1^{n_{i,1}} x_2^{n_{i,2}}) x_3^{n_{i,3}} + R(x_1^{n_{i,1}} x_2^{n_{i,2}}) D(x_3^{n_{i,3}})) \\ =& \sum_i k_i (x_1^{n_{i,1}} D(x_2^{n_{i,2}}) + R(x_2^{n_{i,2}}) D(x_1^{n_{i,1}})) x_3^{n_{i,3}} \\ =& \sum_i k_i (x_1^{n_{i,1}+n_{i,2}-1} + x_1^{n_{i,1}+n_{i,2}-2} x_2 + \ldots + x_1^{n_{i,2}} x_2^{n_{i,1}-1} \\ & - x_1^{n_{i,1}+n_{i,2}-1} - \ldots - x_1^{n_1} x_2^{n_2-1}) x_3^{n_{i,3}}. \end{aligned}$$

Let $[\cdot,\cdot]_R : S \times S \to S$ be the bilinear map defined in Theorem 4. Then,

$$\begin{aligned}
[f,g]_R =&\sum_{i,j} k_i h_j [f_i, g_j]_R \\
=&\sum_{i,j} k_i h_j (f_i D(g_j) - R(g_j) D(f_i)) \\
=&\sum_{i,j} k_i h_j (x_1^{n_{i,1}+m_{j,1}+m_{j,2}-1} x_2^{n_{i,2}} + \ldots + x_1^{n_{i,1}+m_{j,2}} x_2^{n_{i,2}+m_{j,1}-1} \\
&- x_1^{n_{i,1}+m_{j,1}+m_{j,2}-1} x_2^{n_{i,2}} - \ldots - x_1^{m_{j,1}+n_{i,1}} x_2^{n_{i,2}+m_{j,2}-1} \\
&- x_1^{m_{j,2}+n_{i,1}+n_{i,2}-1} x_2^{m_{j,1}} - \ldots - x_1^{n_{i,2}+m_{j,2}} x_2^{n_{i,1}+m_{j,1}-1} \\
&+ x_1^{m_{j,2}+n_{i,1}+n_{i,2}-1} x_2^{m_{j,1}} + \ldots + x_1^{n_{i,1}+m_{j,2}} x_2^{n_{i,2}+m_{j,1}-1}) x_3^{m_{j,3}+n_{i,3}}.
\end{aligned}$$

Since $(S,\cdot)$ is a commutative associative algebra, by Theorem 4 $(S,[\cdot,\cdot]_R)$ is a left-Alia algebra.

Proposition 1. *Let $(A,[\cdot,\cdot])$ be an n-dimensional ($n \geq 2$) left-Alia algebra and $\{e_1,\cdots,e_n\}$ be a basis of A. For all positive integers $1 \leq i,j,t \leq n$ and structural constants $C_{ij}^t \in \mathbb{C}$, set*

$$[e_i, e_j] = \sum_{t=1}^{n} C_{ij}^t e_t. \tag{12}$$

Then, $(A,[\cdot,\cdot])$ is a left-Alia algebra if and only if the structural constants C_{ij}^t satisfy the following equation:

$$\sum_{k,m=1}^{n} \Big((C_{ij}^k - C_{ji}^k) C_{kl}^m + (C_{jl}^k - C_{lj}^k) C_{ki}^m + (C_{li}^k - C_{il}^k) C_{kj}^m \Big) = 0, \ \forall 1 \leq i,j,l \leq n. \tag{13}$$

Proof. By (1), for all $e_i, e_j, e_l \in \{e_1,\cdots,e_n\}$,

$$[[e_i,e_j] - [e_j,e_i], e_l] + [[e_j,e_l] - [e_l,e_j], e_i] + [[e_l,e_i] - [e_i,e_l], e_j] = 0. \tag{14}$$

Set

$$[e_i,e_j] = \sum_{k=1}^{n} C_{ij}^k e_k, \ [e_j,e_l] = \sum_{k=1}^{n} C_{jl}^k e_l, \ [e_l \cdot e_i] = \sum_{k=1}^{n} C_{li}^k e_k, \quad C_{ij}^k, C_{jl}^k, C_{li}^k \in \mathbb{C}.$$

Therefore, Equation (13) holds. □

As a direct consequence, we obtain the following:

Proposition 2. *Let A be a two-dimensional vector space over the complex field $\mathbb{C}$ with a basis $\{e_1,e_2\}$. Then, for any bilinear map $[\cdot,\cdot]$ on A, $(A,[\cdot,\cdot])$ is a left-Alia algebra.*

Next, we give some example of three-dimensional left-Alia algebras.

Example 2. *Let A be a three-dimensional vector space over the complex field $\mathbb{C}$ with a basis $\{e_1,e_2,e_3\}$. Define a bilinear map $[\cdot,\cdot] : A \times A \to A$ by*

$$\begin{aligned}
&[e_1,e_2] = e_1, \ \ [e_1,e_3] = e_1, \ \ [e_2,e_1] = e_2, \ \ [e_3,e_1] = e_3, \\
&[e_1,e_1] = [e_2,e_2] = [e_3,e_3] = [e_2,e_3] = [e_3,e_2] = e_1 + e_2 + e_3.
\end{aligned}$$

Then, $(A,[\cdot,\cdot])$ is a three-dimensional left-Alia algebra.

Remark 5. *A right-Leibniz algebra [10] is a vector space A together with a bilinear operation $[\cdot,\cdot] : A \otimes A \to A$ satisfying*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]],\ \forall\, x,y,z \in A.$$

Then, we have

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = [[x,y] - [y,x],z] + [[y,z] - [z,y],x] + [[z,x] - [x,z],y].$$

Therefore, if a right-Leibniz algebra satisfies

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$$

then $(A,[\cdot,\cdot])$ is a left-Alia algebra.

3.3. From Left-Alia Algebras to Anti-Pre-Lie Algebras

Definition 5 ([8])**.** *Let A be a vector space with a bilinear map $\cdot : A \times A \to A$. $(A,\cdot)$ is called an **anti-pre-Lie algebra** if the following equations are satisfied:*

$$x \cdot (y \cdot z) - y \cdot (x \cdot z) = [y,x] \cdot z, \tag{15}$$
$$[x,y] \cdot z + [y,z] \cdot x + [z,x] \cdot y = 0, \tag{16}$$

where

$$[x,y] = x \cdot y - y \cdot x, \tag{17}$$

for all $x,y,z \in A$.

Remark 6. *Let $(A,[\cdot,\cdot])$ be a left-Alia algebra. If $[\cdot,\cdot] : A \times A \to A$ satisfies*

$$[x,[y,z]] - [y,[x,z]] = [[y,x],z] - [[x,y],z],\ \forall x,y,z \in A, \tag{18}$$

then $(A,[\cdot,\cdot])$ is an anti-pre-Lie algebra.

3.4. From Left-Alia Algebras to Lie Triple Systems

Lie triple systems originated from Cartan's studies on the Riemannian geometry of totally geodesic submanifolds [11], which can be constructed using twisted derivations and left-Alia algebras.

Definition 6 ([30])**.** *A **Lie triple system** is a vector space A together with a trilinear operation $[\cdot,\cdot,\cdot] : A \times A \times A \to A$ such that the following three equations are satisfied, for all $x,\ y,\ z,\ a,\ b$ in A:*

$$[x,x,y] = 0, \tag{19}$$
$$[x,y,z] + [y,z,x] + [z,x,y] = 0 \tag{20}$$

and

$$[a,b,[x,y,z]] = [[a,b,x],y,z] + [x,[a,b,y],z] + [x,y,[a,b,z]]. \tag{21}$$

Proposition 3. *Let $(A,\cdot)$ be a commutative associative algebra and D be a twisted derivation. Define the bilinear map $[\cdot,\cdot]_R : A \times A \to A$ by (10). And define the trilinear map $[\cdot,\cdot,\cdot]_R : A \times A \times A \to A$ by*

$$[x,y,z]_R := \frac{1}{2}[[x,y]_R - [y,x]_R, z]_R,\ \forall x,y,z \in A.$$

If $[\cdot,\cdot,\cdot]_R$ satisfies (21), then $(A,[\cdot,\cdot,\cdot]_R)$ is a Lie triple system.

Proof. For all $x, y \in A$, it is obvious that $[x, x, y]_R = 0$. By the proof of Theorem 4, Equation (20) holds. If, in addition, $[\cdot,\cdot,\cdot]_R$ satisfies (21), then $(A, [\cdot,\cdot,\cdot]_R)$ is a Lie triple system. □

Remark 7. *Let $(A, [\cdot,\cdot])$ be a left-Alia algebra. For all $x, y, z \in A$, set a trilinear map $[\cdot,\cdot,\cdot] : A \times A \times A \to A$ by $[x, y, z] = [[x, y] - [y, x], z]$. If $[\cdot,\cdot,\cdot]$ satisfies (21), then $(A, [\cdot,\cdot,\cdot])$ is a Lie triple system.*

3.5. Representations and Matched Pairs of Left-Alia Algebras

Definition 7. *A **representation** of a left-Alia algebra $(A, [\cdot,\cdot])$ is a triple (l, r, V), where V is a vector space and $l, r : A \to \mathrm{End}(V)$ are linear maps such that the following equation holds:*

$$l([x, y]) - l([y, x]) = r(x)r(y)v - r(y)r(x) + r(y)l(x)v - r(x)l(y), \ \forall x, y \in A, v \in V. \quad (22)$$

*Two representations, (l, r, V) and (l', r', V'), of a left-Alia algebra $(A, [\cdot,\cdot])$ are called **equivalent** if there is a linear isomorphism $\phi : V \to V'$ such that*

$$\phi\big(l(x)v\big) = l'(x)\phi(v), \ \phi\big(r(x)v\big) = r'(x)\phi(v), \ \forall x \in A, v \in V. \quad (23)$$

Example 3. *Let (ρ, V) be a representation of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, that is, $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a linear map such that*

$$\rho([x, y])v = \rho(x)\rho(y)v - \rho(y)\rho(x)v, \ \forall x, y \in \mathfrak{g}, v \in V.$$

Then, both $(\rho, -\rho, V)$ and $(\rho, 2\rho, V)$ satisfy (22) and, hence, are representations of $(\mathfrak{g}, [\cdot,\cdot])$ as a left-Alia algebra.

Proposition 4. *Let $(A, [\cdot,\cdot])$ be a left-Alia algebra, V be a vector space and $l, r : A \to \mathrm{End}(V)$ be linear maps. Then, (l, r, V) is a representation of $(A, [\cdot,\cdot])$ if and only if there is a left-Alia algebra on the direct sum $d = A \oplus V$ of vector spaces (**the semi-direct product**) given by*

$$[x + u, y + v]_d = [x, y] + l(x)v + r(y)u, \ \forall x, y \in A, u, v \in V. \quad (24)$$

In this case, we denote $(A \oplus V, [\cdot,\cdot]_d) = A \ltimes_{l,r} V$.

Proof. This is the special case of matched pairs of left-Alia algebras where $B = V$ is equipped with the zero multiplication in Proposition 6. □

For a vector space A with a bilinear map $[\cdot,\cdot] : A \times A \to A$, we set linear maps $\mathcal{L}_{[\cdot,\cdot]}, \mathcal{R}_{[\cdot,\cdot]} : A \to \mathrm{End}(A)$ using

$$\mathcal{L}_{[\cdot,\cdot]}(x)y = [x, y] = \mathcal{R}_{[\cdot,\cdot]}(y)x, \ \forall x, y \in A.$$

Example 4. *Let $(A, [\cdot,\cdot])$ be a left-Alia algebra. Then, $(\mathcal{L}_{[\cdot,\cdot]}, \mathcal{R}_{[\cdot,\cdot]}, A)$ is a representation of $(A, [\cdot,\cdot])$, which is called an **adjoint representation**. In particular, for a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ with the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by $\mathrm{ad}(x)y = [x, y]$, $\forall x, y \in \mathfrak{g}$,*

$$(\mathcal{L}_{[\cdot,\cdot]}, \mathcal{R}_{[\cdot,\cdot]}, \mathfrak{g}) = (\mathrm{ad}, -\mathrm{ad}, \mathfrak{g})$$

is a representation of $(\mathfrak{g}, [\cdot,\cdot])$ as a left-Alia algebra.

Let A and V be vector spaces. For a linear map $l : A \to \mathrm{End}(V)$, we set a linear map $l^* : A \to \mathrm{End}(V^*)$ using

$$\langle l^*(x)u^*, v\rangle = -\langle u^*, l(x)v\rangle, \ \forall x \in A, u^* \in V^*, v \in V.$$

Proposition 5. *Let (l, r, V) be a representation of a left-Alia algebra $(A, [\cdot, \cdot])$. Then, $(l^*, l^* - r^*, V^*)$ is also a representation of $(A, [\cdot, \cdot])$. In particular, $(\mathcal{L}^*_{[\cdot,\cdot]}, \mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]}, A^*)$ is a representation of $(A, [\cdot, \cdot])$, which is called the* ***coadjoint representation****.*

Proof. Let $x, y \in A, u^* \in V^*, v \in V$. Then, we have

$$\begin{aligned}
&\langle\big(l^*[x,y] - l^*[y,x] + (l^* - r^*)(x)l^*(y) - (l^* - r^*)(x)(l^* - r^*)(y) \\
&+(l^* - r^*)(y)(l^* - r^*)(x) - (l^* - r^*)(y)l^*(x)\big)u^*, v\rangle \\
&= \langle\big(l^*[x,y] - l^*[y,x] + (l^* - r^*)(x)r^*(y) - (l^* - r^*)(y)r^*(x)\big)u^*, v\rangle \\
&= \langle u^*, \big(l[y,x] - l[x,y] + r(y)(l - r)(x) - r(x)(l - r)(y)\big)v\rangle \\
&\overset{(22)}{=} 0.
\end{aligned}$$

Hence, the conclusion follows. □

Example 5. *Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie algebra. Then, the coadjoint representation of $(\mathfrak{g}, [\cdot, \cdot])$ as a left-Alia algebra is*

$$(\mathrm{ad}^*, \mathrm{ad}^* - (-\mathrm{ad}^*), \mathfrak{g}^*) = (\mathrm{ad}^*, 2\mathrm{ad}^*, \mathfrak{g}^*).$$

Hence, there is a left-Alia algebra structure $\mathfrak{g} \ltimes_{\mathrm{ad}^*, 2\mathrm{ad}^*} \mathfrak{g}^*$ on the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ of vector spaces.

Remark 8. *In [5], there is also the notion of a* ***right-Alia algebra****, defined as a vector space A together with a bilinear map $[\cdot, \cdot]' : A \times A \to A$ satisfying*

$$[x, [y, z]']' + [y, [z, x]']' + [z, [x, y]']' = [x, [z, y]']' + [y, [x, z]']' + [z, [y, x]']', \ \forall x, y, z \in A. \tag{25}$$

It is clear that $(A, [\cdot, \cdot]')$ is a right-Alia algebra if and only if the opposite algebra $(A, [\cdot, \cdot])$ of $(A, [\cdot, \cdot]')$, given by $[x, y] = [y, x]'$, is a left-Alia algebra. Thus, our study on left-Alia algebras can straightforwardly generalize a parallel study on right-Alia algebras. Consequently, if (l, r, V) is a representation of a right-Alia algebra $(A, [\cdot, \cdot]')$, then $(r^* - l^*, r^*, V^*)$ is also a representation of $(A, [\cdot, \cdot]')$. Recall [23] that if (l, r, V) is a representation of an anti-pre-Lie algebra $(A, [\cdot, \cdot]_{\mathrm{anti}})$, then $(r^* - l^*, r^*, V^*)$ is also a representation of $(A, [\cdot, \cdot]_{\mathrm{anti}})$. Moreover, admissible Novikov algebras [8] are a subclass of anti-pre-Lie algebras. If (l, r, V) is a representation of an admissible Novikov algebra $(A, [\cdot, \cdot]_{admissible\ Novikov})$, then $(r^* - l^*, r^*, V^*)$ is also a representation of the admissible Novikov algebra $(A, [\cdot, \cdot]_{admissible\ Novikov})$. Therefore, we have the following algebras which preserve the form $(r^* - l^*, r^*, V^*)$ of representations on the dual spaces:

{right-Alia algebras} ⊃ {anti-pre-Lie algebras} ⊃ {admissible Novikov algebras}.

Now, we introduce the notion of matched pairs of left-Alia algebras.

Definition 8. *Let $(A, [\cdot, \cdot]_A)$ and $(B, [\cdot, \cdot]_B)$ be left-Alia algebras and $l_A, r_A : A \to \mathrm{End}(B)$ and $l_B, r_B : B \to \mathrm{End}(A)$ be linear maps. If there is a left-Alia algebra structure $[\cdot, \cdot]_{A \oplus B}$ on the direct sum $A \oplus B$ of vector spaces given by*

$$[x + a, y + b]_{A \oplus B} = [x, y]_A + l_B(a)y + r_B(b)x + [a, b]_B + l_A(x)b + r_A(y)a, \ \forall x, y \in A, a, b \in B,$$

then we say $\big((A, [\cdot, \cdot]_A), (B, [\cdot, \cdot]_B), l_A, r_A, l_B, r_B\big)$ is a ***matched pair of left-Alia algebras****.*

Proposition 6. *Let $(A, [\cdot, \cdot]_A)$ and $(B, [\cdot, \cdot]_B)$ be left-Alia algebras and $l_A, r_A : A \to \mathrm{End}(B)$ and $l_B, r_B : B \to \mathrm{End}(A)$ be linear maps. Then, $\big((A, [\cdot, \cdot]_A), (B, [\cdot, \cdot]_B), l_A, r_A, l_B, r_B\big)$ is a matched* *pair of left-Alia algebras if and only if the triple (l_A, r_A, B) is a representation of $(A, [\cdot,\cdot]_A)$, the triple (l_B, r_B, A) is a representation of $(B, [\cdot,\cdot]_B)$ and the following equations hold:*

$$\begin{aligned} r_B(a)([x,y]_A - [y,x]_A) &= (l_B - r_B)(a)[y,x]_A + (r_B - l_B)(a)[x,y]_A \\ &\quad + l_B\big((r_A - l_A)(y)a\big)x + l_B\big((l_A - r_A)(x)a\big)y, \end{aligned} \tag{26}$$

$$\begin{aligned} r_A(x)([a,b]_B - [b,a]_B) &= (l_A - r_A)(x)[b,a]_B + (r_A - l_A)(x)[a,b]_B \\ &\quad + l_A\big((r_B - l_B)(b)x\big)a + l_A\big((l_B - r_B)(a)x\big)b, \end{aligned} \tag{27}$$

for all $x, y \in A, a, b \in B$.

Proof. The proof follows from a straightforward computation. □

3.6. Quadratic Left-Alia Algebras

Definition 9. *A **quadratic left-Alia algebra** is a triple $(A, [\cdot,\cdot], \mathcal{B})$, where $(A, [\cdot,\cdot])$ is a left-Alia algebra and $\mathcal{B}$ is a nondegenerate symmetric bilinear form on A which is invariant in the sense that*

$$\mathcal{B}([x,y],z) = \mathcal{B}(x, [z,y] - [y,z]), \ \forall x,y,z \in A. \tag{28}$$

Remark 9. *Since $\mathcal{B}$ is symmetric, it follows from Definition 9 that*

$$\mathcal{B}([x,y],z) + \mathcal{B}(y,[x,z]) = 0, \ \forall x,y,z \in A. \tag{29}$$

Lemma 3. *Let $(A, [\cdot,\cdot], \mathcal{B})$ be a quadratic left-Alia algebra. Then, $(\mathcal{L}_{[\cdot,\cdot]}, \mathcal{R}_{[\cdot,\cdot]}, A)$ and $(\mathcal{L}^*_{[\cdot,\cdot]}, \mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]}, A^*)$ are equivalent as representations of $(A, [\cdot,\cdot])$.*

Proof. We set a linear isomorphism $\mathcal{B}^\natural : A \to A^*$ using

$$\langle \mathcal{B}^\natural(x), y \rangle = \mathcal{B}(x,y). \tag{30}$$

Then, by (29) we have

$$\langle \mathcal{B}^\natural(\mathcal{L}_{[\cdot,\cdot]}(x)y), z\rangle = \mathcal{B}([x,y],z) = -\mathcal{B}(y,[x,z]) = -\langle \mathcal{B}^\natural(y), [x,z]\rangle = \langle \mathcal{L}^*_{[\cdot,\cdot]}(x)\mathcal{B}^\natural(y), z\rangle,$$

that is, $\mathcal{B}^\natural(\mathcal{L}_{[\cdot,\cdot]}(x)y) = \mathcal{L}^*_{[\cdot,\cdot]}(x)\mathcal{B}^\natural(y)$. Similarly, by (28), we have $\mathcal{B}^\natural(\mathcal{R}_{[\cdot,\cdot]}(x)y) = (\mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]})(x)\mathcal{B}^\natural(y)$. Hence, the conclusion follows. □

Proposition 7. *Let $(A, \cdot)$ be a commutative associative algebra and $f : A \to A$ be a linear map. Let $\mathcal{B}$ be a nondegenerate symmetric invariant bilinear form on $(A, \cdot)$ and $\hat{f} : A \to A$ be the adjoint map of f with respect to $\mathcal{B}$, given by*

$$\mathcal{B}(\hat{f}(x), y) = \mathcal{B}(x, f(y)), \ \forall x, y \in A.$$

Then, there is a quadratic left-Alia algebra $(A, [\cdot,\cdot], \mathcal{B})$, where $(A, [\cdot,\cdot])$ is the special left-Alia algebra with respect to $(A, \cdot, f, -\hat{f})$, that is,

$$[x,y] = x \cdot f(y) - \hat{f}(x \cdot y). \tag{31}$$

Proof. For all $x, y, z \in A$, we have

$$\begin{aligned} \mathcal{B}([x,y],z) &= \mathcal{B}(x \cdot f(y) - \hat{f}(x \cdot y), z) \\ &= \mathcal{B}(x, z \cdot f(y) - y \cdot f(z)) \\ &= \mathcal{B}\Big(x, z \cdot f(y) - \hat{f}(z \cdot y) - \big(y \cdot f(z) - \hat{f}(y \cdot z)\big)\Big) \\ &= \mathcal{B}(x, [z,y] - [y,z]). \end{aligned}$$

Hence, the conclusion follows. □

Example 6. *Let $(A, [\cdot,\cdot])$ be a left-Alia algebra and $(\mathcal{L}_{[\cdot,\cdot]}, \mathcal{R}_{[\cdot,\cdot]}, A)$ be the adjoint representation of $(A, [\cdot,\cdot])$. By Propositions 4 and 5, there is a left-Alia algebra $A \ltimes_{\mathcal{L}^*_{[\cdot,\cdot]}, \mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]}} A^*$ on $d = A \oplus A^*$, given by (24). There is a natural nondegenerate symmetric bilinear form $\mathcal{B}_d$ on $A \oplus A^*$, given by*

$$\mathcal{B}_d(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \ \forall x, y \in A, a^*, b^* \in A^*. \tag{32}$$

For all $x, y, z \in A, a^*, b^*, c^* \in A^*$, we have

$$\begin{aligned}
\mathcal{B}_d([x + a^*, y + b^*]_d, z + c^*) &= \mathcal{B}_d\big([x, y] + \mathcal{L}^*_{[\cdot,\cdot]}(x)b^* + (\mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]})(y)a^*, z + c^*\big) \\
&= \langle [x, y], c^* \rangle + \langle \mathcal{L}^*_{[\cdot,\cdot]}(x)b^* + (\mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]})(y)a^*, z \rangle \\
&= \langle [x, y], c^* \rangle - \langle [x, z], b^* \rangle + \langle a^*, [z, y] - [y, z] \rangle, \\
\mathcal{B}_d\big(x + a^*, [z + c^*, x + b^*]_d\big) &= \langle [z, y], a^* \rangle - \langle [z, x], b^* \rangle + \langle c^*, [y, x] - [x, y] \rangle, \\
\mathcal{B}_d\big(x + a^*, [y + b^*, z + c^*]_d\big) &= \langle [y, z], a^* \rangle - \langle [y, x], c^* \rangle + \langle b^*, [z, x] - [x, z] \rangle.
\end{aligned}$$

Hence, we have

$$\mathcal{B}_d\big([x + a^*, y + b^*]_d, z + c^*\big) = \mathcal{B}_d\big(x + a^*, [z + c^*, y + b^*]_d - [y + b^*, z + c^*]_d\big),$$

and, thus, $(A \ltimes_{\mathcal{L}^*_{[\cdot,\cdot]}, \mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]}} A^*, \mathcal{B}_d)$ is a quadratic left-Alia algebra.

Remark 10. *By Example 6, an arbitrary Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ renders a quadratic left-Alia algebra $(\mathfrak{g} \ltimes_{\mathrm{ad}^*, 2\mathrm{ad}^*} \mathfrak{g}^*, \mathcal{B}_d)$, where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation of $(\mathfrak{g}, [\cdot,\cdot])$.*

We study the tensor forms of nondegenerate symmetric invariant bilinear forms on left-Alia algebras.

Definition 10. *Let $(A, [\cdot,\cdot])$ be a left-Alia algebra and $h : A \to \mathrm{End}(A \otimes A)$ be a linear map given by*

$$h(x) = (\mathcal{R}_{[\cdot,\cdot]} - \mathcal{L}_{[\cdot,\cdot]})(x) \otimes \mathrm{id} - \mathrm{id} \otimes \mathcal{R}_{[\cdot,\cdot]}(x), \ \forall x \in A. \tag{33}$$

*An element $r \in A \otimes A$ is called **invariant** on $(A, [\cdot,\cdot])$ if $h(x)r = 0$ for all $x \in A$.*

Proposition 8. *Let $(A, [\cdot,\cdot])$ be a left-Alia algebra. Suppose that $\mathcal{B}$ is a nondegenerate bilinear form on A and $\mathcal{B}^\natural : A \to A^*$ is the corresponding map given by (30). Set $\widetilde{\mathcal{B}} \in A \otimes A$ using*

$$\langle \widetilde{\mathcal{B}}, a^* \otimes b^* \rangle = \langle \mathcal{B}^{\natural^{-1}}(a^*), b^* \rangle, \ \forall a^*, b^* \in A^*. \tag{34}$$

Then, $(A, [\cdot,\cdot], \mathcal{B})$ is a quadratic left-Alia algebra if and only if $\widetilde{\mathcal{B}}$ is symmetric and invariant on $(A, [\cdot,\cdot])$.

Proof. It is clear that $\mathcal{B}$ is symmetric if and only if $\widetilde{\mathcal{B}}$ is symmetric. Let $x, y, z \in A$ and $a^* = \mathcal{B}^\natural(x), c^* = \mathcal{B}^\natural(z)$. Under the symmetric assumption, we have

$$\begin{aligned}
&\mathcal{B}([x, y], z) = \langle [x, y], \mathcal{B}^\natural(z) \rangle = \langle [\mathcal{B}^{\natural^{-1}}(a^*), y], c^* \rangle \\
&= -\langle \mathcal{B}^{\natural^{-1}}(a^*), \mathcal{R}^*_{[\cdot,\cdot]}(y)c^* \rangle = -\langle \widetilde{\mathcal{B}}, a^* \otimes \mathcal{R}^*_{[\cdot,\cdot]}(y)c^* \rangle = \langle (\mathrm{id} \otimes \mathcal{R}_{[\cdot,\cdot]}(y))\widetilde{\mathcal{B}}, a^* \otimes c^* \rangle, \\
&\mathcal{B}(x, [z, y] - [y, z]) = \langle \mathcal{B}^\natural(x), [z, y] - [y, z] \rangle = \langle a^*, [\mathcal{B}^{\natural^{-1}}(c^*), y] - [y, \mathcal{B}^{\natural^{-1}}(c^*)] \rangle \\
&= \langle (\mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]})(y)a^*, \mathcal{B}^{\natural^{-1}}(c^*) \rangle = \langle \widetilde{\mathcal{B}}, c^* \otimes (\mathcal{L}^*_{[\cdot,\cdot]} - \mathcal{R}^*_{[\cdot,\cdot]})(y)a^* \rangle \\
&= \langle ((\mathcal{R}_{[\cdot,\cdot]} - \mathcal{L}_{[\cdot,\cdot]})(y) \otimes \mathrm{id})\widetilde{\mathcal{B}}, a^* \otimes c^* \rangle,
\end{aligned}$$

that is, (28) holds if and only if $h(y)\widetilde{\mathcal{B}} = 0$ for all $y \in A$. Hence, the conclusion follows. □

4. Manin Triples of Left-Alia Algebras and Left-Alia Bialgebras

In this section, we introduce the notions of Manin triples of left-Alia algebras and left-Alia bialgebras. We show that they are equivalent structures via specific matched pairs of left-Alia algebras.

4.1. Manin Triples of Left-Alia Algebras

Definition 11. *Let $(A, [\cdot,\cdot]_A)$ and $(A^*, [\cdot,\cdot]_{A^*})$ be left-Alia algebras. Assume that there is a left-Alia algebra structure $(d = A \oplus A^*, [\cdot,\cdot]_d)$ on $A \oplus A^*$ which contains $(A, [\cdot,\cdot]_A)$ and $(A^*, [\cdot,\cdot]_{A^*})$ as left-Alia subalgebras. Suppose that the natural nondegenerate symmetric bilinear form $\mathcal{B}_d$, given by (32), is invariant on $(A \oplus A^*, [\cdot,\cdot]_d)$, that is, $(A \oplus A^*, [\cdot,\cdot]_d, \mathcal{B}_d)$ is a quadratic left-Alia algebra. Then, we say that $\big((A \oplus A^*, [\cdot,\cdot]_d, \mathcal{B}_d), A, A^*\big)$ is a **Manin triple of left-Alia algebras**.*

Recall [19] that a **double construction of commutative Frobenius algebras** $\big((A \oplus A^*, \cdot_d, \mathcal{B}_d), A, A^*\big)$ is a commutative associative algebra $(A \oplus A^*, \cdot_d)$ containing $(A, \cdot_A)$ and $(A^*, \cdot_{A^*})$ as commutative associative subalgebras, such that the natural nondegenerate symmetric bilinear form $\mathcal{B}_d$ given by (32) is invariant on $(A \oplus A^*, \cdot_d)$. Now, we show that double constructions of commutative Frobenius algebras with linear maps naturally give rise to Manin triples of left-Alia algebras.

Corollary 1. *Let $\big((A \oplus A^*, \cdot_d, \mathcal{B}_d), A, A^*\big)$ be a double construction of commutative Frobenius algebras. Suppose that $P : A \to A$ and $Q^* : A^* \to A^*$ are linear maps. Then, there is a Manin triple of left-Alia algebras $\big((A \oplus A^*, [\cdot,\cdot]_d, \mathcal{B}_d), A, A^*\big)$ given by*

$$
\begin{aligned}
&[x + a^*, y + b^*]_d = (x + a^*) \cdot_d \big(P(y) + Q^*(b^*)\big) - (Q + P^*)\big((x + a^*) \cdot_d (y + b^*)\big),\\
&[x, y]_A = x \cdot_A P(y) - Q(x \cdot_A y),\ [a^*, b^*]_{A^*} = a^* \cdot_{A^*} Q^*(b^*) - P^*(a^* \cdot_{A^*} b^*),
\end{aligned}
$$

for all $x, y \in A, a^*, b^* \in A^*$.

Proof. The adjoint map of $P + Q^*$ with respect to $\mathcal{B}_d$ is $Q + P^*$. Hence, the conclusion follows from Proposition 7 by taking $f = P + Q^*$. □

Theorem 5. *Let $(A, [\cdot,\cdot]_A)$ and $(A^*, [\cdot,\cdot]_{A^*})$ be left-Alia algebras. Then, there is a Manin triple of left-Alia algebras $\big((A \oplus A^*, [\cdot,\cdot]_d, \mathcal{B}_d), A, A^*\big)$ if and only if*

$$
\big((A, [\cdot,\cdot]_A), (A^*, [\cdot,\cdot]_{A^*}), \mathcal{L}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}\big)
$$

is a matched pair of left-Alia algebras.

Proof. Let $\big((A \oplus A^*, [\cdot,\cdot]_d, \mathcal{B}_d), A, A^*\big)$ be a Manin triple of left-Alia algebras. For all $x, y \in A, a^*, b^* \in A^*$, we have

$$
\begin{aligned}
\mathcal{B}_d([x, b^*]_d, y) &\overset{(28)}{=} -\mathcal{B}(b^*, [x, y]_A) = -\langle b^*, [x, y]_A\rangle = \langle \mathcal{L}^*_{[\cdot,\cdot]_A}(x)b^*, y\rangle = \mathcal{B}_d\big(\mathcal{L}^*_{[\cdot,\cdot]_A}(x)b^*, y\big),\\
\mathcal{B}_d([x, b^*]_d, a^*) &\overset{(28)}{=} \mathcal{B}_d(x, [a^*, b^*]_{A^*} - [b^*, a^*]_{A^*}) = \langle x, [a^*, b^*]_{A^*} - [b^*, a^*]_{A^*}\rangle\\
&= \langle (\mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}})(b^*)x, a^*\rangle = \mathcal{B}_d\big((\mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}})(b^*)x, a^*\big).
\end{aligned}
$$

Thus,

$$
\mathcal{B}_d([x, b^*]_d, y + a^*) = \mathcal{B}_d\big((\mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}})(b^*)x + \mathcal{L}^*_{[\cdot,\cdot]_A}(x)b^*, y + a^*\big)
$$

and, by the nondegeneracy of $\mathcal{B}_d$, we have

$$
[x, b^*]_d = (\mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}})(b^*)x + \mathcal{L}^*_{[\cdot,\cdot]_A}(x)b^*.
$$

Similarly,
$$[y, a^*]_d = (\mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A})(y)a^* + \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}(a^*)y.$$
Therefore, we have
$$\begin{aligned}[x + a^*, y + b^*]_d = [x, y]_A + \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}(a^*)y + (\mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}})(b^*)x \\ + [a^*, b^*]_{A^*} + \mathcal{L}^*_{[\cdot,\cdot]_A}(x)b^* + (\mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A})(y)a^*.\end{aligned} \tag{35}$$

Hence, $\big((A, [\cdot,\cdot]_A), (A^*, [\cdot,\cdot]_{A^*}), \mathcal{L}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}\big)$ is a matched pair of left-Alia algebras.

Conversely, if $\big((A, [\cdot,\cdot]_A), (A^*, [\cdot,\cdot]_{A^*}), \mathcal{L}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}\big)$ is a matched pair of left-Alia algebras, then it is straightforward to check that $\mathcal{B}_d$ is invariant on the left-Alia algebra $(A \oplus A^*, [\cdot,\cdot]_d)$ given by (35). □

4.2. Left-Alia Bialgebras

Definition 12. *A* ***left-Alia coalgebra*** *is a pair, (A, δ), such that A is a vector space and $\delta : A \to A \otimes A$ is a co-multiplication satisfying*
$$(\mathrm{id}^{\otimes 3} + \xi + \xi^2)(\tau \otimes \mathrm{id} - \mathrm{id}^{\otimes 3})(\delta \otimes \mathrm{id})\delta = 0, \tag{36}$$
where $\tau(x \otimes y) = y \otimes x$ and $\xi(x \otimes y \otimes z) = y \otimes z \otimes x$ for all $x, y, z \in A$.

Proposition 9. *Let A be a vector space and $\delta : A \to A \otimes A$ be a co-multiplication. Let $[\cdot,\cdot]_{A^*} : A^* \otimes A^* \to A^*$ be the linear dual of δ, that is,*
$$\langle [a^*, b^*]_{A^*}, x\rangle = \langle \delta^*(a^* \otimes b^*), x\rangle = \langle a^* \otimes b^*, \delta(x)\rangle, \ \ \forall a^*, b^* \in A^*, x \in A. \tag{37}$$
Then, (A, δ) is a left-Alia coalgebra if and only if $(A^*, [\cdot,\cdot]_{A^*})$ is a left-Alia algebra.

Proof. For all $x \in A, a^*, b^*, c^* \in A^*$, we have
$$\begin{aligned}
\langle [[a^*, b^*]_{A^*}, c^*]_{A^*} - [[b^*, a^*]_{A^*}, c^*]_{A^*}, x\rangle &= \langle \delta^*(\delta^* \otimes \mathrm{id})(\mathrm{id}^{\otimes 3} - \tau \otimes \mathrm{id})a^* \otimes b^* \otimes c^*, x\rangle \\
&= \langle a^* \otimes b^* \otimes c^*, (\mathrm{id}^{\otimes 3} - \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id})\delta(x)\rangle, \\
\langle [[b^*, c^*]_{A^*}, a^*]_{A^*} - [[c^*, b^*]_{A^*}, a^*]_{A^*}, x\rangle &= \langle b^* \otimes c^* \otimes a^*, (\mathrm{id}^{\otimes 3} - \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id})\delta(x)\rangle \\
&= \langle a^* \otimes b^* \otimes c^*, \xi^2(\mathrm{id}^{\otimes 3} - \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id})\delta(x)\rangle, \\
\langle [[c^*, a^*]_{A^*}, b^*]_{A^*} - [[a^*, c^*]_{A^*}, b^*]_{A^*}, x\rangle &= \langle c^* \otimes a^* \otimes b^*, (\mathrm{id}^{\otimes 3} - \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id})\delta(x)\rangle \\
&= \langle a^* \otimes b^* \otimes c^*, \xi(\mathrm{id}^{\otimes 3} - \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id})\delta(x)\rangle.
\end{aligned}$$

Hence, (1) holds for $(A^*, [\cdot,\cdot]_{A^*})$ if and only if (36) holds. □

Definition 13. *A* ***left-Alia bialgebra*** *is a triple $(A, [\cdot,\cdot], \delta)$, such that $(A, [\cdot,\cdot])$ is a left-Alia algebra, (A, δ) is a left-Alia coalgebra and the following equation holds:*
$$(\tau - \mathrm{id}^2)\big(\delta([x, y] - [y, x]) + (\mathcal{R}_{[\cdot,\cdot]}(x) \otimes \mathrm{id})\delta(y) - (\mathcal{R}_{[\cdot,\cdot]}(y) \otimes \mathrm{id})\delta(x)\big) = 0, \ \forall x, y \in A. \tag{38}$$

Theorem 6. *Let $(A, [\cdot,\cdot]_A)$ be a left-Alia algebra. Suppose that there is a left-Alia algebra structure $(A^*, [\cdot,\cdot]_{A^*})$ on the dual space A^*, and $\delta : A \to A \otimes A$ is the linear dual of $[\cdot,\cdot]_{A^*}$. Then, $\big((A, [\cdot,\cdot]_A), (A^*, [\cdot,\cdot]_{A^*}), \mathcal{L}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}\big)$ is a matched pair of left-Alia algebras if and only if $(A, [\cdot,\cdot]_A, \delta)$ is a left-Alia bialgebra.*

Proof. For all $x, y \in A, a^*, b^* \in A^*$, we have

$$\begin{aligned}
\langle (\mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}})(a^*)([x,y]_A - [y,x]_A), b^* \rangle &= \langle [x,y]_A - [y,x]_A, [b^*,a^*]_{A^*} - [a^*,b^*]_{A^*} \rangle \\
&= \langle (\tau - \mathrm{id}^{\otimes 2})\delta([x,y]_A - [y,x]_A), a^* \otimes b^* \rangle, \\
\langle [\mathcal{R}^*_{[\cdot,\cdot]_{A^*}}(a^*)y, x]_A, b^* \rangle &= -\langle \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}(a^*)y, \mathcal{R}^*_{[\cdot,\cdot]_A}(x)b^* \rangle \\
&= \langle y, [\mathcal{R}^*_{[\cdot,\cdot]_A}(x)b^*, a^*]_{A^*} \rangle \\
&= -\langle \big(\mathcal{R}_{[\cdot,\cdot]_A}(x) \otimes \mathrm{id}\big)\delta(y), b^* \otimes a^* \rangle \\
&= -\langle \tau\big(\mathcal{R}_{[\cdot,\cdot]_A}(x) \otimes \mathrm{id}\big)\delta(y), a^* \otimes b^* \rangle, \\
-\langle [\mathcal{R}^*_{[\cdot,\cdot]_{A^*}}(a^*)y, x]_A, b^* \rangle &= \langle \tau\big(\mathcal{R}_{[\cdot,\cdot]_A}(y) \otimes \mathrm{id}\big)\delta(x), a^* \otimes b^* \rangle, \\
-\langle \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}\big(\mathcal{R}^*_{[\cdot,\cdot]_A}(y)a^*\big)x, b^* \rangle &= \langle x, [\mathcal{R}^*_{[\cdot,\cdot]_A}(y)a^*, b^*]_{A^*} \rangle \\
&= -\langle \big(\mathcal{R}_{[\cdot,\cdot]_A}(y) \otimes \mathrm{id}\big)\delta(x), a^* \otimes b^* \rangle, \\
\langle \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}\big(\mathcal{R}^*_{[\cdot,\cdot]_A}(x)a^*\big)y), b^* \rangle &= \langle \big(\mathcal{R}_{[\cdot,\cdot]_A}(x) \otimes \mathrm{id}\big)\delta(y), a^* \otimes b^* \rangle.
\end{aligned}$$

Thus, (38) holds if and only if (26) holds for $l_A = \mathcal{L}^*_{[\cdot,\cdot]_A}$, $r_A = \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}$, $l_B = \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}$, $r_B = \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}$. Similarly, (38) holds if and only if (27) holds for $l_A = \mathcal{L}^*_{[\cdot,\cdot]_A}$, $r_A = \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}$, $l_B = \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}$, $r_B = \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}$. Hence, the conclusion follows. □

Summarizing Theorems 5 and 6, we have the following corollary:

Corollary 2. *Let $(A, [\cdot,\cdot]_A)$ be a left-Alia algebra. Suppose that there is a left-Alia algebra structure $(A^*, [\cdot,\cdot]_{A^*})$ on the dual space A^*, and $\delta : A \to A \otimes A$ is the linear dual of $[\cdot,\cdot]_{A^*}$. Then, the following conditions are equivalent:*

(a) There is a Manin triple of left-Alia algebras $\big((d = A \oplus A^*, [\cdot,\cdot]_d, \mathcal{B}_d), A, A^*\big)$.
(b) $\big((A, [\cdot,\cdot]_A), (A^*, [\cdot,\cdot]_{A^*}), \mathcal{L}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}\big)$ is a matched pair of left-Alia algebras.
(c) $(A, [\cdot,\cdot]_A, \delta)$ is a left-Alia bialgebra.

Example 7. *Let $(A, [\cdot,\cdot]_A)$ be the three-dimensional left-Alia algebra given in Example 2.*

Then, there is a left-Alia bialgebra $(A, [\cdot,\cdot]_A, \delta)$ with a non-zero co-multiplication δ on A, given by

$$\delta(e_1) = e_1 \otimes e_1. \tag{39}$$

Then, by Corollary 2, there is a Manin triple $\big((A \oplus A^*, [\cdot,\cdot], \mathcal{B}_d), A, A^*\big)$. Here, the multiplication $[\cdot,\cdot]_{A^*}$ on A^* is given through δ by (39), that is,

$$[e_1^*, e_1^*]_{A^*} = e_1^*,$$

and the multiplication $[\cdot,\cdot]$ on $A \oplus A^*$ is given by (35). Moreover, $\big((A, [\cdot,\cdot]_A), (A^*, [\cdot,\cdot]_{A^*}), \mathcal{L}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_A} - \mathcal{R}^*_{[\cdot,\cdot]_A}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}}, \mathcal{L}^*_{[\cdot,\cdot]_{A^*}} - \mathcal{R}^*_{[\cdot,\cdot]_{A^*}}\big)$ is a matched pair of left-Alia algebras.

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