Article

The Convergence and Boundedness of Solutions to SFDEs with the G-Framework

Rahman Ullah [1], Faiz Faizullah [2,*] and Quanxin Zhu [3]

1   School of Mathematics and Physics, Hubei Polytechnic University, Huangshi 435003, China;
    rahman@wus.edu.pk or rahman_yzi@yahoo.com
2   College of Electrical and Mechanical Engineering, National University of Sciences and Technology (NUST),
    Islamabad 44000, Pakistan
3   MOE-LCSM, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, China;
    zqx22@hunnu.edu.cn
*   Correspondence: faiz_math@ceme.nust.edu.pk

Abstract: Generally, stochastic functional differential equations (SFDEs) pose a challenge as they often lack explicit exact solutions. Consequently, it becomes necessary to seek certain favorable conditions under which numerical solutions can converge towards the exact solutions. This article aims to delve into the convergence analysis of solutions for stochastic functional differential equations by employing the framework of G-Brownian motion. To establish the goal, we find a set of useful monotone type conditions and work within the space $\mathbb{C}_r((-\infty, 0]; R^n)$. The investigation conducted in this article confirms the mean square boundedness of solutions. Furthermore, this study enables us to compute both $\mathbb{L}_G^2$ and exponential estimates.

Keywords: G-Brownian motion; exponential and $\mathbb{L}_G^2$ estimates; boundedness; convergence

MSC: 39B52; 60H10; 60H35; 60G65

1. Introduction

Stochastic functional differential equations (SFDEs) find applications in various fields of engineering and science, such as neural networks [1], financial assets [2–4], population dynamics [5,6], and gene expression [7]. These dynamic systems are also used in the study of turbulent flow analysis and modeling [8]. A vast amount of literature exists on moment estimates, convergence, stability, and existence of solutions for SFDEs [9–13]. The study of SFDEs driven by G-Brownian motion is relatively new, dating back to the invention of G-Brownian theory in 2006 [14]. In [2,15], the classical Lipschitz condition and linear growth condition were used to establish the existence-uniqueness theorem for SFDEs in the space $BC((-\infty, 0]; R^n)$. The study of SFDEs under the G-framework with non-Lipschitz conditions and mean square stability was investigated in [16]. The work in [17–19] provides insights into pth moment estimates, the Cauchy–Maruyama approximation scheme, and exponential estimates for solutions to SFDEs within the framework of G-Brownian motion. Asymptotic estimates were studied in [20], while SFDEs under the G-Lévy processes were investigated in [21].

In this article, we introduce some useful monotone type conditions to study SFDEs under the G-framework within the space $\mathbb{C}_r((-\infty, 0]; R^n)$. Our findings contribute to the growing body of research on SFDEs driven by G-Brownian motion and deepen our understanding of the role of G-framework in stochastic analysis. We study the convergence of solutions for a SFDEs using the framework of G-Brownian motion. Our analysis results in the mean square boundedness of solutions and allows us to compute both $\mathbb{L}_G^2$ and exponential estimates. Consider a matrix A; its transpose is denoted by A^T. Let

$\mathbb{C}((-\infty, 0]; R^n)$ denotes the set of continuous mappings from $(-\infty, 0]$ to R^n. Define the space $\mathbb{C}_r((-\infty, 0]; R^n)$, $r > 0$ as

$$\mathbb{C}_r((-\infty, 0]; R^n) = \{\alpha \in C((-\infty, 0]; R^n) : \lim_{\sigma \to -\infty} e^{r\sigma} \alpha(\sigma) \text{ exists in } R^n\},$$

Associated with norm $\|\alpha\|_r = \sup_{-\infty < \sigma \leq 0} e^{r\sigma} |\alpha(\sigma)| < \infty$, the space $\mathbb{C}_r((-\infty, 0]; R^n)$ is a Banach space of bounded continuous mappings. For each $0 < r_1 \leq r_2 < \infty$, $\mathbb{C}_{r_1} \subseteq \mathbb{C}_{r_2}$ [22,23]. Represent the σ-algebra of $\mathbb{C}_r$ by $\mathcal{B}(\mathbb{C}_r)$ and $\mathbb{C}_r^0 = \{\alpha \in \mathbb{C}_r : \lim_{\sigma \to -\infty} e^{r\sigma} \alpha(\sigma) = 0\}$. Let $\mathbb{L}^2(\mathbb{C}_r)$ denote the space of all $\mathcal{F}$-measurable stochastic processes α taking values in $\mathbb{C}_r$, such that $\hat{\mathbb{E}}|\alpha|_r^2 < \infty$. Similarly, let $\mathbb{L}^2(\mathbb{C}_r^0)$ denote the space of all $\mathcal{F}$-measurable stochastic processes α taking values in $\mathbb{C}_r^0$, such that $\mathbb{E}|\alpha|_r^2 < \infty$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where $\mathcal{F}$ is a sigma-algebra of subsets. The natural filtration $\mathcal{F}_t$ on $(\Omega, \mathcal{F}, P)$ is defined as the sigma-algebra, denoted by $\mathcal{F}_t = \sigma\mathcal{B}(v) : 0 \leq v \leq t$, where $\mathcal{B}(v)$ represents the Borel sigma-algebra of $\mathbb{C}_r$. We use $\mathcal{P}$ to represent the set of all probability measures on $(\mathbb{C}_r, \mathcal{B}(\mathbb{C}_r))$. Additionally, $\mathbb{L}_b(\mathbb{C}_r)$ denotes the collection of continuous bounded functionals on $\mathbb{C}_r$. Finally, let Λ_0 be the collection of probability measures on $(-\infty, 0]$ satisfying $\int_{-\infty}^0 \mu(d\sigma) = 1$ for every $\mu \in \Lambda_0$. We define

$$\Lambda_k = \{\mu \in \Lambda_0 : \mu^{(k)} = \int_{-\infty}^0 e^{-k\sigma} \mu(d\sigma) < \infty\}, \tag{1}$$

where $\Lambda_{k_0} \subset \Lambda_k \subset \Lambda_0$ for any $k \in (0, k_0)$ [23]. Let $\kappa : \mathbb{C}_r((-\infty, 0]; R^n) \to R^n$, $\eta : \mathbb{C}_r((-\infty, 0]; R^n) \to R^n$ and $\gamma : \mathbb{C}_r((-\infty, 0]; R^n) \to R^n$ be Borel measurable. Consider the SFDEs driven by G-Brownian motion of the form

$$dz(t) = \kappa(z_t)dt + \eta(z_t)d\langle B, B \rangle(t) + \gamma(z_t)dB(t), \tag{2}$$

on $t \geq 0$ where $z_t = \{z(t + \theta) : -\infty < \theta \leq 0\}$. Equation (2) has the starting value $z_0 = \zeta \in \mathbb{C}_r((-\infty, 0]; R^n)$. Let $\langle B, B \rangle(t)$ denote the quadratic variation process of the G-Brownian motion $B(t)$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $B(t)$ is a one-dimensional process under the filtration $\mathcal{F}_{t \geq 0}$ satisfying the usual conditions. This paper presents the analysis of solutions for equations of the type (2). The remaining paper is arranged as follows. Section 2 presents the basic results. In Section 3, some useful lemmas are given. Section 4 investigates the mean square boundedness and convergence of solutions. In Section 5, we first study the $\mathbb{L}_G^2$ estimate and then derive the exponential estimate. Section 6 contains conclusions.

2. Basic Notions and Results

This section presents some fundamental concepts and results that we utilize in the following research work of this article [14,24,25]. The following two basic lemmas can be utilized in forthcoming sections of this paper [9].

Lemma 1. *Let $c, l \geq 0$ and $\delta \in (0, 1)$. Then*

$$(c + l)^2 \leq \frac{c^2}{\delta} + \frac{l^2}{1 - \delta}.$$

Lemma 2. *Let $p \geq 2$ and $\hat{\delta}, c, l > 0$. Then,*

(i) $c^{p-1} l \leq \frac{(p-1)\hat{\delta}c^p}{p} + \frac{l^p}{p\hat{\delta}^{p-1}}$.

(ii) $c^{p-2} l^2 \leq \frac{(p-2)\hat{\delta}c^p}{p} + \frac{2l^p}{p\hat{\delta}^{\frac{p-2}{2}}}$.

Let $\mathcal{H}$ be a space of real mappings defined on a non-empty set Ω.

Definition 1. $\forall\ x, y \in \mathcal{H}$, a functional $\hat{\mathbb{E}} : \mathcal{H} \to R$ assuring the below given features are called a G-expectation

1. $\hat{\mathbb{E}}[y] \geq \hat{\mathbb{E}}[x]$ whenever $y \geq x$.
2. $\hat{\mathbb{E}}[m_1] = m_1$, for any $m_1 \in R$.
3. $\hat{\mathbb{E}}[m_2 y] = m_2 \hat{\mathbb{E}}[y]$, for any $m_2 \in R^+$.
4. $\hat{\mathbb{E}}[y + x] \leq \hat{\mathbb{E}}[y] + \hat{\mathbb{E}}[x]$.

Suppose that Ω is the space of R^n-valued continuous paths $(w(t))_{t \geq 0}$ such that $w(0) = 0$ associated with the norm

$$\mathbb{D}(w^1, w^2) = \sum_{j=1}^{\infty} 2^{-j} \left(\max_{t \in [0,j]} |w^1(t) - w^2(t)| \wedge 1 \right).$$

Choose $\Omega_T = \{\omega \wedge T : \omega \in \Omega\}$ [14]. Assume the canonical process $B(t) = B(t, w)$ where $t \geq 0$ and $w \in \Omega$. Let $\lambda \in C_{b.Lip}(R^{n \times d})$ and $t_1, t_2, \ldots, t_n \in [0, T]$, then

$$\mathbb{L}_{ip}^0(\Omega_T) = \left\{ \lambda(B(t_1), B(t_2), \ldots, B(t_n)) : n \geq 1 \right\}.$$

Notice that $\mathbb{L}_{ip}^0(\Omega_t) \subseteq \mathbb{L}_{ip}^0(\Omega_T)$, $\mathbb{L}_{ip}^0(\Omega) = \cup_{m=1}^{\infty} \mathbb{L}_{ip}^0(\Omega_m)$ and the completion of $\mathbb{L}_{ip}^0(\Omega)$ associated with $\hat{\mathbb{E}}[|.|^p]^{\frac{1}{p}}$, $p \geq 1$ is $\mathbb{L}_G^p(\Omega)$. Related to $\{B(t)\}_{t \geq 0}$, we can express the filtration as $\mathcal{F}_t = \sigma\{B(u), 0 \leq u \leq t\}$ where $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Let $0 \leq t_0 \leq t_1 \leq \ldots \leq t_N < \infty$ and $\triangle_T = \{t_0, t_1, \ldots, t_N\}$ is a partition of $[0, T]$. Let $p \geq 1$, then $\mathbb{M}_G^{p,0}(0, T)$ is given by

$$\mathbb{M}_G^{p,0}(0, T) = \{\rho_t(w) = \sum_{o=0}^{N-1} \vartheta_j(w) I_{[t_o, t_{o+1}]}(t) : \vartheta_o \in \mathbb{L}_G^p(\Omega_{t_o}), o = 0, 1, \ldots, N-1\},$$

The space $\mathbb{M}_G^p(0, T)$ is the completion of $\mathbb{M}_G^{p,0}(0, T)$ under the norm $\|\rho\| = \left\{ \int_0^T \mathbb{E}[|\rho_s|^p] ds \right\}^{1/p}$, $p \geq 1$.

Definition 2. The G-Brownian motion is an n-dimensional stochastic process $\{B(t)\}_{t \geq 0}$ fulfilling the following characteristics:

(i) $B(0) = 0$.
(ii) $B(t + v) - B(t)$ is G-normally distributed and independent of $B(t_1), B(t_2), \ldots \ldots B(t_n)$ for any $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq, \ldots, \leq t_n \leq t$.

Definition 3. Let $\rho(t) \in \mathbb{M}_G^{2,0}(0, T)$. The G-Itô integral $\mathcal{I}(\rho)$ is defined as the stochastic integral of a function ρ with respect to G-Brownian motion given by

$$\mathcal{I}(\rho) = \int_0^T \rho(s) dB^a(s) = \sum_{k=0}^{N-1} \xi_k \left(B^a(t_{k+1}) - B^a(t_k) \right),$$

One can extend $\mathcal{I} : \mathbb{M}_G^{2,0}(0, T) \mapsto \mathbb{L}_G^2(\mathcal{F}_T)$ to $\mathcal{I} : \mathbb{M}_G^2(0, T) \mapsto \mathbb{L}_G^2(\mathcal{F}_T)$, where for $\rho \in \mathbb{M}_G^2(0, T)$ we have

$$\int_0^T \rho(s) dB^a(s) = \mathcal{I}(\rho).$$

Definition 4. Let $\langle B^a \rangle(0) = 0$. The G-quadratic variation process $\{\langle B^a \rangle(t)\}_{t \geq 0}$ is given as follows:

$$\langle B^a \rangle(t) = \lim_{N \to \infty} \sum_{j=0}^{N-1} \left(B^a(t_{j+1}^N) - B^a(t_j^N) \right)^2 = B^a(t)^2 - 2 \int_0^t B^a(s) dB^a(s).$$

Consider a function $\mathcal{K}_{0,T} : \mathbb{M}_G^{0,1}(0,T) \mapsto \mathbb{L}_G^2(\mathcal{F}_T)$ given as

$$\mathcal{K}_{0,T}(\rho) = \int_0^T \rho(s)d\langle B^a \rangle(s) = \sum_{i=0}^{N-1} \xi_i \Big(\langle B^a \rangle (t_{i+1}) - \langle B^a \rangle (t_i) \Big).$$

One can extend $\mathcal{K}_{0,T}$ to $\mathbb{M}_G^1(0,T)$. For $\rho \in \mathbb{M}_G^1(0,T)$, and it is given by

$$\int_0^T \rho(s)d\langle B^a \rangle(s) = \mathcal{K}_{0,T}(\rho).$$

Lemma 3 ([14]). *Assume that $\gamma \in \mathbb{M}_G^p(0,T)$ and $p \geq 2$. Then*

$$\hat{\mathbb{E}} \Big[\sup_{0 \leq t \leq T} \Big| \int_0^t \gamma(s)dB(s) \Big|^p \Big] \leq a_3 \hat{\mathbb{E}} \Big[\int_0^t |\gamma(s)|^2 ds \Big]^{\frac{p}{2}},$$

where $a_3 \in (0, \infty)$ is a p dependent constant.

Lemma 4 ([14]). *Assume that $\gamma \in \mathbb{M}_G^p(0,T)$, $p \geq 1$. Then,*

$$\hat{\mathbb{E}} \Big[\sup_{0 \leq t \leq T} \Big| \int_0^t \gamma(s)d\langle B, B \rangle(s) \Big|^p \Big] \leq a_2 \hat{\mathbb{E}} \Big[\int_0^t |\gamma(s)|^2 ds \Big]^{\frac{p}{2}},$$

where $a_2 \in (0, \infty)$ depends on p.

Lemma 5 ([26]). *Let $p \geq 0$, $z \in \mathbb{L}_G^p$. For each $m_1 > 0$,*

$$\nu(|z| > m_1) \leq \frac{\hat{\mathbb{E}}[|z|^p]}{m_1},$$

where $\hat{\mathbb{E}}|z|^p < \infty$.

3. Some Useful Results

In this section, we introduce and discuss some important assumptions and establish two lemmas. We consider the following hypotheses:

(H) *Let $a_i > 0$, $i = 1, 2, \ldots, 5$ and $\alpha(\sigma) - \beta(\sigma) = \Psi(\sigma)$. For any $\alpha, \beta \in \mathbb{C}_r((-\infty, 0]; R^n)$ and for any probability measure $\mu_1, \mu_2, \mu_3 \in \Lambda_{2r}$ the following inequalities hold*

$$[\Psi(0)]^T[\kappa(\alpha) - \kappa(\beta)] \leq -a_1|\Psi(0)|^2 + a_2 \int_{-\infty}^0 |\Psi(\sigma)|^2 \mu_1(d\sigma), \tag{3}$$

$$[\Psi(0)]^T[\eta(\alpha) - \eta(\beta)] \leq -a_3|\Psi(0)|^2 + a_4 \int_{-\infty}^0 |\Psi(\sigma)|^2 \mu_2(d\sigma), \tag{4}$$

$$|\gamma(\alpha) - \gamma(\beta)|^2 \leq a_5 \int_{-\infty}^0 |\Psi(\sigma)|^2 \mu_3(d\sigma). \tag{5}$$

Lemma 6. *Let $a < pr$ and $p \geq 1$. Then,*

$$\|z_t\|_r^p \leq e^{-at}\|\zeta\|_r^p + \sup_{0 < u \leq t} |z(u)|^p,$$

where $\zeta \in \mathbb{C}_r((-\infty, 0]; R^n)$.

Proof. Suppose $pr > a$. We can deduce the following by employing the definition of norm $\| \cdot \|_r$:

$$
\begin{aligned}
\|z_t\|_r^p &= \left[\sup_{-\infty < \sigma \leq 0} e^{r\sigma} |z(t + \sigma)| \right]^p \leq \sup_{-\infty < \sigma \leq 0} e^{a\sigma} |z(t + \sigma)|^p \\
&\leq \sup_{0 < u \leq t} e^{-a(t-u)} |z(u)|^p + \sup_{-\infty < u \leq 0} e^{-a(t-u)} |z(u)|^p \\
&= e^{-at} \|\zeta\|_r^p + e^{-at} \sup_{0 < u \leq t} e^{as} |z(u)|^p \\
&\leq \sup_{0 < u \leq t} |z(u)|^p + e^{-at} \|\zeta\|_r^p.
\end{aligned}
$$

The proof stands completed. $\square$

Throughout this paper, we let that for any $p \geq 1$, $a < pr$.

Lemma 7. *Let $a < pr$, $p \geq 2$ and $\mu_i \in \Lambda_k$, $\forall\, i \in N$. Then,*

$$
\int_0^t \int_{-\infty}^0 |z(\sigma + s)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr} \|\zeta\|_r^p + \int_0^t |z(s)|^p ds, \tag{6}
$$

$$
\int_0^t \int_{-\infty}^0 e^{as} |z(\sigma + s)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr - a} \|\zeta\|_r^p + \mu_i^{(pr)} \int_0^t e^{as} |z(s)|^p ds, \tag{7}
$$

where $\zeta \in \mathbb{C}_r((-\infty, 0]; R^n)$.

Proof. As for each $i \in Z^+$, $\mu_i \in \Lambda_{pr}$ and $\zeta \in C_r((-\infty, 0]; R^n)$, by using the Fubini theorem and the definition of norm, it follows that

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^0 |z(\sigma + s)|^p \mu_i(d\sigma) ds \\
&= \int_0^t \left[\int_{-\infty}^{-s} e^{pr(\sigma + s)} |z(\sigma + s)|^p e^{-pr(s+\sigma)} \mu_i(d\sigma) + \int_{-s}^0 |z(\sigma + s)|^p \mu_i(d\sigma) \right] ds \\
&\leq \|\zeta\|_r^p \int_0^t e^{-prs} ds \int_{-\infty}^0 e^{-pr\sigma} \mu_i(d\sigma) + \int_{-\infty}^0 \mu_i(d\sigma) \int_0^t |z(s)|^p ds.
\end{aligned}
$$

Observing that $\int_{-\infty}^0 \mu_i(d\sigma) = 1$ and $\int_{-\infty}^0 e^{-pr\sigma} \mu_i(d\sigma) = \mu_i^{(pr)}$, $i \in N$, it follows that

$$
\int_0^t \int_{-\infty}^0 |z(s + \sigma)|^p \mu_i(d\sigma) ds \leq \frac{\mu_i^{(pr)}}{pr} \|\zeta\|_r^p + \int_0^t |z(s)|^p ds.
$$

The proof of (6) is complete. Using similar arguments as used above we determine

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^0 e^{as} |z(\sigma + s)|^p \mu_i(d\sigma) ds \\
&= \int_0^t e^{as} ds \left[\int_{-\infty}^{-s} |z(\sigma + s)|^p \mu_i(d\sigma) + \int_{-s}^0 |z(\sigma + s)|^p \mu_i(d\sigma) \right] \\
&= \int_0^t e^{as} ds \int_{-\infty}^{-s} |z(\sigma + s)|^p \mu_i(d\sigma) + \int_{-t}^0 \mu_i(d\sigma) \int_{-\sigma}^t e^{as} |z(\sigma + s)|^p ds \\
&\leq \int_0^t e^{as} ds \int_{-\infty}^{-s} e^{pr(\sigma + s)} |z(s + \sigma)|^p e^{-pr(\sigma + s)} \mu_i(d\sigma) + \int_{-\infty}^0 \mu_i(d\sigma) \int_0^t e^{a(s - \sigma)} |z(s)|^p ds \\
&\leq \|\zeta\|_r^p \int_0^t e^{-(pr - a)s} ds \int_{-\infty}^0 e^{-pr\sigma} \mu_i(d\sigma) + \int_{-\infty}^0 e^{-a\sigma} \mu_i(d\sigma) \int_0^t e^{as} |z(s)|^p ds.
\end{aligned}
$$

With reference to Equation (1), and taking note that $pr > a$, we can conclude that

$$\int_0^t \int_{-\infty}^0 e^{as}|z(\sigma+s)|^p \mu_i(d\sigma)ds \leq \frac{\mu_i^{(pr)}}{pr-a}\|\zeta\|_r^p + \mu_i^{(pr)}\int_0^t e^{as}|z(s)|^p ds.$$

The proof of (7) is complete. □

4. Convergence and Mean Square Boundedness

Firstly, let us derive the mean square boundedness for the solutions to Equation (2).

Theorem 1. *Let the inequalities* (3)–(5) *be satisfied. Assume Equation* (2) *with initial condition* $\zeta \in C_r((-\infty, 0]; R^n)$ *has just one solution* $z(t)$. *Let* a_i, $i = 1, 2, \ldots, 5$ *assure* $2a_1 > 2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)} - 2b_1a_3$. *Then there is* $a \in (0, (2a_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \wedge 2r)$ *so that*

$$\hat{\mathbb{E}}[|z(t)|^2] \leq c_1 + c_2 e^{-at}, \tag{8}$$

where

$$c_1 = \frac{1}{a}\left(\frac{1}{\delta}|\kappa(0)|^2 + \frac{b_1}{\delta_1}|\eta(0)|^2 + \frac{b_1}{\delta_2}|\gamma(0)|^2\right)$$

and

$$c_2 = \hat{\mathbb{E}}|z(0)|^2 + \frac{2a_2\mu_1^{(2r)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{2b_1a_4\mu_2^{(2r)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{b_1a_5\mu_3^{(2r)}}{(2r-a)(1-\delta_2)}\hat{\mathbb{E}}\|\zeta\|_r^2.$$

The values of δ, δ_1 *and* δ_2 *are sufficiently small so that*

$$2a_1 - \delta - a - b_1\delta_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - \frac{b_1a_5}{1-\delta_2}\mu_3^{(2r)} > 0.$$

Proof. By using the G-Itô formula, G-Itô integral and Lemma 4, it follows

$$\hat{\mathbb{E}}[e^{at}|z(t)|^2] \leq \hat{\mathbb{E}}|z(0)|^2 + \hat{\mathbb{E}}\int_0^t e^{as}\left[a|z(s)|^2 + 2z^T(s)\kappa(z_s)\right]ds$$
$$+ b_1\hat{\mathbb{E}}\int_0^t e^{as}\left[2z^T(s)\eta(z_s) + |\gamma(z_s)|^2\right]ds. \tag{9}$$

Utilizing (3), (4) and Lemma 2, we determine

$$z^T(t)\kappa(z_t) \leq (\frac{\delta}{2} - a_1)|z(t)|^2 + \frac{1}{2\delta}|\kappa(0)|^2 + a_2\int_{-\infty}^0 |z(t+\sigma)|^2\mu_1(d\sigma),$$

$$z^T(t)\eta(z_t) \leq (\frac{\delta_1}{2} - a_3)|z(t)|^2 + \frac{1}{2\delta_1}|\eta(0)|^2 + a_4\int_{-\infty}^0 |z(t+\sigma)|^2\mu_2(d\sigma).$$

It follows from Lemma 1 and the condition given in (5) that

$$|\gamma(z_t)|^2 \leq \frac{1}{\delta_2}|\gamma(0)|^2 + \frac{a_5}{1-\delta_2}\int_{-\infty}^0 |z(t+\sigma)|^2\mu_3(d\sigma).$$

Using the above inequalities, (9) becomes

$$
\begin{aligned}
\hat{\mathbb{E}}[e^{at}|z(t)|^2] \leq{}& \hat{\mathbb{E}}|z(0)|^2 + \frac{1}{a}\Big(\frac{1}{\delta}|\kappa(0)|^2 + \frac{b_1}{\delta_1}|\eta(0)|^2 + \frac{b_1}{\delta_2}|\gamma(0)|^2\Big)(e^{at}-1) \\
&+ (\delta + a - 2a_1 + b_1\delta_1 - 2b_1a_3)\hat{\mathbb{E}}\int_0^t e^{as}|z(s)|^2 ds \\
&+ 2a_2\hat{\mathbb{E}}\int_0^t e^{as}\int_{-\infty}^0 |z(s+\sigma)|^2\mu_1(d\sigma)ds \\
&+ 2b_1a_4\hat{\mathbb{E}}\int_0^t e^{as}\int_{-\infty}^0 |z(s+\sigma)|^2\mu_2(d\sigma)ds \\
&+ b_1\frac{a_5}{1-\delta_2}\hat{\mathbb{E}}\int_0^t e^{as}\int_{-\infty}^0 |z(s+\sigma)|^2\mu_3(d\sigma)ds.
\end{aligned}
\tag{10}
$$

In view of Lemma 7, it follows

$$
\int_0^t \int_{-\infty}^0 e^{as}|z(s+\sigma)|^2\mu_i(d\sigma)ds \leq \frac{1}{2r-a}\|\zeta\|_r^2 \mu_i^{(2r)} + \mu_i^{(2r)}\int_0^t e^{as}|z(s)|^2 ds.
\tag{11}
$$

Substituting (11) in (10), we derive

$$
\begin{aligned}
\hat{\mathbb{E}}[e^{at}|z(t)|^2] \leq{}& \hat{\mathbb{E}}|z(0)|^2 + \frac{2a_2\mu_1^{(2r)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{2b_1a_4\mu_2^{(2r)}}{2-ra}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{b_1a_5\mu_3^{(2r)}}{(2r-a)(1-\delta_2)}\hat{\mathbb{E}}\|\zeta\|_r^2 \\
&+ \frac{1}{\lambda}\Big(\frac{1}{\delta}|\kappa(0)|^2 + \frac{b_1}{\delta_1}|\eta(0)|^2 + \frac{b_1}{\delta_2}|\gamma(0)|^2\Big)(e^{at}-1) \\
&- \Big(2a_1 - \delta - a - b_1\delta_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - \frac{b_1a_5}{1-\delta_2}\mu_3^{(2r)}\Big)\hat{\mathbb{E}}\int_0^t e^{as}|z(s)|^2 ds.
\end{aligned}
$$

As $2a_1 > 2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)} - 2b_1a_3$ and $a \in (0, (2a_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \wedge 2r)$. Selecting δ, δ_1 and δ_2 sufficiently small so that

$$
2a_1 - \delta - a - b_1\delta_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - \frac{b_1a_5}{1-\delta_2}\mu_3^{(2r)} > 0,
$$

we obtain the desired result:
$$
\hat{\mathbb{E}}[|z(t)|^2] \leq c_1 + c_2 e^{-at},
$$

where
$$
c_1 = \frac{1}{a}\Big(\frac{1}{\delta}|\kappa(0)|^2 + \frac{b_1}{\delta_1}|\eta(0)|^2 + \frac{b_1}{\delta_2}|\gamma(0)|^2\Big)
$$

and
$$
c_2 = \hat{\mathbb{E}}|z_0|^2 + \frac{2a_2\mu_1^{(2q)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{2b_1a_4\mu_2^{(2r)}}{2r-a}\hat{\mathbb{E}}\|\zeta\|_r^2 + \frac{b_1a_5\mu_3^{(2r)}}{(2r-a)(1-\delta_2)}\hat{\mathbb{E}}\|\zeta\|_r^2.
$$

$\square$

Theorem 1 describes that Equation (2) has a mean square bounded solution. The following Theorem 2 expresses that any two distinct solutions of Equation (2) are convergent.

Theorem 2. *Assuming that all hypotheses of Theorem 1 are satisfied, let $z(t)$ and $y(t)$ be two solutions of Equation (2) associated with initial values ζ and ξ, respectively. Then, we have*

$$
\hat{\mathbb{E}}[|z(t) - y(t)|^2] \leq c_3\hat{\mathbb{E}}|\xi - \zeta|_r^2 e^{-at},
\tag{12}
$$

where $c_3 = 1 + \frac{1}{2r-a}(2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)})$.

Proof. Define $\phi(t) = z(t) - y(t)$, $\hat{\gamma}(t) = \gamma(z_t) - \gamma(y_t)$, $\hat{\kappa}(t) = \kappa(z_t) - \kappa(y_t)$, and $\hat{\eta}(t) = \eta(z_t) - \eta(y_t)$. Utilizing the G-Itô integral, Lemma 4, and G-Itô formula, it follows that

$$\hat{\mathbb{E}}[e^{at}|\phi(t)|^2] \leq \hat{\mathbb{E}}|\xi(0) - \zeta(0)|^2 + \hat{\mathbb{E}} \int_0^t e^{as}[a|\phi(s)|^2 + 2\phi^T(s)\hat{\kappa}(s)]ds \tag{13}$$
$$+ b_1 \hat{\mathbb{E}} \int_0^t e^{\phi s}[2\phi^T(s)\hat{\eta}(s) + |\hat{\gamma}(s)|^2]ds.$$

From hypothesis H, we derive

$$\phi^T(t)\hat{\kappa}(t) \leq -a_1|\phi(t)|^2 + a_2 \int_{-\infty}^0 \phi(\sigma + t)\mu_1(d\sigma),$$
$$\phi^T(t)\hat{\eta}(t) \leq -a_3|\phi(t)|^2 + a_4 \int_{-\infty}^0 \phi(\sigma + t)\mu_2(d\sigma)$$

and

$$|\hat{\gamma}(t)|^2 \leq a_5 \int_{-\infty}^0 \phi(t + \sigma)\mu_3(d\sigma).$$

Utilizing the above inequalities, (13) becomes

$$\hat{\mathbb{E}}[e^{at}|\phi(t)|^2] \leq \hat{\mathbb{E}}|\xi(0) - \zeta(0)|^2 + (a - 2a_1 - 2b_1a_3)\hat{\mathbb{E}} \int_0^t e^{as}|\phi(s)|^2 ds$$
$$+ 2a_2\hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 e^{as}\phi(s + \sigma)\mu_1(d\sigma)ds + 2b_1a_4\hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 e^{as}\phi(s + \sigma)\mu_2(d\sigma)ds \tag{14}$$
$$+ k_1a_5\hat{\mathbb{E}} \int_0^t \int_{-\infty}^0 e^{as}\phi(s + \sigma)\mu_3(d\sigma)ds.$$

From Lemma 7 for $i = 1, 2, 3$, it follows

$$\int_0^t \int_{-\infty}^0 e^{as}|\phi(s + \sigma)|^2 \mu_i(d\sigma)ds \leq \frac{1}{2q - a}\|\zeta - \xi\|_r^2 \mu_i^{(2r)} + \mu_i^{(2r)} \int_0^t e^{as}|\phi(s)|^2 ds. \tag{15}$$

Plugging (15) in (14), we determine

$$e^{at}\hat{\mathbb{E}}|\phi(t)|^2 \leq \hat{\mathbb{E}}|\xi(0) - \zeta(0)|^2 + \frac{1}{2r - a}[2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)}]\hat{\mathbb{E}}\|\zeta - \xi\|_r^2$$
$$- (2a_1 + 2b_1a_3 - a - 2a_2\mu_1^{(2r)} - 2k_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)})\hat{\mathbb{E}} \int_0^t e^{as}|\phi(s)|^2 ds.$$

As $2a_1 > 2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)} - 2b_1a_3$ and $a \in (0, (2a_1 + 2b_1a_3 - 2a_2\mu_1^{(2r)} - 2b_1a_4\mu_2^{(2r)} - b_1a_5\mu_3^{(2r)}) \wedge 2r)$ it follows that

$$\hat{\mathbb{E}}|\phi(t)|^2 \leq [1 + \frac{1}{2r - a}(2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)})]\hat{\mathbb{E}}\|\zeta - \xi\|_r^2 e^{-at},$$

consequently, we derive the following required result

$$\hat{\mathbb{E}}|z(t) - y(t)|^2 \leq c_3\hat{\mathbb{E}}\|\zeta - \xi\|_r^2 e^{-at},$$

where $c_3 = 1 + \frac{1}{2r-a}(2a_2\mu_1^{(2r)} + 2b_1a_4\mu_2^{(2r)} + b_1a_5\mu_3^{(2r)})$. $\square$

If $\kappa(0) = \eta(0) = \gamma(0) = 0$, then from Theorem 2 we can obtain that the trivial solution of Equation (2) is mean square exponentially stable.

Example 1. *Consider $z(t)$ and $y(t)$ as two solutions of the equation*

$$dz(t) = z_t dt + \sin(z_t) d\langle B\rangle(t) + z_t dB(t)$$

with initial values ζ and ξ, respectively. Define $\phi(t) = z(t) - y(t)$, $\hat{\eta}(t) = \hat{\gamma}(t) = z_t - y_t$ and $\hat{\kappa}(t) = \sin(z_t) - \sin(y_t)$. Under the given hypothesis one can easily derive that $y(t)$ is mean square convergent to $z(t)$.

5. The Exponential Estimate

Firstly, let us determine the $\mathcal{L}_G^2$ estimates. Let Equation (2) with initial condition $\zeta \in C_r((-\infty, 0]; R^n)$ has just one solution $z(t)$ on $t \in [0, \infty)$.

Theorem 3. *Assume that the hypothesis H holds and $E\|\zeta\|_r^2 < \infty$. For every $t \geq 0$,*

$$\hat{\mathbb{E}}\Big[\sup_{-\infty < s \leq t} |z(t)|^2 \Big] \leq [\hat{\mathbb{E}}\|\zeta\|_r^2 + m_1]e^{m_2 t},$$

where $m_1 = c + \frac{2}{r}[r + a_2\mu_1^{(2r)} + b_1(a_5\mu_3^{(2r)} + \mu_2^{(2r)}) + 2b_3 a_5 \mu_3^{(2r)}]\hat{\mathbb{E}}\|\zeta\|_r^2$, $c = 2[|\kappa(0)|^2 + b_1(|\eta(0)|^2 + 2|\gamma(0)|^2) + 4b_3|\gamma(0)|^2]T$ and $m_2 = 2[2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3 a_5]$.

Proof. Using the G-Itô formula and properties of the G-expectation, it follows that

$$\begin{aligned}
\hat{\mathbb{E}}\Big[\sup_{0 \leq s \leq t} |z(t)|^2 \Big] \leq{}& \hat{\mathbb{E}}|z(0)|^2 + 2\hat{\mathbb{E}}\Big[\sup_{0 \leq s \leq t} \int_0^t z^T(s)\kappa(z_s)ds \Big] \\
&+ \hat{\mathbb{E}}\Big[\sup_{0 \leq s \leq t} \int_0^t (2z^T(s)\eta(z_s) + |\gamma(z_s)|^2)d\langle B, B\rangle(s) \Big] \qquad (16) \\
&+ 2\hat{\mathbb{E}}\Big[\sup_{0 \leq s \leq t} \int_0^t z^T(s)\gamma(z_s)dB(s) \Big].
\end{aligned}$$

From our assumption H, $2a_1 a_2 \leq \sum_{i=1}^2 a_i^2$ and $(\sum_{i=1}^2 a_i)^2 \leq 2\sum_{i=1}^2 a_i^2$, it follows that

$$z^T(t)\kappa(z_t) \leq -(a_1 - \frac{1}{2})|z(t)|^2 + \frac{1}{2}|\kappa(0)|^2 + a_2 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_1(d\sigma), \qquad (17)$$

$$z^T(t)\eta(z_t) \leq -(a_3 - \frac{1}{2})|z(t)|^2 + \frac{1}{2}|\eta(0)|^2 + a_4 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_2(d\sigma), \qquad (18)$$

$$|\gamma(z_t)|^2 \leq 2|\gamma(0)|^2 + 2a_5 \int_{-\infty}^0 |z(t+\sigma)|^2 \mu_3(d\sigma). \qquad (19)$$

In view of (17) and (11), we determine that

$$2\hat{\mathbb{E}}\Big[\sup_{0 < s \leq t} \int_0^t z^T(s)\kappa(z_s)ds \Big] \leq |\kappa(0)|^2 T + \frac{a_2}{r}\hat{\mathbb{E}}\|\zeta\|_r^2 \mu_1^{(2r)} + (2a_2 - 2a_1 + 1)\hat{\mathbb{E}}\int_0^t |z(s)|^2 ds.$$

Utilizing (18), (19), (11) and Lemma 4, it follows that

$$\begin{aligned}
\hat{\mathbb{E}}\Big[\sup_{0 \leq s \leq t} \int_0^t (2z^T(s)\eta(z_s) + |\gamma(z_s)|^2)d\langle B, B\rangle(s) \Big] &\leq b_1\hat{\mathbb{E}}\Big[\int_0^t (2z^T(s)\eta(z_s) + |\gamma(z_s)|^2) \Big]ds \\
&\leq b_1[|\eta(0)|^2 + 2|\gamma(0)|^2]T + b_1\frac{1}{r}(a_5\mu_3^{(2r)} + \mu_2^{(2r)})\hat{\mathbb{E}}\|\zeta\|_r^2 \\
&+ b_1(2a_5 - 2a_3 + 3)\hat{\mathbb{E}}\int_0^t |z(s)|^2 ds.
\end{aligned}$$

The inequality $a_1 a_2 \leq \frac{1}{2}\sum_{i=1}^{2} a_i$, Lemma 3 and (19) give

$$2\hat{\mathbb{E}}\Big[\sup_{0<s\leq t}\int_0^t z^T(s)\gamma(z_s)dB(t)\Big] \leq 2b_2\hat{\mathbb{E}}\Big[\int_0^t |z^T(s)\gamma(z_s)|^2 ds\Big]^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\hat{\mathbb{E}}\Big[\sup_{0<s\leq t}|z(s)|^2\Big] + 4b_2^2|\gamma(0)|^2 T + 4b_2^2 a_5\hat{\mathbb{E}}\int_0^t \int_{-\infty}^0 |z(s+\sigma)|^2\mu_3(d\sigma)ds.$$

Using Lemma 7, we derive

$$2\hat{\mathbb{E}}\Big[\sup_{0<s\leq t}\int_0^t z^T(s)\gamma(z_s)dB(s)\Big]$$
$$\leq \frac{1}{2}\hat{\mathbb{E}}\Big[\sup_{0<s\leq t}|z(s)|^2\Big] + 4b_3|\gamma(0)|^2 T + \frac{4b_3 a_5}{2r}\mu_3^{(2r)}\hat{\mathbb{E}}\|\zeta\|_r^2 + 4b_3 a_5\hat{\mathbb{E}}\int_0^t |z(s)|^2 ds, \tag{20}$$

where $b_3 = b_2^2$. By substituting the aforementioned inequalities into Equation (16), and letting $m_1 = c + \frac{2}{r}[r + a_2\mu_1^{(2r)} + b_1(a_5\mu_3^{(2r)} + \mu_2^{(2r)}) + 2b_3 a_5\mu_3^{(2r)}]E\|\zeta\|_r^2$, $m_2 = 2[2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3 a_5]$, we can evaluate the result:

$$\hat{\mathbb{E}}[\sup_{0\leq s\leq t}|z(t)|^2] \leq m_1 + m_2\int_0^t \hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t}|z(s)|^2\Big]ds, \tag{21}$$

where $c = 2\Big[|\kappa(0)|^2 + b_1(|\eta(0)|^2 + 2|\gamma(0)|^2) + 4b_3|\gamma(0)|^2\Big]T$. By observing that

$$\hat{\mathbb{E}}[\sup_{-\infty<s\leq t}|z(s)|^2] \leq \hat{\mathbb{E}}\|\zeta\|_r^2 + \hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t}|z(s)|^2\Big],$$

it follows that

$$\hat{\mathbb{E}}\Big[\sup_{-\infty<s\leq t}|z(s)|^2\Big] \leq \hat{\mathbb{E}}\|\zeta\|_r^2 + m_1 + m_2\int_0^t \hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t}|z(s)|^2\Big]ds$$

$$\leq \hat{\mathbb{E}}\|\zeta\|_r^2 + m_1 + m_2\int_0^t \hat{\mathbb{E}}\Big[\sup_{-\infty<s\leq t}|z(s)|^2\Big]ds.$$

Finally, the required result is obtained by using the Grownwall inequality. □

Theorem 4. *Under the conditions of Theorem 3, it follows that*

$$\limsup_{t\to\infty}\frac{1}{t}log|z(t)| \leq \alpha,$$

where $\alpha = 2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3 a_5$ *and* $a_1, a_2, \ldots$ *are positive constants.*

Proof. Assuming that $m_1 = m + \frac{2}{r}[r + a_2\mu_1^{(2r)} + b_1(a_5\mu_3^{(2r)} + \mu_2^{(2r)}) + 2b_3 a_5\mu_3^{(2r)}]E\|\zeta\|_r^2$ and $m_2 = 2[2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3 a_5]$ then from the inequality (21), we can conclude that

$$\hat{\mathbb{E}}\Big[\sup_{0\leq s\leq t}|z(s)|^2\Big] \leq m_1 e^{m_2 t}, \tag{22}$$

where $m = 2\Big[|\kappa(0)|^2 + b_1(|\eta(0)|^2 + 2|\gamma(0)|^2) + 4b_3|\gamma(0)|^2\Big]T$. For each $q = 1, 2, 3, \ldots$, from (22) it follows that

$$\hat{\mathbb{E}}\Big[\sup_{q-1\leq t\leq q}|z(t)|^2\Big] \leq m_1 e^{m_2 q}.$$

By utilizing Lemma 5 for every given $\delta > 0$, we obtain

$$\nu\left\{w: \sup_{q-1\leq t\leq q} |z(t)|^2 > e^{(m_2+\delta)q}\right\} \leq \frac{\hat{\mathbb{E}}\left[\sup_{q-1\leq t\leq q} |z(t)|^2\right]}{e^{(m_2+\delta)q}}$$

$$\leq \frac{m_1 e^{m_2 q}}{e^{(m_2+\delta)q}}$$

$$= m_1 e^{-\delta q}.$$

But the Borel–Cantelli lemma gives that for almost every $w \in \Omega$ there is a random number $q_0 = q_0(w) \in Z$ in a manner that when $q \geq q_0$, then

$$\sup_{q-1\leq t\leq q} |z(t)|^2 \leq e^{(m_2+\delta)q},$$

which implies

$$\limsup_{t\to\infty} \frac{1}{t} log|z(t)| \leq \frac{m_2 + \delta}{2}$$

$$= 2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3 a_5 + \frac{\delta}{2},$$

as δ is arbitrary and letting $\alpha = 2a_2 - 2a_1 + 1 + b_1(2a_5 - 2a_3 + 3) + 4b_3 a_5$, we can conclude that

$$\alpha \geq \limsup_{t\to\infty} \frac{1}{t} log|z(t)|.$$

The proof stands completed. □

The lemma above expresses that the second moment of the Lyapunov exponent, as defined in [27] as $\lim_{t\to\infty} \sup \frac{1}{t} \log |z(t)|$, is bounded above by α.

6. Conclusions

Several stochastic functional differential equations (SFDEs) in financial mathematics do not hold the standard Lipschitz assumption such as the Cox–Ingersoll–Ross, Heston and Ait–Sahalia models. In this article, some useful monotone-type conditions have been introduced. We have proved that any two solutions of SFDEs in the G-framework under distinct initial conditions are convergent. The solutions are mean square bounded. The $\mathcal{L}_G^2$ and exponential estimates have been calculated. We anticipate that the findings presented in this article will offer valuable insights into the analysis of equations, even when not under the constraints of standard assumptions. This contribution is poised to have a substantial positive impact on the examination of various unresolved inquiries, including the investigation into the existence, uniqueness, convergence and stability of solutions for backward and forward stochastic dynamic systems driven by G-Brownian motion with conditions of a monotone nature.

Author Contributions: Conceptualization, supervision, F.F.; methodology, R.U.; formal analysis, Q.Z.; investigation, F.F. and R.U. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: All authors have no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Li, X.; Fu, X. Stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks. *J. Comput. Appl. Math.* **2010**, *234*, 407–417. [CrossRef]
2. Chang, M.H.; Youree, R.K. The European option with hereditary price structures: Basic theory. *Appl. Math. Comput.* **1999**, *102*, 279–296. [CrossRef]
3. Takeuchi, A. Joint distributions for stochastic functional differential equations. *Int. J. Probab. Stoch. Process.* **2016**, *5*, 711–736. [CrossRef]
4. Zheng, Y. Asset Pricing Based on Stochastic Delay Differential Equations. Ph.D. Thesis, Iowa State University, Ames, IA, USA, 2015.
5. Bahar, A.; Mao, X. Stochastic delay Lotka-Volterra model. *J. Math. Anal. Appl.* **2004**, *292*, 364–380. [CrossRef]
6. Mao, X.; Yuan, C.; Zou, J. Stochastic differential delay equations of population dynamics. *J. Math. Anal. Appl.* **2005**, *304*, 296–320. [CrossRef]
7. Miekisz, J.; Poleszczuk, J.; Bodnar, M.; Fory's, U. Stochastic models of gene expression with delayed degradation. *Bull. Math. Biol.* **2011**, *73*, 2231–2247. [CrossRef]
8. Waclawczyk, M.; Grebenev, N.V.; Oberlack, M. Conformal invariance of the 1-point statistics of the zero-isolines of 2d scalar fields in inverse turbulent cascades. *Phys. Rev. Fluids* **2021**, *6*, 084610. [CrossRef]
9. Mao, X. *Stochastic Differential Equations and Their Applications*; Horwood Publishing: Chichester, UK, 1997.
10. Mao, X. *Exponential Stability of Stochastic Differential Equations*; Marcel Dekker, Inc.: New York, NY, USA, 1994.
11. Mao, X.; Yuan, C. *Stochastic Differential Equations with Markovian Switching*; Imperial College Press: London, UK, 2006.
12. Mohammed, S.-E.A. *Stochastic Functional Differential Equations*; Research Notes in Mathematics 99; Pitman Advanced Publishing Program: Boston, MA, USA, 1984.
13. Mohammed, S.-E.A.; Scheutzow, M.K.R. The stable manifold theorem for non-linear stochastic systems with memory. I. Existence of the semiflow. *J. Funct. Anal.* **2003**, *205*, 271–305. [CrossRef]
14. Peng, S. Multi-dimentional G-Brownian motion and related stochastic calculus under G-expectation. *Stoch. Proc. Appl.* **2008**, *118*, 2223–2253 [CrossRef]
15. Ren, Y.; Bi, Q.; Sakthivel, R. Stochastic functional differential equations with infinite delay driven by G-Brownian motion. *Math. Method Appl Sci.* **2013**, *36*, 1746–1759. [CrossRef]
16. Faizullah, F. Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions. *J. Comput. Anal. Appl.* **2017**, *2*, 344–354.
17. Faizullah, F.; Zhu, Q.; Ullah, R. The existence-uniqueness and exponential estimate of solutions for stochastic functional differential equations driven by G-Brownian motion. *Math. Methods Appl. Sci.* **2020**, *2020*, 1–12. [CrossRef]
18. Faizullah, F. A note on p-th moment estimates for stochastic functional differential equations in the framework of G-Brownian motion. *Iran. J. Sci. Technol. Trans. Sci.* **2017**, *41*, 1131–1138. [CrossRef]
19. Faizullah, F. Existence of solutions for G-SFDEs with Cauchy-Maruyama Approximation Scheme. *Abst. Appl. Anal.* **2014**, *2014*, 809431. [CrossRef]
20. Wei, W.; Zhang, M.; Luo, P. Asympyotic estimates for the solution of stochastic differential equations driven by G-Brownian motion. *Appl. Anal.* **2017**, *97*, 2025–2036. [CrossRef]
21. Faizullah, F.; Ullah, R.; Zhu, Q. On the existence and mean square stability of neutral stochastic functional differential equations driven by G-Levy process. *Math. Methods Appl. Sci.* **2022**, *2022*, 1–12. [CrossRef]
22. Kuang, Y.; Smith, H.L. Global stability for infinite delay Lotka-Volterra type system. *J. Differ. Equ.* **1993**, *103*, 221–246. [CrossRef]
23. Wu, F.; Yin, G.; Mei, H. Stochastic functional differential equations with infinite delay: Existence and uniqueness of solutions, solution maps, Markov properties, and ergodicity. *J. Differ. Equ.* **2017**, *262*, 1226–1252. [CrossRef]
24. Bai, X.; Lin, Y. On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with Integral-Lipschitz coefficients. *Acta Math. Appl. Sin. Engl. Ser.* **2014**, *3*, 589–610. [CrossRef]
25. Luo, P.; Wang, F. On the comparison theorem for multi-dimensional G-SDEs. *Stat. Probab. Lett.* **2015**, *96*, 38–44. [CrossRef]
26. Denis, L.; Hu, M.; Peng, S. Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths. *Potential Anal.* **2011**, *34*, 139–161. [CrossRef]
27. Kim, Y.H. On the pth moment extimats for the slutions of stochastic differential equations. *J. Inequal. Appl.* **2014**, *2014*, 395. [CrossRef]