mathematics

Article

Trajectory and Global Attractors for the Kelvin–Voigt Model Taking into Account Memory along Fluid Trajectories

Mikhail Turbin * and **Anastasiia Ustiuzhaninova**

Research Institute of Mathematics, Voronezh State University, Universitetskaya pl. 1, 394018 Voronezh, Russia; nastyzhka@gmail.com
* Correspondence: mrmike@mail.ru

Abstract: This article is devoted to the study of the existence of trajectory and global attractors in the Kelvin–Voigt fluid model, taking into account memory along the trajectories of fluid motion. For the model under study, the concept of a weak solution on a finite segment and semi-axis is introduced and the existence of their solutions is proved. The necessary exponential estimates for the solutions are established. Then, based on these estimates, the existence of trajectory and global attractors in the problem under study is proved.

Keywords: trajectory attractor; global attractor; Kelvin–Voigt model; regular Lagrangian flow; boundary value problem; non-Newtonian fluid; a priory estimate; existence theorem

MSC: 35B41; 35Q35; 76A10

check for updates

Citation: Turbin, M.; Ustiuzhaninova, A. Trajectory and Global Attractors for the Kelvin–Voigt Model Taking into Account Memory along Fluid Trajectories. *Mathematics* **2024**, *12*, 266. https://doi.org/10.3390/math12020266

Academic Editor: Marco Pedroni

Received: 23 November 2023
Revised: 6 January 2024
Accepted: 12 January 2024
Published: 14 January 2024

1. Introduction

The study of mathematical fluid dynamics problems is one of the most important problems in the field of mathematics. Of particular interest in the study of such problems is the study of the limiting behavior of solutions, namely, the behavior of solutions as time tends to infinity. In some problems, it is possible to prove that solutions can tend to a certain set in the phase space. Here, a phase space is understood as a set whose elements are identified with the states of the system. That is, regardless of the initial conditions of the problem, its solutions turn out to be in this set, possibly after a sufficiently long time. Such sets are called attractors because solutions are attracted to them. Thus, it is natural to study the attractors of the system, since states that do not belong to attractors do not affect the system.

One of the first papers on the theory of the attractors of fluid dynamics equations is an article by Ladyzhenskaya [1]. In this work, the existence of a global attractor for the two-dimensional Navier–Stokes system was proved. The proof is based on the theory of dynamical systems. The idea of using the theory of dynamical systems for the study of attractors for equations of mathematical physics was further developed in a large number of papers; see, for more details, the review paper by Ladyzhenskaya [2] and monograph by Temam [3]. We especially note paper [4], in which the existence of a global attractor for the 2D Bingham model was proved.

However, the theory of dynamical systems requires the uniqueness of a global solution to the problem under consideration. But for fluid dynamics equations, this property turns out to be limiting and is often not satisfied. For example, for the 3D Navier–Stokes system, the uniqueness of weak solutions has not been established, and for strong solutions, non-local existence theorems have not been proved. Just for the 3D Navier–Stokes system, in order to overcome these difficulties, Vishik and Chepyzhov created the theory of trajectory attractors [5,6]. Around the same time, independently of these authors, a similar theory for the 3D Navier–Stokes system was created by Sell [7].

In the theory of trajectory attractors, instead of a semi-group of evolutionary operators, a certain set of functions that depend on time and take values in the phase space is considered. This set of functions is called the trajectory space, and the functions belonging to it are called trajectories. Each trajectory represents a certain version of the system development. The theory of trajectory spaces makes it possible to bypass the requirement of uniqueness of the solution. In the case under consideration, several trajectories can emerge from a certain point in the phase space. Or, what is the same, for the same initial condition, there may be several solutions.

Subsequently, the theory of trajectory attractors was developed in the papers of Zvyagin and Vorotnikov [8,9]. In particular, they managed to abandon the condition of the translational invariance of the trajectory space. This condition is unnecessarily restrictive and is often not satisfied in fluid dynamics problems. The point is that the trajectory spaces in the theory under consideration are usually constructed on the basis of energy estimates. It is not always possible to obtain the required translation-invariant estimate. But it is often possible to establish an exponential estimate, which, thanks to the results of Zvyagin and Vorotnikov, turns out to be quite sufficient.

In real applications in chemistry, medicine and the pharmaceutical industry, fluid models that do not satisfy the Newtonian rheological relation often arise. At the moment, there is a fairly large number of models of such fluids, called non-Newtonian. This article deals with one of such models, namely, the Kelvin–Voigt model of fluid motion of order $L, L = 1, 2, \ldots$ with the full derivative with respect to time in the rheological relation:

$$\left(1+\sum_{i=1}^{L}\lambda_i\frac{d^i}{dt^i}\right)\sigma(t,x)=2\left(\nu+\sum_{i=1}^{L+1}\varkappa_i\frac{d^i}{dt^i}\right)\mathcal{E}(v)(t,x),\quad (t,x)\in[0,T]\times\Omega. \tag{1}$$

Here, $\Omega\subset\mathbb{R}^n, n=2,3$, is a bounded domain with a smooth boundary $\partial\Omega$, $[0,T]$ is a time interval, σ is the deviator of the stress tensor, $\mathcal{E}(v)$ is the strain rate tensor, $\lambda_i, i=\overline{1,L}$ are the relaxation times, ν is the fluid viscosity, $\varkappa_i, i=\overline{1,L+1}$ are the retardation times, and $\frac{d}{dt}=\frac{\partial}{\partial t}+\sum_{i=1}^{n}v_i\frac{\partial}{\partial x_i}$ is the full (substantial) derivative with respect to time.

This model describes the motion of various solutions and melts of polymers, such as solutions of polyethylene oxide, polyacrylamide and guar gum [10], and has been confirmed experimentally [11,12]. It is one of the models of linear viscoelastic fluids with a finite number of discretely distributed relaxation and retardation times. The general theory of such fluids, including the Kelvin–Voigt model, was built on the basis of the Boltzmann superposition principle. According to this principle, all influences on the medium are independent and additive and the reactions of the medium to external influences are linear [13].

From a mathematical point of view, various simplifications of this model began to be studied in Oskolkov's papers (see review paper [14] and references therein). Oskolkov's works considered a model in which the full time derivative in rheological relation (1) was replaced with a partial one:

$$\left(1+\sum_{i=1}^{L}\lambda_i\frac{\partial^i}{\partial t^i}\right)\sigma(t,x)=2\left(\nu+\sum_{i=1}^{L+1}\varkappa_i\frac{\partial^i}{\partial t^i}\right)\mathcal{E}(v)(t,x). \tag{2}$$

From rheological relation (2), the stress tensor deviator σ can be expressed through $\mathcal{E}$ as follows (see, for example, [15] for more details):

$$\sigma(t,x)=2\mu_2\frac{\partial}{\partial t}\mathcal{E}(v)(t,x)+2\mu_1\mathcal{E}(v)(t,x)+2\int_0^t\sum_{i=1}^{L}\beta_i e^{\alpha_i(t-s)}\mathcal{E}(v)(s,x)\,ds+\sigma_0(x). \tag{3}$$

In this case, it is additionally assumed that the polynomial $Q(p)=1+\sum_{i=1}^{L}\lambda_i p^i$, defined by the left-hand side of (2) has real, negative and distinct roots $\alpha_i, i=\overline{1,L}$. Note that this condition is completely consistent with the physical meaning of the problem and is not burdensome. The coefficients μ_2, μ_1 and $\beta_i, i = \overline{1, L}$ are real and are determined through the coefficients of relation (2). The function σ_0 is an expression of the initial conditions on $\sigma, \mathcal{E}$ and their time derivatives. It is usually assumed for simplicity that these initial conditions are chosen in such a way that $\sigma_0 \equiv 0$. The integral term in (3) is responsible for memory effects in the fluid. That is, the value of the stress tensor deviator σ at time t depends on the values of the strain rate tensor $\mathcal{E}$ for all $s \in [0, t]$.

For the system of equations obtained by substituting (3) into the system of equations of an incompressible fluid motion, various initial boundary value problems were studied [14–17]. Also, for the resulting system of equations, questions about the existence of attractors [18] were investigated. Let us also note that the model with rheological relation (2) for $L = 1$ is often called the Navier–Stokes–Voigt model. A large number of works are devoted to the study of its solvability and questions about the limiting behavior of solutions [19–23]. Let us also note papers [24,25] devoted to the study of the solvability of the modified Kelvin–Voigt model. It is also necessary to mention papers [26–29] devoted to the study of inhomogeneous incompressible Kelvin–Voigt fluid with rheological relation (1) for various values of L and its generalizations.

Models with rheological relation (1) have not been studied in such detail at the moment. The solvability of the initial boundary value problem for a model with a substantial time derivative (1) for $L = 1$ was proved in [15,30]. For an arbitrary L, the solvability of the initial boundary value problem for this model could not be proved for a long time. The point is that in this case, under similar assumptions regarding the roots of the polynomial determined from the left-hand side of (1), the deviator of the stress tensor σ can be expressed as follows (see, for more details, [31]):

$$\sigma(t, x) = 2\mu_2 \frac{d}{dt}\mathcal{E}(v)(t, x) + 2\mu_1 \mathcal{E}(v)(t, x) + 2\int_0^t \sum_{i=1}^{L} \beta_i e^{\alpha_i(t-s)} \mathcal{E}(v)(s, z(s; t, x))\, ds + \sigma_0(x). \quad (4)$$

Here, z is the solution to the following Cauchy problem in integral form:

$$z(\tau; t, x) = x + \int_t^\tau v(s, z(s; t, x))ds, \quad 0 \leqslant t, \tau \leqslant T, \quad x \in \Omega. \quad (5)$$

In this case, the integral in (4) is taken along the trajectories of fluid motion, which is of much greater interest from a physical point of view. Such models more accurately describe the behavior of a fluid. In (4), the integral is taken along the trajectory $z(s; t, x)$. Let us explain that $z(s; t, x)$ is the position of a particle at time s, provided that at time t it was at point x. Thus, the deviator of the stress tensor at the moment t depends not only on the value of the strain rate tensor at the moment t and its values on the interval $[0, t]$, but also on the trajectories of fluid particles. The dependence of stresses at the current moment on the behavior of the fluid in the past is understood as the memory of the fluid.

But this integral term is precisely the main problem in proving the existence of weak solutions to the corresponding initial boundary value problem. In order to find the trajectories of motion of fluid particles, it is necessary to solve the Cauchy problem (5). Since the weak solution belongs to $W_2^1(\Omega)^n$, this is not enough for the classical solvability of (5). The way out of this situation is to use the theory of regular Lagrangian flows, created in [32]. Based on this theory, in a recent paper [31], the solvability in the weak sense of the initial boundary value problem for the Kelvin–Voigt model with rheological relation (1) was established.

In this paper, the existence of trajectory and global attractors is established for a system of equations corresponding to the Kelvin–Voigt model with rheological relation (1) under certain conditions on the coefficients of the problem. The proof is carried out using the approximation-topological approach to the study of fluid dynamic problems (see, for example, [15]), as well as the theory of trajectory and global attractors for non-invariant trajectory spaces [8,33]. Namely, some problem that approximates the original one is considered. Then, the operator interpretation of the considered and the approximation problems is introduced. Using the Leray–Schauder fixed point theorem, the solvability of the approximation problem on a finite segment is established. Then, conditions are obtained for the coefficients, under which the exponential estimates necessary for applying the abstract theory of attractors take place. Based on these estimates, the solvability of the approximation problem on the semi-axis is proved. Then, using passage to the limit, the weak solvability of the original problem on the semi-axis is proved. As a result, based on the exponential estimates, a trajectory space is constructed and the existence of the required attractors is established.

This paper consists of an introduction and seven sections. Section 2 provides a statement of the problem under study. Section 3 contains necessary facts from the abstract theory of attractors for non-invariant trajectory spaces. Section 4 gives necessary notations and statements used in this paper. Section 5 contains a weak formulation of the considered problem and an approximation problem. Section 6 is devoted to obtaining the necessary estimates for solutions. In Section 7, theorems on the solvability of the considered problem on the semi-axis are proved. Finally, in Section 8, theorems on the existence of attractors for the problem under consideration are proved.

2. Problem Statement

Substituting σ from (4) into the equation of fluid motion, we obtain the following system that describes the motion of the incompressible Kelvin–Voigt fluid with

$$\frac{\partial v}{\partial t} - \nu\Delta v + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \varkappa \frac{\partial \Delta v}{\partial t}$$

$$-2\varkappa \mathrm{Div}\left(v_k \frac{\partial \mathcal{E}(v)}{\partial x_k}\right) - 2\mathrm{Div} \int_0^t \sum_{i=1}^{L} \beta_i e^{-\alpha_i(t-s)} \mathcal{E}(v)(s, z(s,t,x))\, ds + \nabla p = f; \tag{6}$$

$$\mathrm{div}\, v = 0, \quad (t,x) \in Q_T = [0,T] \times \Omega; \tag{7}$$

$$z(\tau;t,x) = x + \int_t^\tau v(s, z(s;t,x))ds, \quad 0 \leqslant t, \tau \leqslant T, \quad x \in \Omega. \tag{8}$$

Here, Ω is a convex bounded domain from $\mathbb{R}^n, n = 2,3$, with a smooth boundary $\partial\Omega$, v is the velocity vector of a fluid particle, p is the fluid pressure, f is the density of external forces, $\nu > 0, \varkappa > 0$ are fluid viscosity and retardation time, respectively, and $\beta_i, \alpha_i, i = \overline{1,L}$ are some constants. Based on the physical meaning, it is assumed that $\alpha_i > 0, i = \overline{1,L}$. For convenience, we denoted the negative roots of the polynomial $Q(p)$ (see Introduction for more details) by $-\alpha_1, -\alpha_2, \dots, -\alpha_L$, where $\alpha_i > 0, i = \overline{1,L}$ are positive numbers. Function $z(\tau;t,x)$ is the trajectory of fluid particles corresponding to the velocity field v.

System (6)–(8) is supplemented with the following initial and boundary conditions:

$$v|_{t=0} = a(x), \quad x \in \Omega, \quad v|_{\partial\Omega\times[0,T]} = 0. \tag{9}$$

3. Necessary Definitions and Statements from Attractor Theory

Let us present some facts from the theory of trajectory attractors. This presentation does not pretend to be complete and contains only those facts that we will directly need (for more details, see monograph [8], as well as articles [9,33]).

Let E, E_0 be two Banach spaces. We will assume that the space E is reflexive and the embedding $E \subset E_0$ is continuous. Let $\mathbb{R}_+$ denote the non-negative semi-axis of $\mathbb{R}$.

The space $C(\mathbb{R}_+; E_0)$ consists of continuous functions defined on $\mathbb{R}_+$ and taking values in E_0. Since the semi-axis $\mathbb{R}_+$ is non-compact, then in $C(\mathbb{R}_+; E_0)$ it is impossible to specify the usual norm of the space of continuous functions. Consider in the space $C(\mathbb{R}_+; E_0)$ the following family of semi-norms:

$$\|u\|_n = \|u\|_{C([0,n],E_0)}, \quad n = 1, 2, \ldots . \tag{10}$$

Let us define the topology in $C(\mathbb{R}_+; E_0)$ by determining the convergence of sequences with respect to the introduced semi-norms. Namely, the sequence $\{u_m\}$ from $C(\mathbb{R}_+; E_0)$ converges to the function u as $m \to \infty$, if $\|u_m - u\|_n \to 0$ for any $n = 1, 2, \ldots$. The space $C(\mathbb{R}_+; E_0)$ with family of semi-norms (10) is a countably normed space. The topology of local uniform convergence in the space $C(\mathbb{R}_+; E_0)$ is metrizable with respect to the metric

$$\rho(u, v) = \|u - v\|_{C(\mathbb{R}_+;E_0)} = \sum_{n=1}^{\infty} 2^{-n} \frac{\|v\|_{C([0,n],E_0)}}{1 + \|v\|_{C([0,n],E_0)}}.$$

The resulting metric space is a Fréchet space.

We use the already-traditional notation $\|u - v\|_{C(\mathbb{R}_+;E_0)}$ for the metric in $C(\mathbb{R}_+; E_0)$. This is due to the use of abstract concepts and statements from [8,33], in which this notation is used. Note that the functional $\|\cdot\|_{C(\mathbb{R}_+;E_0)}$ is not a norm, since $\|\lambda v\|_{C(\mathbb{R}_+;E_0)} \neq |\lambda|\|v\|_{C(\mathbb{R}_+;E_0)}$ for $\lambda \neq \pm 1$.

Denote by Π_M ($M \geqslant 0$) the operator of restriction of functions defined on $\mathbb{R}_+$, to the interval $[0, M]$. The following criterion holds for the relative compactness of sets from $C(\mathbb{R}_+; E_0)$.

Lemma 1. *The set $P \subset C(\mathbb{R}_+; E_0)$ is relatively compact in $C(\mathbb{R}_+; E_0)$ iff for any $M > 0$ the set $\Pi_M P$ is relatively compact in $C([0, M], E_0)$.*

$L_\infty(\mathbb{R}_+; E)$ is the space of essentially bounded functions defined on $\mathbb{R}_+$ and taking values in E with the norm $\|u\|_{L_\infty(\mathbb{R}_+;E)} = \operatorname*{ess\,sup}_{t\in\mathbb{R}_+} \|u(t)\|_E$. The space $L_\infty(\mathbb{R}_+; E)$ is a Banach space (see, for example, [34]).

Definition 1. *Let J be a finite or infinite interval of the real axis, $\bar{J}$ be its closure and Y be a Banach space. A function $u : \bar{J} \to Y$ is called weakly continuous if for any $t_n \to t$, $t_n \in \bar{J}$, the sequence $u(t_n)$ converges weakly to $u(t)$ in Y. We will denote the set of weakly continuous functions $u : \bar{J} \to Y$ by $C_w(\bar{J}, Y)$.*

We also need one well-known theorem (see, for example, [35]).

Theorem 1. *Let E and E_0 be two Banach spaces such that $E \subset E_0$ and the embedding is continuous. Let a function v belong to $L_\infty(0, T; E)$ and be continuous as a function with values in E_0. Then, v is weakly continuous as a function with values in E, i.e., $v \in C_w([0, T], E)$.*

Therefore, the function $v \in C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is weakly continuous as a function with values in E. Therefore, $v(t) \in E$ for all $t \in \mathbb{R}_+$, and the following equality holds:

$$\|v\|_{C(\mathbb{R}_+;E_0)\cap L_\infty(\mathbb{R}_+;E)} = \sup_{t\in\mathbb{R}_+} \|v(t)\|_E.$$

Let us consider the shift operators $T(h)$ ($h \geqslant 0$), which assign function f to a function $T(h)f$ such that $T(h)f(t) = f(t + h)$. Let us note that $T(h_1)T(h_2) = T(h_1 + h_2)$ and $T(0)$ is the identity operator.

Consider a non-empty family of functions

$$\mathcal{H}^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E).$$

The set $\mathcal{H}^+$ is called the trajectory space, and the elements of $\mathcal{H}^+$ are called trajectories. A natural condition is imposed on $\mathcal{H}^+$ that it is non-empty.

Let us give the main definitions.

Definition 2. *The set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called an attracting set for the trajectory space $\mathcal{H}^+$ if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(\mathbb{R}_+; E)$ it holds that*

$$\sup_{u \in B} \inf_{v \in P} \|T(h)u - v\|_{C(\mathbb{R}_+; E_0)} \to 0 \quad as \quad h \to \infty.$$

Definition 3. *The set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called an absorbing set for the trajectory space $\mathcal{H}^+$ if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(\mathbb{R}_+; E)$ there exists $h \geqslant 0$, such that for all $t \geqslant h$ it holds that $T(t)B \subset P$.*

Every absorbing set is an attracting set.

Definition 4. *The set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called a trajectory semi-attractor of the trajectory space $\mathcal{H}^+$ if the following are true:*

(i) *The set P is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$;*
(ii) *$T(t)P \subset P$ for all $t \geqslant 0$;*
(iii) *P is the attracting set for $\mathcal{H}^+$.*

Definition 5. *The set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called a trajectory attractor of the trajectory space $\mathcal{H}^+$ if the following are true:*

(i) *The set P is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$;*
(ii) *$T(t)P = P$ for all $t \geqslant 0$;*
(iii) *P is the attracting set for $\mathcal{H}^+$.*

Definition 6. *The minimal trajectory attractor of the trajectory space $\mathcal{H}^+$ is the smallest trajectory attractor with respect to inclusion.*

Definition 7. *The set $\mathcal{A} \subset E$ is called a global attractor (in E_0) of the trajectory space $\mathcal{H}^+$, if it satisfies the following conditions:*

(i) *The set $\mathcal{A}$ is compact in E_0 and bounded in E;*
(ii) *For every set $B \subset \mathcal{H}^+$ bounded in $L_\infty(\mathbb{R}_+; E)$, the attraction condition is satisfied:*

$$\sup_{u \in B} \inf_{y \in \mathcal{A}} \|u(t) - y\|_{E_0} \to 0 \quad as \quad t \to \infty;$$

(iii) *The set $\mathcal{A}$ is the smallest due to the inclusion set satisfying conditions (i) and (ii).*

Remark 1. *If there is a minimal trajectory attractor or a global attractor, then it is unique.*

Let us give one more statement ([33] (Lemma 4.2)) that we need to prove the main result.

Lemma 2. *For the trajectory space $\mathcal{H}_+$, let P be an attracting (respectively, absorbing) set that is relatively compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$. Then, its closure $\overline{P}$ in the space $C(\mathbb{R}_+; E_0)$ is the attracting (respectively, absorbing) set for $\mathcal{H}_+$, that is, $\overline{P}$ is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$. If, in addition, the inclusion $T(t)P \subset P$ holds for all $t \geqslant 0$, then $\overline{P}$ is a semi-attractor.*

The following theorems hold on the existence of a minimal trajectory and global attractor.

Theorem 2. *Let there be a trajectory semi-attractor P of the trajectory space $\mathcal{H}^+$. Then, there is a minimal trajectory attractor $\mathcal{U}$ of the trajectory space $\mathcal{H}^+$.*

Theorem 3. *Let there be a minimal trajectory attractor $\mathcal{U}$ of the trajectory space $\mathcal{H}^+$. Then, there is a global attractor $\mathcal{A}$ of the trajectory space $\mathcal{H}^+$.*

4. Preliminaries

In what follows, we will need the definitions of some spaces. Denote by $C_0^\infty(\Omega)^n$ the space of C^∞ functions on Ω with values in $\mathbb{R}^n$ with compact support in Ω. Let us set $\mathcal{V} = \{v : v \in C_0^\infty(\Omega)^n,\ \mathrm{div} v = 0\}$ and define V^0 and V^1 as the completion of $\mathcal{V}$ with respect to the norms of $L_2(\Omega)^n$ and $H^1(\Omega)^n$, respectively. Let $V^2 = H^2(\Omega)^n \cap V^1$.

Due to the Weyl decomposition of vector fields from $L_2(\Omega)^n$ (see, for example, [35,36]), $L_2(\Omega)^n = V^0 \oplus \nabla H^1(\Omega)$. Here, $\nabla H^1(\Omega) = \{\nabla p : p \in H^1(\Omega)\}$. Consider in $\mathcal{V}$ the operator $A = -\pi\Delta$. As it is well known (see [37,38]), the operator A extends in the space V^0 to a closed operator, which is a self-adjoint positive operator with a completely continuous inverse. The domain of A coincides with V^2. By the Hilbert Theorem on the spectral decomposition of completely continuous operators, the eigenfunctions $\{e_j\}$ of the operator A form an orthonormal basis in V^0.

Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_k \leq \ldots$ be the eigenvalues of the operator A. Denote by E_∞ the set of finite linear combinations of e_j. Define the space V^α, $\alpha \in \mathbb{R}$, as the completion of E_∞ with respect to the norm $\|v\|_{V^\alpha} = \left(\sum_{k=1}^\infty \lambda_k^\alpha |v_k|^2\right)^{1/2}$.

In [39], it was proved that the norm in V^α is equivalent to the norm $\|v\|_{V^\alpha} = \|A^{\alpha/2} v\|_{V^0}$.

To introduce a notion of a weak solution for the original and approximation problems on an interval $[0, T]$, we introduce the following spaces:

$$W_1[0,T] = \{v\colon v \in L_\infty(0,T;V^1),\ v' \in L_\infty(0,T;V^{-1})\};$$
$$W_2[0,T] = \{v\colon v \in C([0,T],V^5),\ v' \in L_\infty(0,T;V^5)\}$$

with corresponding norms:

$$\|v\|_{W_1[0,T]} = \|v\|_{L_\infty(0,T;V^1)} + \|v'\|_{L_\infty(0,T;V^{-1})};$$
$$\|v\|_{W_2[0,T]} = \|v\|_{C([0,T],V^5)} + \|v'\|_{L_\infty(0,T;V^5)}.$$

To determine the weak solution on the semi-axis $\mathbb{R}_+$, we consider the spaces $W_1^{\mathrm{loc}}(\mathbb{R}_+)$ and $W_2^{\mathrm{loc}}(\mathbb{R}_+)$. The space $W_1^{\mathrm{loc}}(\mathbb{R}_+)$ consists of functions v, defined almost everywhere on $\mathbb{R}_+$ and taking values in V^1, such that the restriction of v to any interval $[0, T]$ belongs to $W_1[0, T]$. The space $W_2^{\mathrm{loc}}(\mathbb{R}_+)$ consists of functions $v \in C(\mathbb{R}_+, V^5)$, such that the restriction of v to any interval $[0, T]$ belongs to $W_2[0, T]$.

We need also the Aubin–Dubinsky–Simon Theorem [40].

Theorem 4. *Let $X \subset E \subset Y$ be Banach spaces such that the embedding $X \subset E$ is compact and the embedding $E \subset Y$ is continuous. Let $F \subset L_p(0, T; X)$, $1 \leqslant p \leqslant \infty$. We will assume that for any $f \in F$ its generalized derivative in space $D'(0, T; Y)$ belongs to $L_r(0, T; Y)$, $1 \leqslant r \leqslant \infty$. Next, let the following hold:*

(i) The set F is bounded in $L_p(0, T; X)$;
(ii) The set $\{f' : f \in F\}$ is bounded in $L_r(0, T; Y)$.

Then, for $p < \infty$ the set F is relatively compact in $L_p(0, T; E)$, and for $p = \infty$ and $r > 1$, the set F is relatively compact in $C([0, T], E)$.

Let us give the necessary statements about the solvability of the problem

$$z(\tau;t,x) = x + \int_t^\tau v(s, z(s;t,x)),\, ds, \quad 0 \leq t, \tau \leq T, \quad x \in \overline{\Omega}. \tag{11}$$

Let $v \in L_1(0, T; C(\overline{\Omega})^n)$. The solution of (11) is defined as the function $z(\tau) \equiv z(\tau; t; x)$, $\tau, t \in [0, T]$, $x \in \overline{\Omega}$, such that $z(\tau) \in C([0, T], \overline{\Omega})$ and satisfies (11).

Let $\overset{\circ}{C}(\overline{\Omega})$ be the set of continuous functions that vanish on $\partial\Omega$. The following lemma [41] holds:

Lemma 3. *Let $v \in L_1(0, T; C^1(\overline{\Omega})^n \cap \overset{\circ}{C}(\overline{\Omega}))$ and $\partial\Omega \in C^1$. Then, (11) has a unique solution z. Moreover, z, $\frac{\partial z}{\partial x}$ are continuous in the variables $\tau, t \in [0, T], x \in \overline{\Omega}$.*

However, in the case of only a summable function v, the situation becomes much more complicated and it requires a much more general concept for the solution to problem (11).

Definition 8. *Function $z(\tau; t, x) : [0, T] \times [0, T] \times \overline{\Omega} \to \mathbb{R}^n$ is called a regular Lagrangian flow, corresponding to v, if the following conditions are satisfied:*

(i) *For almost all x and any $t \in [0, T]$, the function $\gamma(\tau) = z(\tau; t, x)$ is absolutely continuous and satisfies (11);*

(ii) *For any $\tau, t \in [0, T]$ and an arbitrary Lebesgue measurable set $B \subset \overline{\Omega}$ with Lebesgue measure $m(B)$, it holds that $m(z(\tau, t, B)) = m(B)$;*

(iii) *For any $t_1, t_2, t_3 \in [0, T]$ and almost all $x \in \overline{\Omega}$, it holds that $z(t_3, t_1, x) = z(t_3, t_2, z(t_2, t_1, x))$.*

For the concept of regular Lagrangian flows, see, for example, [32]. Here, we consider a special case of a bounded domain Ω and a divergence-free function v. Note also that in the case of a smooth function v, the regular Lagrangian flow coincides with the classical solution of (11).

We need the following theorem [32].

Theorem 5. *Let $v \in L_1(0, T; W_p^1(\Omega)^n)$, $1 \le p \le +\infty$, $\operatorname{div} v(t, x) = 0$ and $v(t, x)|_{\partial\Omega} = 0$. Then, there is a unique regular Lagrangian flow z, corresponding to v. Moreover,*

$$\frac{\partial}{\partial\tau} z(\tau; t, x) = v(\tau, z(\tau; t, x)), \quad t, \tau \in \Omega, \quad \text{for allmost all} \quad x \in \Omega,$$
$$z(\tau, t, \overline{\Omega}) = \overline{\Omega} \quad \text{(up to zero measure)}.$$

We also give one lemma [42], which is used in this paper.

Lemma 4. *Let the sequence v^m weakly converge to v in $L_2(0, T; V^1)$ as $m \to \infty$. Then,*

$$\int_0^t \mathcal{E}(v^m)(s, z^m(s; t, x))ds \to \int_0^t \mathcal{E}(v)(s, z(s; t, x))ds$$

weakly in $L_2(0, T; L_2(\Omega)^{n^2})$ as $m \to \infty$. Here, z^m is a regular Lagrangian flow generated by v^m and z is a regular Lagrangian flow generated by v.

5. Weak Problem Statement and Approximation

Let $a \in V^1, f \in V^{-1}$. Let us give the definition of a weak solution to problem (6)–(9) on a finite segment $[0, T]$ and on $\mathbb{R}_+$.

Definition 9. *A weak solution to problem (6)–(9) on $[0, T]$ is a function $v \in W_1[0, T]$ such that the identity*

$$\langle (J + \varkappa A)v', \varphi \rangle - \int_\Omega \sum_{i,j=1}^n v_i v_j \frac{\partial \varphi_j}{\partial x_i} dx + \nu \int_\Omega \nabla v : \nabla \varphi \, dx$$
$$- \varkappa \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_i}{\partial x_j} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} dx - \varkappa \int_\Omega \sum_{i,j,k=1}^n v_k \frac{\partial v_j}{\partial x_i} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} dx$$
$$+ 2 \int_0^t \sum_{i=1}^L \beta_i e^{-\alpha_i(t-s)} \int_\Omega \mathcal{E}(v)(s, z(s, t, x)) : \mathcal{E}(\varphi) dx ds = \langle f, \varphi \rangle \quad (12)$$

is satisfied almost everywhere on $(0,T)$ for any function $\varphi \in V^3$ and the function v satisfies the initial condition

$$v(0) = a \tag{13}$$

Here, z is a solution to problem (8), which exists due to Theorem 5.

Definition 10. *The function $v \in W_1^{loc}(\mathbb{R}_+)$ is called the weak solution of problem (6)–(9) on the semi-axis $\mathbb{R}_+$ if for every $T > 0$ the restriction of v to the interval $[0,T]$ is a weak solution to problem (6)–(9) on $[0,T]$.*

Let $\varepsilon > 0$. Consider the following approximation problem:

$$\frac{\partial v}{\partial t} - \nu\Delta v + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \varkappa\frac{\partial \Delta v}{\partial t} + \varepsilon e^{-\gamma t}\frac{\partial \Delta^4 v}{\partial t}$$

$$-2\varkappa \mathrm{Div}\left(v_k \frac{\partial \mathcal{E}(v)}{\partial x_k}\right) - 2\mathrm{Div}\int_0^t \sum_{i=1}^{L}\beta_i e^{-\alpha_i(t-s)}\mathcal{E}(v)(s,z(s,t,x))\,ds + \nabla p = f; \tag{14}$$

$$\mathrm{div}\, v = 0, \quad (t,x) \in Q_T; \tag{15}$$

$$z(\tau;t,x) = x + \int_t^\tau v(s,z(s;t,x))ds, \quad 0 \leqslant t,\tau \leqslant T, \quad x \in \Omega; \tag{16}$$

$$v|_{t=0} = b(x), \quad x \in \Omega; \tag{17}$$

$$v|_{\partial\Omega\times[0,T]} = \Delta v|_{\partial\Omega\times[0,T]} = \Delta^2 v|_{\partial\Omega\times[0,T]} = \Delta^3 v|_{\partial\Omega\times[0,T]} = 0. \tag{18}$$

Here, γ is a constant for which the following inequality holds:

$$0 < \gamma \leqslant \min\left(\frac{\nu}{K_0 + \varkappa}, \alpha_1, \alpha_2, \ldots \alpha_L\right). \tag{19}$$

Here, K_0 is a constant from Poincaré's inequality:

$$\|u\|_{V^0}^2 \leqslant K_0\|u\|_{V^1}^2. \tag{20}$$

The exact choice of γ is described in the proof of Theorem 7.
Let us assume $b \in V^5, f \in V^{-1}$.

Definition 11. *A function $v \in W_2$ is called a solution to approximation problem (14)–(18) if it satisfies, for any function $\varphi \in V^3$ and for almost all $t \in (0,T)$, the identity*

$$\int_\Omega v'\varphi dx + \varkappa\int_\Omega \nabla(v') : \nabla\varphi dx - \varepsilon e^{-\gamma t}\int_\Omega \nabla\left(\Delta^2 v'\right) : \nabla(\Delta\varphi)dx - \int_\Omega \sum_{i,j=1}^{n} v_i v_j \frac{\partial \varphi_j}{\partial x_i}dx$$

$$+\nu\int_\Omega \nabla v : \nabla\varphi\, dx - \varkappa\int_\Omega \sum_{i,j,k=1}^{n} v_k \frac{\partial v_i}{\partial x_j}\frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}dx - \varkappa\int_\Omega \sum_{i,j,k=1}^{n} v_k \frac{\partial v_j}{\partial x_i}\frac{\partial^2 \varphi_j}{\partial x_i \partial x_k}dx$$

$$+2\int_0^t \sum_{i=1}^{L}\beta_i e^{-\alpha_i(t-s)}\int_\Omega \mathcal{E}(v)(s,z(s,t,x)) : \mathcal{E}(\varphi)dxds = \langle f,\varphi\rangle \tag{21}$$

and the initial condition

$$v(0) = b. \tag{22}$$

Here, z is the solution to problem (16). Due to the continuous embedding $V^5 \subset C^1(\overline{\Omega})^n$, problem (16) has a unique solution z, which exists by Lemma 3.

Definition 12. *A function $v \in W_2^{loc}(\mathbb{R}_+)$ is called a solution to approximation problem (14)–(18) on the semi-axis $\mathbb{R}_+$ if for every $T > 0$ the restriction of v to the interval $[0, T]$ is a solution to approximation problem (14)–(18) on $[0, T]$.*

The following theorem holds.

Theorem 6. *There is at least one solution to approximation problem (14)–(18).*

Let us introduce operators using the following equalities:

$$A : V^1 \to V^{-1}, \quad \langle Au, \varphi\rangle = \int_\Omega \nabla u : \nabla\varphi dx, \quad \forall u, \varphi \in V^1;$$
$$A^4 : V^5 \to V^{-3}, \quad \langle A^4 u, \varphi\rangle = -\int\limits_\Omega \nabla(\Delta^2 u) : \nabla(\Delta\varphi)\, dx, \quad u \in V^5,\ \varphi \in V^3;$$
$$J : V^1 \to V^{-1}, \quad \langle Ju, \varphi\rangle = \int_\Omega u\varphi dx, \quad \forall u, \varphi \in V^1;$$
$$B_1 : L_4(\Omega)^n \to V^{-1}, \quad \langle B_1(u)(t), \varphi\rangle = \int\limits_\Omega \sum_{i,j=1}^n u_i(t)u(t)_j \frac{\partial\varphi_j}{\partial x_i}\, dx, \quad u \in L_4(\Omega)^n,\ \varphi \in V^1;$$
$$B_2 : V^1 \to V^{-3}, \quad \langle B_2(u)(t), \varphi\rangle = \int\limits_\Omega \sum_{i,j,k=1}^n u_k(t)\frac{\partial u(t)_i}{\partial x_j}\frac{\partial^2\varphi_j}{\partial x_i \partial x_k}\, dx, \quad u \in V^1,\ \varphi \in V^3;$$
$$B_3 : V^1 \to V^{-3}, \quad \langle B_3(u)(t), \varphi\rangle = \int\limits_\Omega \sum_{i,j,k=1}^n u_k(t)\frac{\partial u(t)_j}{\partial x_i}\frac{\partial^2\varphi_j}{\partial x_i \partial x_k}\, dx, \quad u \in V^1,\ \varphi \in V^3;$$
$$N : L_2(0, T; V^1) \to L_2(0, T; V^{-1}),$$
$$\langle N(u)(t), \varphi\rangle = 2\int_0^t \sum_{i=1}^L \beta_i e^{-\alpha_i(t-s)} \int_\Omega \mathcal{E}(u)(s, z(s, t, x)) : \mathcal{E}(\varphi) dx ds,$$
$$u \in L_2(0, T; V^1),\ \varphi \in V^1.$$

Then, the solvability of problem (6)–(9) is equivalent to the existence of a function $v \in W_1$ which satisfies the operator equation

$$(J + \varkappa A)v' + \nu Av - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v) + N(v) = f, \tag{23}$$

as well as initial condition (13).

And the problem of finding a function $v \in W_2$, satisfying for any test function $\varphi \in V^3$ for almost all $t \in (0, T)$ identity (21) and initial condition (22), is equivalent to the problem of finding a function $v \in W_2$ that is a solution to the operator equation

$$(J + \varepsilon e^{-\gamma t} A^4 + \varkappa A)v' + \nu Av - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v) + N(v) = f, \tag{24}$$

and that satisfies initial condition (22).

We need the following lemma about the properties of operators. Proof of these properties can be found in [31].

Lemma 5.

(1) For any function $v \in L_2(0, T; V^1)$, it holds that $Av \in L_2(0, T; V^{-1})$, the operator $A : L_2(0, T; V^1) \to L_2(0, T; V^{-1})$ is continuous and for almost all $t \in (0, T)$ the following estimate holds:

$$\|Av(t)\|_{V^{-1}} \leqslant \|v(t)\|_{V^1}. \tag{25}$$

(2) *For any function $v \in L_2(0,T;V^5)$, it takes place that $A^4 v \in L_2(0,T;V^{-3})$, the operator $A^4 : L_2(0,T;V^5) \to L_2(0,T;V^{-3})$ is continuous and for almost all $t \in (0,T)$ the following estimate holds:*

$$\|A^4 v(t)\|_{V^{-3}} \leqslant \|v(t)\|_{V^5}. \tag{26}$$

(3) *For any function $v \in L_2(0,T;V^{-1})$, we obtain $(J + \varkappa A)v \in L_2(0,T;V^{-3})$, the operator $(J+\varkappa A) : L_2(0,T;V^{-1}) \to L_2(0,T;V^{-3})$ is continuous and for almost all $t \in (0,T)$ the following estimate holds:*

$$C_0\|v(t)\|_{V^{-1}} \leqslant \|(J+\varkappa A)v(t)\|_{V^{-3}}. \tag{27}$$

(4) *For any function $v \in L_2(0,T;V^5)$, it holds that $(J + \varepsilon e^{-\gamma t}A^4 + \varkappa A)v \in L_2(0,T;V^{-3})$, the operator $(J + \varepsilon e^{-\gamma t}A^4 + \varkappa A) : L_2(0,T;V^5) \to L_2(0,T;V^{-3})$ is continuous and invertible and for almost all $t \in (0,T)$ it satisfies the estimate*

$$\varepsilon\|v(t)\|_{V^5} \leqslant \|(J + \varepsilon e^{-\gamma t}A^4 + \varkappa A)v(t)\|_{V^{-3}} \leqslant (C_1 + \varepsilon + \varkappa C_2)\|v(t)\|_{V^5}. \tag{28}$$

(5) *For any function $v \in L_2(0,T;V^1)$, we obtain $B_1(v) \in L_2(0,T;V^{-3})$, the mapping $B_1 : L_2(0,T;V^1) \to L_2(0,T;V^{-3})$ is continuous and for almost all $t \in (0,T)$ the following estimate holds:*

$$\|B_1(v)(t)\|_{V^{-3}} \leqslant C_3\|v(t)\|_{V^1}. \tag{29}$$

(6) *For any function $v \in L_2(0,T;V^1)$, the value $B_2(v)$ belongs to $L_2(0,T;V^{-3})$, the mapping $B_2 : L_2(0,T;V^1) \to L_2(0,T;V^{-3})$ is continuous and for almost all $t \in (0,T)$ the following estimate takes place:*

$$\|B_2(v)(t)\|_{V^{-3}} \leqslant C_4\|v(t)\|_{V^1}. \tag{30}$$

(7) *For any function $v \in L_2(0,T;V^1)$, it holds that $B_3(v) \in L_2(0,T;V^{-3})$. The mapping $B_3 : L_2(0,T;V^1) \to L_2(0,T;V^{-3})$ is continuous and for almost all $t \in (0,T)$ it satisfies the inequality*

$$\|B_3(v)(t)\|_{V^{-3}} \leqslant C_4\|v(t)\|_{V^1}. \tag{31}$$

The inequality obtained in the following lemma is important for obtaining the necessary a priori estimates for solutions.

Lemma 6. *The operator $N : L_2(0,T;V^1) \to L_2(0,T;V^{-1})$ is continuous, and for all $t \in [0,T]$ the following inequality holds:*

$$\|N(v)(t)\|_{V^{-1}} \leqslant C_5\left(\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2_{V^1}ds\right)^{1/2}. \tag{32}$$

Proof. The continuity of the operator is proved similarly to [31] (Lemma 10).

Let us prove the validity of the required inequality. For any $t \in [0,T]$, by definition of the operator N, due to Hölder's inequality, we have

$$|\langle N(v)(t),\varphi\rangle| = \left|2\int_0^t \sum_{i=1}^L \beta_i e^{-\alpha_i(t-s)}\int_\Omega \mathcal{E}(v)(s,z(s,t,x)) : \mathcal{E}(\varphi)dxds\right|$$
$$\leqslant 2\sum_{i=1}^L|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\left(\int_\Omega|\mathcal{E}(v)(s,z(s,t,x))|^2dx\right)^{1/2}\left(\int_\Omega|\mathcal{E}(\varphi)|^2dx\right)^{1/2}ds.$$

In the first of these two integrals, we make the change of variables $y = z(s,t,x)$. Since $\mathrm{div} v = 0$, then $\det\frac{\partial z}{\partial x} = 1$. Therefore, for this integral, we have

$$\int_\Omega|\mathcal{E}(v)(s,z(s,t,x))|^2dx = \int_\Omega|\mathcal{E}(v)(s,y))|^2dy = \|\mathcal{E}(v)(s)\|^2_{L_2(\Omega)^{n^2}}.$$

Thus,

$$|\langle N(v)(t),\varphi\rangle| \leqslant 2\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|\mathcal{E}(v)(s)\|_{L_2(\Omega)^{n^2}}ds\|\mathcal{E}(\varphi)\|_{L_2(\Omega)^{n^2}}$$
$$\leqslant 2\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1}ds\|\varphi\|_{V^1}.$$

Here, we used the inequality $\|\mathcal{E}(v)\|_{L_2(\Omega)^{n^2}} \leqslant \|v\|_{V^1}$.
Therefore, for each $t \in [0,T]$, the following inequality holds:

$$\|N(v)(t)\|_{V^{-1}} \leqslant 2\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1}ds. \tag{33}$$

For each $i=\overline{1,L}$, due to Hölder's inequality and (19), we have

$$\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1}ds = \int_0^t e^{-(\alpha_i-\gamma/2)(t-s)}e^{-(t-s)\gamma/2}\|v(s)\|_{V^1}ds$$
$$\leqslant \left(\int_0^t e^{-2(\alpha_i-\gamma/2)(t-s)}ds\right)^{1/2}\left(\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2_{V^1}ds\right)^{1/2}$$
$$= \left(\frac{1-e^{-2(\alpha_i-\gamma/2)t}}{2\alpha_i-\gamma}\right)^{1/2}\left(\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2_{V^1}ds\right)^{1/2}$$
$$\leqslant \frac{1}{\sqrt{2\alpha_i-\gamma}}\left(\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2_{V^1}ds\right)^{1/2}.$$

From here and from (33), the required inequality (32) follows. □

6. Estimates of Solutions

In this section, we establish the estimates necessary to determine trajectory spaces. Conditions are also formulated for the coefficients of problem (14)–(18) under which these estimates take place.

Theorem 7. *Let v be the solution to approximation Equation (24), and let the coefficients of problem (14)–(18) satisfy the following conditions:*

$$\nu\alpha_i > 4L|\beta_i|, \quad i=\overline{1,L}. \tag{34}$$

Then, for all $\tau \in [0,T]$, the following estimate holds:

$$\varkappa\|v(\tau)\|^2_{V^1} + \varepsilon e^{-\gamma\tau}\|v(\tau)\|^2_{V^4} + \lambda_1\nu\int_0^\tau e^{-\gamma(\tau-t)}\|v(t)\|^2_{V^1}dt$$
$$\leqslant C_7 + e^{-\gamma\tau}\Big(C_6\|v(0)\|^2_{V^1} + \varepsilon\|v(0)\|^2_{V^4}\Big). \tag{35}$$

Here, $C_7 = \dfrac{\|f\|^2_{V^{-1}}}{\gamma\nu(1-\lambda_1-\lambda_2)}$, $C_6 = (K_0+\varkappa)$, K_0 is a constant from Poincaré's inequality (20), and λ_1, λ_2 are some constants such that $\lambda_1 > 0, \lambda_2 > 0, 0 < \lambda_1+\lambda_2 < 1$.

Proof. Let v be the solution to Equation (24). Apply (24) to v. We obtain

$$\langle Jv' + \varepsilon e^{-\gamma t}A^4v' + \varkappa Av' + \nu Av - B_1(v) - \varkappa B_2(v) - \varkappa B_3(v) + N(v), v\rangle = \langle f, v\rangle. \tag{36}$$

Let us transform the terms from the last equality as follows:

$$\langle Jv', v\rangle = \int_\Omega v'v dx = \frac{1}{2}\frac{d}{dt}\|v\|^2_{V^0};$$
$$\nu\langle Av, v\rangle = \nu\|v\|^2_{V^1};$$
$$\varepsilon e^{-\gamma t}\langle A^4 v', v\rangle = -\varepsilon e^{-\gamma t}\int_\Omega \nabla\left(\Delta^3 v'\right) : \nabla v dx = \varepsilon e^{-\gamma t}\int_\Omega \Delta^3 v' \Delta v\, dx$$
$$= -\varepsilon e^{-\gamma t}\int_\Omega \nabla(\Delta^2 v') : \nabla\Delta v dx = \varepsilon e^{-\gamma t}\int_\Omega \Delta^2 v' \Delta^2 v dx = \frac{\varepsilon e^{-\gamma t}}{2}\frac{d}{dt}\|v\|^2_{V^4};$$
$$\varkappa\langle Av', v\rangle = \varkappa\int_\Omega \nabla(v') : \nabla v dx = \frac{\varkappa}{2}\frac{d}{dt}\|v\|^2_{V^1}.$$

The following equalities also hold (see [31], for example, for a complete proof):

$$\langle B_1(v), v\rangle = 0; \quad \langle B_2(v) + B_3(v), v\rangle = 0.$$

Similar to the proof of Lemma 6 for the last term on the left-hand side of (36), we have

$$|\langle Nv, v\rangle| \leqslant 2\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1} ds\|v(t)\|_{V^1}.$$

Let λ_1, λ_2 be numbers such that $\lambda_1 > 0, \lambda_2 > 0, 0 < \lambda_1 + \lambda_2 < 1$, and the exact value of λ_1 and λ_2 will be indicated below. Let us estimate the right-hand side of (36) as follows:

$$\langle f, v\rangle \leqslant \|f\|_{V^{-1}}\|v(t)\|_{V^1} \leqslant \frac{\nu(1-\lambda_1-\lambda_2)\|v(t)\|^2_{V^1}}{2} + \frac{\|f\|^2_{V^{-1}}}{2\nu(1-\lambda_1-\lambda_2)}.$$

Here, we used the elementary inequality $ab \leqslant \frac{\delta a^2}{2} + \frac{b^2}{2\delta}$, which holds for any non-negative a, b and positive δ. Namely, we set $\delta = \nu(1-\lambda_1-\lambda_2)$.

Then, from (36), we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{V^0} + \nu\|v(t)\|^2_{V^1} + \frac{\varkappa}{2}\frac{d}{dt}\|v(t)\|^2_{V^2} + \frac{\varepsilon}{2}e^{-\gamma t}\frac{d}{dt}\|v(t)\|^2_{V^4}$$
$$- 2\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1} ds\|v(t)\|_{V^1}$$
$$\leqslant \frac{\nu(1-\lambda_1-\lambda_2)\|v(t)\|^2_{V^1}}{2} + \frac{\|f\|^2_{V^{-1}}}{2\nu(1-\lambda_1-\lambda_2)}.$$

Multiply both sides of the last inequality by 2 and collect similar ones. We obtain

$$\frac{d}{dt}\|v(t)\|^2_{V^0} + \nu\|v(t)\|^2_{V^1} + \varkappa\frac{d}{dt}\|v(t)\|^2_{V^2} + \varepsilon e^{-\gamma t}\frac{d}{dt}\|v(t)\|^2_{V^4} + \lambda_1\nu\|v(t)\|^2_{V^1}$$
$$+ \lambda_2\nu\|v(t)\|^2_{V^1} - 4\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1} ds\|v(t)\|_{V^1} \leqslant \frac{\|f\|^2_{V^{-1}}}{\nu(1-\lambda_1-\lambda_2)}.$$

For brevity, we denote

$$F = \frac{\|f\|^2_{V^{-1}}}{\nu(1-\lambda_1-\lambda_2)}; \quad G(t) = \lambda_2\nu\|v(t)\|^2_{V^1} - 4\sum_{i=1}^{L}|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1} ds\|v(t)\|_{V^1}.$$

Then,

$$\frac{d}{dt}\|v(t)\|^2_{V^0} + \nu\|v(t)\|^2_{V^1} + \varkappa\frac{d}{dt}\|v(t)\|^2_{V^2} + \varepsilon e^{-\gamma t}\frac{d}{dt}\|v(t)\|^2_{V^4} + \lambda_1\nu\|v(t)\|^2_{V^1} + G(t) \leqslant F. \tag{37}$$

Let us introduce on V^1 an auxiliary norm

$$\|u\|^2 = \|u\|^2_{V^0} + \varkappa\|u\|^2_{V^1}. \tag{38}$$

Its equivalence to the V^1 norm follows from Poincaré's inequality (20), due to which the following inequality holds:

$$\varkappa\|u\|^2_{V^1} \leqslant \|u\|^2 \leqslant (K_0 + \varkappa)\|u\|^2_{V^1}. \tag{39}$$

Then, due to condition (19) on γ, we obtain

$$\nu\|u\|^2_{V^1} \geqslant \frac{\nu}{(K_0 + \varkappa)}\|u\|^2 \geqslant \gamma\|u\|^2. \tag{40}$$

From (37), using (38) and (40), we have

$$\frac{d}{dt}\|v(t)\|^2 + \gamma\|v(t)\|^2 + \varepsilon e^{-\gamma t}\frac{d}{dt}\|v(t)\|^2_{V^4} + \lambda_1\nu\|v(t)\|^2_{V^1} + G(t) \leqslant F.$$

In the first two terms on the left-hand side, we make the change $v(t) = e^{-\gamma t/2}\overline{v}(t)$.

$$\frac{d}{dt}\|e^{-\gamma t/2}\overline{v}(t)\|^2 + \gamma\|e^{-\gamma t/2}\overline{v}(t)\|^2 + \varepsilon e^{-\gamma t}\frac{d}{dt}\|v(t)\|^2_{V^4} + \lambda_1\nu\|v(t)\|^2_{V^1} + G(t) \leqslant F.$$

Therefore,

$$\begin{aligned} -\gamma e^{-\gamma t}\|\overline{v}(t)\|^2 + e^{-\gamma t}\frac{d}{dt}\|\overline{v}(t)\|^2 + \gamma e^{-\gamma t}\|\overline{v}(t)\|^2 \\ + \varepsilon e^{-\gamma t}\frac{d}{dt}\|v(t)\|^2_{V^4} + \lambda_1\nu\|v(t)\|^2_{V^1} + G(t) \leqslant F. \end{aligned}$$

Let us multiply both sides of the inequality by $e^{\gamma t}$.

$$\frac{d}{dt}\|\overline{v}(t)\|^2 + \varepsilon\frac{d}{dt}\|v(t)\|^2_{V^4} + \lambda_1\nu e^{\gamma t}\|v(t)\|^2_{V^1} + e^{\gamma t}G(t) \leqslant e^{\gamma t}F.$$

Let us integrate the last inequality over t from 0 to τ, where $\tau \in [0, T]$.

$$\begin{aligned} \|\overline{v}(\tau)\|^2 + \varepsilon\|v(\tau)\|^2_{V^4} + \lambda_1\nu\int_0^\tau e^{\gamma t}\|v(t)\|^2_{V^1}dt + \int_0^\tau e^{\gamma t}G(t)dt \\ \leqslant \int_0^\tau e^{\gamma t}Fdt + \|\overline{v}(0)\|^2 + \varepsilon\|v(0)\|^2_{V^4}. \end{aligned}$$

We multiply both sides by $e^{-\gamma\tau}$ and estimate the right-hand side from above as follows:

$$\begin{aligned} e^{-\gamma\tau}\|\overline{v}(\tau)\|^2 + \varepsilon e^{-\gamma\tau}\|v(\tau)\|^2_{V^4} + \lambda_1\nu\int_0^\tau e^{-\gamma(\tau-t)}\|v(t)\|^2_{V^1}dt + \int_0^\tau e^{-\gamma(\tau-t)}G(t)dt \leqslant \\ \leqslant \int_0^\tau e^{-\gamma(\tau-t)}Fdt + e^{-\gamma\tau}\Big(\|\overline{v}(0)\|^2 + \varepsilon\|v(0)\|^2_{V^4}\Big) \\ = \frac{F}{\gamma}(1 - e^{-\gamma\tau}) + e^{-\gamma\tau}\Big(\|\overline{v}(0)\|^2 + \varepsilon\|v(0)\|^2_{V^4}\Big) \leqslant \frac{F}{\gamma} + e^{-\gamma\tau}\Big(\|\overline{v}(0)\|^2 + \varepsilon\|v(0)\|^2_{V^4}\Big). \end{aligned}$$

Let us make the reverse change $\overline{v}(t) = e^{\gamma t/2} v(t)$. Due to the definition of the auxiliary norm, we obtain

$$\begin{aligned}
&\|v(\tau)\|_{V^0}^2 + \varkappa\|v(\tau)\|_{V^1}^2 + \varepsilon e^{-\gamma\tau}\|v(\tau)\|_{V^4}^2 \\
&\qquad + \lambda_1\nu\int_0^\tau e^{-\gamma(\tau-t)}\|v(t)\|_{V^1}^2 dt + \int_0^\tau e^{-\gamma(\tau-t)}G(t)dt \\
&\qquad\qquad \leqslant \frac{F}{\gamma} + e^{-\gamma\tau}\Big((K_0+\varkappa)\|v(0)\|_{V^1}^2 + \varepsilon\|v(0)\|_{V^4}^2\Big). \qquad (41)
\end{aligned}$$

Let us now show that the last term on the left side of this inequality is non-negative. Recalling the previously introduced notation, we have

$$\begin{aligned}
&\int_0^\tau e^{-\gamma(\tau-t)}G(t)dt \\
&\qquad = \int_0^\tau e^{-\gamma(\tau-t)}\left(\lambda_2\nu\|v(t)\|_{V^1}^2 - 4\sum_{i=1}^L|\beta_i|\int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1}ds\|v(t)\|_{V^1}\right)dt.
\end{aligned}$$

Let us introduce auxiliary functions

$$h(t) = \|v(t)\|_{V^1}; \quad g_i(t) = \int_0^t e^{-\alpha_i(t-s)}\|v(s)\|_{V^1}ds = \int_0^t e^{-\alpha_i(t-s)}h(s)ds \quad i=\overline{1,L}.$$

The function h is continuous on $[0,T]$, and the functions $g_i, i=\overline{1,L}$, are continuously differentiable on this interval. Direct calculation gives

$$g_i'(t) = h(t) - \alpha_i\int_0^t e^{-\alpha_i(t-s)}h(s)ds = h(t) - \alpha_i g_i(t), \quad i=\overline{1,L}.$$

Then,

$$g_i'(t) + \alpha_i g_i(t) = h(t); \quad g_i(0) = 0, \quad i=\overline{1,L}.$$

Therefore,

$$\begin{aligned}
G(t) &= \lambda_2\nu h^2(t) - 4h(t)\sum_{i=1}^L|\beta_i|g_i(t) \\
&= \sum_{i=1}^L\left(\frac{\lambda_2\nu}{L}(g_i'(t)+\alpha_i g_i(t))^2 - 4|\beta_i|g_i(t)(g_i'(t)+\alpha_i g_i(t))\right) \\
&= \sum_{i=1}^L\left(\frac{\lambda_2\nu}{L}(g_i'(t))^2 + 2\left(\frac{\lambda_2\nu\alpha_i}{L} - 2|\beta_i|\right)g_i'(t)g_i(t) + \left(\frac{\lambda_2\nu\alpha_i^2}{L} - 4|\beta_i|\alpha_i\right)g_i^2(t)\right).
\end{aligned}$$

By virtue of the integration by parts formula, for any $i=\overline{1,L}$, we have

$$\begin{aligned}
2\int_0^\tau e^{-\gamma(\tau-t)}g_i'(t)g_i(t)dt &= e^{-\gamma\tau}\left(e^{\gamma t}g_i^2(t)\right)\Big|_0^\tau - \gamma e^{-\gamma\tau}\int_0^\tau e^{\gamma t}g_i^2(t)dt \\
&= g_i^2(\tau) - \gamma\int_0^\tau e^{-\gamma(\tau-t)}g_i^2(t)dt.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\tau e^{-\gamma(\tau-t)}G(t)dt &= \sum_{i=1}^L\left(\frac{\lambda_2\nu}{L}\int_0^\tau e^{-\gamma(\tau-t)}(g_i'(t))^2dt + \left(\frac{\lambda_2\nu\alpha_i}{L} - 2|\beta_i|\right)g_i^2(t)\right. \\
&\quad \left. + \left(\alpha_i\left(\frac{\lambda_2\nu\alpha_i}{L} - 4|\beta_i|\right) - \gamma\left(\frac{\lambda_2\nu\alpha_i}{L} - 2|\beta_i|\right)\right)\int_0^\tau e^{-\gamma(\tau-t)}g_i^2(t)dt\right).
\end{aligned}$$

Let us show that for each $i = \overline{1,L}$, if the conditions in (34) are satisfied, then it is possible to choose positive number μ_i such that the expression

$$\left(\frac{\lambda_2 \nu}{L} \int_0^\tau e^{-\gamma(\tau-t)} (g_i'(t))^2 dt + \left(\frac{\lambda_2 \nu \alpha_i}{L} - 2|\beta_i| \right) g_i^2(t) \right. \\ \left. + \left(\left(\frac{\lambda_2 \nu \alpha_i^2}{L} - 4|\beta_i|\alpha_i \right) - \mu_i \left(\frac{\lambda_2 \nu \alpha_i}{L} - 2|\beta_i| \right) \right) \int_0^\tau e^{-\gamma(\tau-t)} (g_i(t))^2 dt \right) \tag{42}$$

is non-negative.

Since $\lambda_2 > 0, \nu > 0$ and $L \geqslant 1$, then for each $i = \overline{1,L}$, we have

$$\frac{\lambda_2 \nu}{L} \int_0^\tau e^{-\gamma(\tau-t)} (g_i'(t))^2 dt \geqslant 0.$$

Further, due to (34), we have $\frac{\nu \alpha_i}{L} > 4|\beta_i|$, $i = \overline{1,L}$. Therefore, we can choose λ_2, possibly close enough to 1, such that

$$\frac{\lambda_2 \nu \alpha_i}{L} - 4|\beta_i| > 0, \quad i = \overline{1,L}. \tag{43}$$

Consequently, the second term in (42) is non-negative.

Let us move on to the coefficient before the third term in (42). Due to (43), we obtain

$$\left(\frac{\lambda_2 \nu \alpha_i}{L} - 4|\beta_i| \right) > 0, \quad \left(\frac{\lambda_2 \nu \alpha_i}{L} - 2|\beta_i| \right) > 0.$$

Therefore, one can always choose $\mu_i, 0 < \mu_i \leqslant \gamma$, such that

$$\alpha_i \left(\frac{\lambda_2 \nu \alpha_i}{L} - 4|\beta_i| \right) - \mu_i \left(\frac{\lambda_2 \nu \alpha_i}{L} - 2|\beta_i| \right) > 0.$$

Let us put

$$\gamma = \min_{i=\overline{1,L}} \mu_i.$$

Therefore,

$$\int_0^\tau e^{-\gamma(\tau-t)} G(t) dt \geqslant 0.$$

Estimating the left-hand side of inequality (41) from below, we obtain the required inequality (35). □

Remark 2. *Note that the conditions in (34) cannot be weakened. In fact, in the case $\lambda_i = \lambda, i = \overline{1,L}$, $\alpha_i = \alpha, i = \overline{1,L}$, $\beta_i = \beta, i = \overline{1,L}$, we obtain*

$$\int_0^\tau e^{-\gamma(\tau-t)} G(t) dt = \lambda_2 \nu \int_0^\tau e^{-\gamma(\tau-t)} (g'(t))^2 dt + (\lambda_2 \nu \alpha - 2L|\beta|) g^2(t) \\ + (\alpha(\lambda_2 \nu \alpha - 4L|\beta|) - \gamma(\lambda_2 \nu \alpha - 2L|\beta|)) \int_0^\tau e^{-\gamma(\tau-t)} g^2(t) dt. \tag{44}$$

Here, $g(t) = \int_0^t e^{-\alpha(t-s)} \|v(s)\|_{V^1} ds$.

Therefore, if the condition $\nu\alpha > 4L|\beta|$ is not satisfied, then the coefficient before the last term in (44) is not non-negative.

Theorem 8. *Let v be a solution to Equation (24) on $[0,T], T > 0$, for some $\varepsilon > 0$, and let the coefficients $\nu, \alpha_i, \beta_i, i = \overline{1,L}$ satisfy the conditions in (34). Then, for almost all $t \in (0,T)$, the following estimates hold:*

$$\varepsilon e^{-\gamma t}\|v'(t)\|_{V^5} \leqslant C_{12} + C_{13}e^{-\gamma t}\Big(C_6\|v(0)\|^2_{V^1} + \varepsilon\|v(0)\|^2_{V^4}\Big); \tag{45}$$

$$C_0\|v'(t)\|_{V^{-1}} \leqslant 2C_{12} + 2C_{13}e^{-\gamma t}\Big(C_6\|v(0)\|^2_{V^1} + \varepsilon\|v(0)\|^2_{V^4}\Big). \tag{46}$$

And for all $t \in [0,T]$, the estimate

$$e^{-\gamma t}\|v(t)\|_{V^5} \leqslant e^{-\gamma t}\|v(0)\|_{V^5} + \frac{1}{\varepsilon\gamma}\Big(C_{12} + C_{13}\Big(C_6\|v(0)\|^2_{V^1} + \varepsilon\|v(0)\|^2_{V^4}\Big)\Big) \tag{47}$$

is valid.

Proof. Since v is a solution to Equation (24), then for almost all $t \in (0,T)$ the following equality of norms holds:

$$\begin{aligned}&\Big\|(J + \varepsilon e^{-\gamma t}A^4 + \varkappa A)v'(t)\Big\|_{V^{-3}} \\ &\qquad = \|-\nu Av(t) + B_1(v)(t) + \varkappa B_2(v)(t) + \varkappa B_3(v)(t) - N(v)(t) + f\|_{V^{-3}}.\end{aligned} \tag{48}$$

By virtue of (28), the left-hand side of the last equality can be estimated as follows:

$$\varepsilon e^{-\gamma t}\|v'(t)\|_{V^5} \leqslant \Big\|(J + \varepsilon e^{-\gamma t}A^4 + \varkappa A)v'(t)\Big\|_{V^{-3}}. \tag{49}$$

Due to inequalities (25), (29)–(32) and the continuity of the embedding $V^{-1} \subset V^{-3}$ for the right-hand side of equality (48), we have

$$\begin{aligned}&\|-\nu Av(t) + B_1(v)(t) + \varkappa B_2(v)(t) + \varkappa B_3(v)(t) - N(v)(t) + f\|_{V^{-3}} \\ &\leqslant \|\nu Av(t)\|_{V^{-3}} + \|B_1(v)(t)\|_{V^{-3}} + \varkappa\|B_2(v)(t)\|_{V^{-3}} \\ &\quad + \varkappa\|B_3(v)(t)\|_{V^{-3}} + \|N(v)(t)\|_{V^{-3}} + \|f\|_{V^{-3}} \\ &\leqslant \nu C_8\|v(t)\|_{V^1} + C_3\|v(t)\|^2_{V^1} + 2\varkappa C_4\|v(t)\|^2_{V^1} \\ &\quad + C_5C_8\Big(\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2_{V^1}ds\Big)^{1/2} + C_8\|f\|_{V^{-1}} \\ &\leqslant C_9\|v(t)\|^2_{V^1} + C_{10}\int_0^t e^{-\gamma(t-s)}\|v(s)\|^2_{V^1}dt + C_{11}.\end{aligned}$$

Here, we used the elementary inequality $a \leqslant 1 + a^2$, which holds for any $a \geqslant 0$.

Thus, the required inequality (45) follows from the last estimate, in addition to equality (48), inequality (49) as well as inequality (35).

Similarly, if v is a solution to (24), then the following equality of norms holds:

$$\|(J + \varkappa A)v'(t)\|_{V^{-3}} = \Big\|-\varepsilon e^{-\gamma t}A^4v'(t) - \nu Av(t) + B_1(v)(t) + \varkappa B_2(v)(t) + \varkappa B_3(v)(t) - N(v)(t) + f\Big\|_{V^{-3}}. \tag{50}$$

By virtue of (27), the left-hand side of the last equality from below can be estimated as follows:

$$C_0\|v(t)\|_{V^{-1}} \leqslant \|(J + \varkappa A)v'(t)\|_{V^{-3}}. \tag{51}$$

Similar to the previous one, the right-hand side of (50) due to inequality (26) can be estimated as follows:

$$\begin{aligned}\Big\|-\varepsilon e^{-\gamma t}A^4 v'(t)-\nu A v(t)+B_1(v)(t)+\varkappa B_2(v)(t)+\varkappa B_3(v)(t)-N(v)(t)+f\Big\|_{V^{-3}}\\ \leqslant \Big\|-\varepsilon e^{-\gamma t}A^4 v'(t)\Big\|_{V^{-3}}\\ +\|-\nu A v(t)+B_1(v)(t)+\varkappa B_2(v)(t)+\varkappa B_3(v)(t)-N(v)(t)+f\|_{V^{-3}}\\ \leqslant \varepsilon e^{-\gamma t}\|v'(t)\|_{V^5}\\ +\|-\nu A v(t)+B_1(v)(t)+\varkappa B_2(v)(t)+\varkappa B_3(v)(t)-N(v)(t)+f\|_{V^{-3}}\\ \leqslant 2C_{12}+2C_{13}e^{-\gamma t}\Big(C_6\|v(0)\|^2_{V^1}+\varepsilon\|v(0)\|^2_{V^4}\Big).\end{aligned}$$

Thus, due to the last inequality and estimate (51), using (50), we obtain the required inequality (46).

In order to obtain estimate (47), we note that for all $t\in[0,T]$ the following equality holds:

$$v(t)=v(0)+\int_0^t v'(s)ds.$$

Let us multiply both sides of this equality by $e^{-\gamma t}$. We obtain

$$e^{-\gamma t}v(t)=e^{-\gamma t}\Big(v(0)+\int_0^t v'(s)ds\Big).$$

Then, by virtue of (45), we have

$$\begin{aligned}\|e^{-\gamma t}v(t)\|_{V^5}=\Big\|e^{-\gamma t}\Big(v(0)+\int_0^t v'(s)ds\Big)\Big\|_{V^5}=\Big\|e^{-\gamma t}\Big(v(0)+\int_0^t e^{-\gamma s}e^{\gamma s}v'(s)ds\Big)\Big\|_{V^5}\\ \leqslant e^{-\gamma t}\|v(0)\|_{V^5}+\int_0^t e^{-\gamma(t-s)}e^{-\gamma s}\|v'(s)\|_{V^5}ds\\ \leqslant e^{-\gamma t}\|v(0)\|_{V^5}+\frac{1}{\varepsilon}\int_0^t e^{-\gamma(t-s)}\Big(C_{12}+C_{13}e^{-\gamma s}\Big(C_6\|v(0)\|^2_{V^1}+\varepsilon\|v(0)\|^2_{V^4}\Big)\Big)ds\\ \leqslant e^{-\gamma t}\|v(0)\|_{V^5}+\frac{C_{12}}{\varepsilon}\int_0^t e^{-\gamma(t-s)}ds\\ +\frac{1}{\varepsilon}\int_0^t e^{-\gamma(t-s)}e^{-\gamma s}\Big(C_{13}\Big(C_6\|v(0)\|^2_{V^1}+\varepsilon\|v(0)\|^2_{V^4}\Big)\Big)ds\\ \leqslant e^{-\gamma t}\|v(0)\|_{V^5}+\frac{C_{12}}{\varepsilon\gamma}(1-e^{-\gamma t})+\frac{C_{13}}{\varepsilon\gamma}(1-e^{-\gamma t})\Big(C_6\|v(0)\|^2_{V^1}+\varepsilon\|v(0)\|^2_{V^4}\Big)\\ \leqslant e^{-\gamma t}\|v(0)\|_{V^5}+\frac{1}{\varepsilon\gamma}\Big(C_{12}+C_{13}\Big(C_6\|v(0)\|^2_{V^1}+\varepsilon\|v(0)\|^2_{V^4}\Big)\Big).\end{aligned}$$

Therefore, the required estimate (47) is proved. □

Corollary 1. *Let v be a solution to Equation (24) on the interval $[0,T]$, $T>0$, for some $\varepsilon>0$, and let the coefficients $\nu,\alpha_i,\beta_i,i=\overline{1,L}$ satisfy the conditions in (34). Then, for almost all $t\in(0,T)$, the following estimates hold:*

$$\|v(t)\|_{V^1}+\|v'(t)\|_{V^{-1}}\leqslant C_{14}+C_{15}e^{-\gamma t}\Big(C_6\|v(0)\|^2_{V^1}+\varepsilon\|v(0)\|^2_{V^4}\Big); \tag{52}$$

$$e^{-\gamma t}(\|v(t)\|_{V^5}+\|v'(t)\|_{V^5})\leqslant C_{16}. \tag{53}$$

Here, the constant C_{16} depends on $\frac{1}{\varepsilon}$.

7. Existence Theorems for Solutions

The following theorem holds on the existence of solutions to the approximation problem on the interval $[0,T], T>0$.

Theorem 9. *Let the coefficients $\nu, \alpha_i, \beta_i, i=\overline{1,L}$ satisfy the conditions in (34). Then, on any interval $[0,T], T>0$, there exists a solution to Equation (24) that satisfies initial condition (22), and this solution satisfies estimates (52) and (53).*

The proof of this theorem is similar to the proof of Theorem 6 in [31], and is based on the Leray–Schauder fixed point theorem. The presence of restrictions on the coefficients of the approximation problem does not have any effect on the progress of the proof.

We also need the following technical lemma. We can establish convergence in spaces with better properties. Namely, for the approximation problem, due to the obtained estimates, it is possible to establish convergence in smoother spaces. But for our purposes, the convergences indicated below are sufficient.

Lemma 7. *Let the sequence $\{v_m\}$ be bounded in $W_1[0,T]$. Then, the following hold:*

(1) There is a subsequence $\{v_{m_k}\}$ converging to the limit function v_* in $C([0,T];L_4(\Omega)^n)$, and the following relations hold:

$$Jv'_{m_k} \rightharpoonup Jv'_* \text{ weakly in } L_2(0,T;V^{-1}) \quad \text{as } m_k\to\infty; \tag{54}$$

$$\nu Av_{m_k} \rightharpoonup \nu Av_* \text{ weakly in } L_2(0,T;V^{-1}) \quad \text{as } m_k\to\infty; \tag{55}$$

$$\varkappa Av'_{m_k} \rightharpoonup \varkappa Av'_* \text{ weakly in } L_2(0,T;V^{-3}) \quad \text{as } m_k\to\infty; \tag{56}$$

$$B_1(v_{m_k}) \to B_1(v_*) \text{ strongly in } C([0,T],V^{-1}) \quad \text{as } m_k\to\infty; \tag{57}$$

$$\varkappa B_2(v_{m_k}) \rightharpoonup \varkappa B_2(v_*) \text{ weakly in } L_2(0,T;V^{-3}) \quad \text{as } m_k\to\infty; \tag{58}$$

$$\varkappa B_3(v_{m_k}) \rightharpoonup \varkappa B_3(v_*) \text{ weakly in } L_2(0,T;V^{-3}) \quad \text{as } m_k\to\infty; \tag{59}$$

$$N(v_{m_k}) \rightharpoonup N(v_*) \text{ weakly in } L_2(0,T;V^{-1}) \quad \text{as } m_k\to\infty. \tag{60}$$

(2) Let $\{\varepsilon_m\}$ be a number sequence, with $\varepsilon_m\to 0$ as $m\to\infty$, and let a sequence $\{\varepsilon_m v'_m\}$ be bounded in $L_\infty(0,T,V^5)$. Then, there exists a subsequence $\{\varepsilon_{m_k}v'_{m_k}\}$ such that $\varepsilon_{m_k}e^{-\gamma t}A^4v'_{m_k} \rightharpoonup 0$ weakly in $L_2(0,T;V^{-3})$ as $m_k\to\infty$.

(3) Let a sequence $\{v'_m\}$ be bounded in $L_\infty(0,T;V^5)$. Then, there is a subsequence $\{v'_{m_k}\}$ such that $(J+\varepsilon e^{-\gamma t}A^4+\varkappa A)v'_{m_k} \rightharpoonup (J+\varepsilon e^{-\gamma t}A^4+\varkappa A)v'_*$ weakly in $L_2(0,T;V^{-3})$ as $m_k\to\infty$.

Proof. (1) By Theorem 4, the embedding $W_1[0,T]\subset C([0,T],L_4(\Omega)^n)$ is compact. Since the sequence $\{v_m\}$ is bounded in $W_1[0,T]$, it is relatively compact in $C([0,T],L_4(\Omega)^n)$. Therefore, there is a subsequence $\{v_{m_k}\}$, converging in $C([0,T],L_4(\Omega)^n)$ to some function v_*. That is,

$$v_{m_k}\to v_* \quad \text{strongly in } C([0,T],L_4(\Omega)^n) \quad \text{as } m_k\to+\infty. \tag{61}$$

Let us move from non-reflexive spaces L_∞ to reflexive spaces L_p, in order to take advantage of the weak compactness of bounded sets. Since the space L_∞ is continuously embedded in L_p with $p\geqslant 1$, then the sequences $\{v_m\}$ and $\{v'_m\}$ are bounded in $L_2(0,T;V^1)$ and $L_2(0,T;V^{-1})$, respectively. Therefore, without loss of generality, we obtain

$$v_{m_k}\rightharpoonup v_* \qquad \text{weakly in } L_2(0,T;V^1) \quad \text{as } m_k\to+\infty; \tag{62}$$

$$v'_{m_k}\rightharpoonup v'_* \qquad \text{weakly in } L_2(0,T;V^{-1}) \quad \text{as } m_k\to+\infty. \tag{63}$$

Convergence (63) directly implies (54). By Lemma 5, the linear operator A is continuous. Therefore, convergences (55) and (56) follow from (62) and (63), respectively.

Due to strong convergence (61) and the definition of the operator B_1, we obtain the validity of convergence (57).

Since v_m converges to v_* strongly in $C([0,T], L_4(\Omega)^n)$ and ∇v_m converges weakly to ∇v_* in $L_4(0,T; L_2(\Omega)^{n^2})$, then their product converges weakly to the product of limits. Using the definitions of B_2 and B_3, we obtain the validity of convergences (58) and (59).

Convergence (60) is valid due to Lemma 4.

(2) Since the sequence $\{\varepsilon_m v_m'\}$ is bounded in $L_\infty(0,T,V^5)$, then the sequence $\{\varepsilon_m e^{-\gamma t} v_m'\}$ is also bounded in the same space. Therefore, there is a subsequence $\{\varepsilon_{m_k} e^{-\gamma t} v_{m_k}'\}$, which weakly converges to some function w in $L_2(0,T;V^5)$ as $m_k \to \infty$. But in the sense of distributions on the interval $[0,T]$ with values in V^{-7}, this subsequence converges to zero. In fact, for any $\chi \in \mathfrak{D}([0,T])$, $\varphi \in V^7$, using Green's formula and weak convergence (62), we obtain

$$\lim_{m_k\to\infty}\left|\varepsilon_{m_k}\int_0^T\int_\Omega \nabla\left(\Delta^2 v_{m_k}'\right) : \nabla(\Delta\varphi)dx\chi(t)dt\right|$$
$$= \lim_{m_k\to\infty}\varepsilon_{m_k}\lim_{m_k\to\infty}\left|\int_0^T\int_\Omega \nabla v_{m_k}(t) : \nabla\left(\Delta^3\varphi\right)dx\frac{\partial\chi(t)}{\partial t}dt\right|$$
$$= \left|\int_0^T\int_\Omega \nabla v(t) : \nabla\left(\Delta^3\varphi\right)dx\frac{\partial\chi(t)}{\partial t}dt\right|\lim_{m_k\to\infty}\varepsilon_{m_k} = 0.$$

Therefore, due to the uniqueness of the weak limit,

$$\varepsilon_{m_k}\int_\Omega \nabla\left(\Delta^2 v_{m_k}'\right) : \nabla(\Delta\varphi)dx \to 0 \quad \text{as } m_k \to \infty.$$

This implies the required convergence.

(3) Due to the boundedness of $\{v_m'\}$, there exists a subsequence $\{v_{m_k}'\}$ which converges to v_*' weakly in $L_2(0,T,V^5)$. Therefore, the required convergence follows from the continuity of the linear operator $(J + \varepsilon e^{-\gamma t}A^4 + \varkappa A) : L_2(0,T;V^5) \to L_2(0,T;V^{-3})$. □

The following theorem establishes the solvability of approximation problem (24), (22) on the semi-axis $\mathbb{R}_+$.

Theorem 10. *Let the coefficients $\nu, \alpha_i, \beta_i, i = \overline{1,L}$ satisfy the conditions in (34). Then, problem (24), (22) on $\mathbb{R}_+$ has a solution $v \in W_2^{loc}(\mathbb{R}_+)$ satisfying for almost all $t \in \mathbb{R}_+$ the following inequalities:*

$$\|v(t)\|_{V^1} + \|v'(t)\|_{V^{-1}} \leqslant C_{18} + C_{19}e^{-\gamma t}\left(C_6\|v(0)\|^2_{V^1} + \varepsilon\|v(0)\|^2_{V^4}\right); \tag{64}$$
$$\varepsilon e^{-\gamma t}\|v'(t)\|_{V^5} \leqslant C_{12} + C_{13}e^{-\gamma t}\left(C_6\|v(0)\|^2_{V^1} + \varepsilon\|v(0)\|^2_{V^4}\right). \tag{65}$$

Proof. Let v_m be a solution to problem (24), (22) on the interval $[0,m]$ $(m = 1,2,\ldots)$, which exists by Theorem 9. Let us extend the functions v_m onto the semi-axis $\mathbb{R}_+$ as follows:

$$\hat{v}_m(t) = \begin{cases} v(t), & 0 \leqslant t \leqslant m, \\ v(m), & t \geqslant m. \end{cases}$$

Based on the continuation on $\mathbb{R}_+$, the functions $\hat{v}_m$ belong to the space $W_2^{\mathrm{loc}}(\mathbb{R}_+)$. Let us show that the sequence $\{\hat{v}_m\}$ is relatively compact in $C(\mathbb{R}_+, V^1)$. By Lemma 1, it is sufficient to establish that for any $T > 0$ the sequence $\{\Pi_T\hat{v}_m\}$ is relatively compact in the space $C([0,T], V^1)$.

Let us take an arbitrary $T > 0$. Having possibly discarded the first few terms of the sequence, we can assume that the functions $\{\Pi_T \hat{v}_m\}$ are solutions to problem (24), (22) on $[0, T]$. Since the functions $\Pi_T \hat{v}_m$ have the same value for $t = 0$, then by Corollary 1 these functions satisfy for almost all $t \in (0, T)$ the estimate

$$e^{-\gamma t}(\|\Pi_T \hat{v}_m(t)\|_{V^5} + \|\Pi_T \hat{v}'_m(t)\|_{V^5}) \leqslant C_{16}.$$

Hence,

$$\|\Pi_T \hat{v}_m(t)\|_{L_\infty(0,T;V^5)} + \|\Pi_T \hat{v}'_m(t)\|_{L_\infty(0,T;V^5)} \leqslant C_{17}. \quad (66)$$

Here, C_{17} depends on T and $\frac{1}{\varepsilon}$ and does not depend on m.

Thus, the sequence $\{\Pi_T \hat{v}_m\}$ is bounded in $L_\infty(0, T; V^5)$, and the sequence $\{\Pi_T \hat{v}'_m\}$ is bounded in $L_\infty(0, T; V^5)$. Due to the compactness of the embedding $V^5 \subset V^1$ by Theorem 4, the sequence $\{\Pi_T \hat{v}_m\}$ is relatively compact in $C([0, T], V^1)$.

Due to the arbitrariness of T, the sequence $\{\hat{v}_m\}$ contains a subsequence $\{\hat{v}_{m_k}\}$ which converges in $C(\mathbb{R}_+, V^1)$ to some function v_*. Let us show that v_* is the solution to problem (24), (22) on $\mathbb{R}_+$.

Let us show that v_* belongs to the space $W_2^{\mathrm{loc}}(\mathbb{R}_+)$. From estimate (66), it follows that for every $T > 0$ the sequences $\{\Pi_T \hat{v}_{m_k}\}$ and $\{\Pi_T \hat{v}'_{m_k}\}$ are bounded in $L_\infty(0, T; V^5)$. Consequently, without loss of generality, we can assume that these sequences converge $*$-weakly in $L_\infty(0, T; V^5)$, respectively, to v_* and some function $u \in L_\infty(0, T; V^5)$. However, in the sense of distributions on $(0, T)$ with values in V^5, the sequence $\{\Pi_T \hat{v}'_{m_k}\}$ converges to v'_*. So, $u = \Pi_T v'_*$. Therefore, $\Pi_T v_* \in L_\infty(0, T; V^5)$ and $\Pi_T v'_* \in L_\infty(0, T; V^5)$. Since

$$\Pi_T v_*(t) = \Pi_T v_*(0) + \int_0^t \Pi_T v'_*(s) ds,$$

then $\Pi_T v_* \in C([0, T], V^5)$. Hence, $\Pi_T v_* \in W_2[0, T]$. Due to the arbitrariness of T, the function v_* belongs to $W_2^{\mathrm{loc}}(\mathbb{R}_+)$.

The convergence in $C([0, T], V^1)$ implies pointwise convergence. Since all functions $\{\hat{v}_{m_k}\}$ satisfy the same initial condition and the sequence $\{\Pi_T \hat{v}_{m_k}\}$ converges pointwise, then v_* also satisfies initial condition (22).

Let us show that the function v_* is a solution to equation (24). We need to show that for every $T > 0$ the restriction of the function v_* to segment $[0, T]$ is a solution to equation (24) on $[0, T]$.

Since the sequence $\{\hat{v}_{m_k}\}$ converges to v_* in $C(\mathbb{R}_+, V^1)$, then for every $T > 0$ the sequence of restrictions $\{\Pi_T \hat{v}_{m_k}\}$ converges to $\Pi_T v_*$ in $C([0, T], V^1)$ as $m_k \to \infty$. Starting from a certain number, every function $\Pi_T \hat{v}_{m_k}$ is the solution to (24). That is, each $\Pi_T \hat{v}_{m_k}$ satisfies the equality

$$\begin{aligned}(J + \varepsilon e^{-\gamma t} A^4 + \varkappa A)\Pi_T \hat{v}'_{m_k} + \nu A \Pi_T \hat{v}_{m_k} - B_1(\Pi_T \hat{v}_{m_k})& \\ - \varkappa B_2(\Pi_T \hat{v}_{m_k}) - \varkappa B_3(\Pi_T \hat{v}_{m_k}) + N(\Pi_T \hat{v}_{m_k}) = f.& \quad (67)\end{aligned}$$

From inequality (66), it follows that the conditions of Lemma 7 (the first and third points) are satisfied. By this lemma, passing in (67) to the weak limit in $L_2(0, T; V^{-3})$, we obtain that the limit function satisfies the following relation:

$$\begin{aligned}(J + \varepsilon e^{-\gamma t} A^4 + \varkappa A)\Pi_T v'_* + \nu A \Pi_T v_* - B_1(\Pi_T v_*)& \\ - \varkappa B_2(\Pi_T v_*) - \varkappa B_3(\Pi_T v_*) + N(\Pi_T v_*) = f.&\end{aligned}$$

Thus, the function $\Pi_T v_*$ is the solution to equation (24) on $[0, T]$. Due to the arbitrariness of T, the function v_* is the solution to equation (24) on $\mathbb{R}_+$.

Let us prove estimate (64). Due to Corollary 1, the following inequality holds:

$$\|v_{m_k}(t)\|_{V^1} + \|v'_{m_k}(t)\|_{V^{-1}} \leqslant C_{14} + C_{15} e^{-\gamma t}\Big(C_6 \|v(0)\|^2_{V^1} + \varepsilon \|v(0)\|^2_{V^4}\Big). \quad (68)$$

For each m_k, this inequality holds for all $t \in \mathbb{R}_+ \setminus Q_{m_k}$, where Q_{m_k} is some set of zero measure. Let $Q = \cup_{m_k} Q_{m_k}$. Then, Q is a set of zero measure. Therefore, for all $t \in \mathbb{R}_+ \setminus Q$, inequality (68) holds for each m_k.

Due to the above-mentioned strong convergence $v_{m_k} \to v_*$ in $C(\mathbb{R}_+, V^1)$, for any $t \in \mathbb{R}_+ \setminus Q$, it holds that $v_{m_k}(t) \to v_*(t)$ in V^1. Due to inequality (68), the sequence $\{v_{m_k}(t)\}$ is bounded in V^1, and the sequence $\{v'_{m_k}(t)\}$ is bounded in V^{-1}. Therefore, without loss of generality and, if necessary, passing to a subsequence, we obtain that $v_{m_k}(t)$ converges weakly to $v_*(t)$ in V^1 and $v'_{m_k}(t)$ converges weakly to $v'_*(t)$ in V^{-1}. Consequently,

$$\|v_*(t)\|_{V^1} \leqslant \varliminf_{m_k \to \infty} \|v_{m_k}(t)\|_{V^1} \leqslant C_{14} + C_{15} e^{-\gamma t}\Big(C_6 \|v(0)\|^2_{V^1} + \varepsilon \|v(0)\|^2_{V^4}\Big);$$
$$\|v'_*(t)\|_{V^{-1}} \leqslant \varliminf_{m_k \to \infty} \|v'_{m_k}(t)\|_{V^{-1}} \leqslant C_{14} + C_{15} e^{-\gamma t}\Big(C_6 \|v(0)\|^2_{V^1} + \varepsilon \|v(0)\|^2_{V^4}\Big).$$

Thus, for almost all $t \in \mathbb{R}_+$, we have the estimates

$$\|v_*(t)\|_{V^1} \leqslant C_{14} + C_{15} e^{-\gamma t}\Big(C_6 \|v(0)\|^2_{V^1} + \varepsilon \|v(0)\|^2_{V^4}\Big);$$
$$\|v'_*(t)\|_{V^{-1}} \leqslant C_{14} + C_{15} e^{-\gamma t}\Big(C_6 \|v(0)\|^2_{V^1} + \varepsilon \|v(0)\|^2_{V^4}\Big).$$

Adding these estimates, we obtain the required estimate (64).

Estimate (65) is obtained in a similar way. □

Theorem 11. *Let the coefficients $\nu, \alpha_i, \beta_i, i = \overline{1, L}$ satisfy the conditions in (34). Then, problem (23), (13) has a weak solution on the semi-axis $\mathbb{R}_+$, satisfying for almost all $t > 0$ the inequality*

$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^{-1}} \leqslant C_{20}(1 + e^{-\gamma t}\|v(0)\|^2_{V^1}). \tag{69}$$

Here, C_{20} is a constant that depends on $\nu, \varkappa, f$ and does not depend on v and ε.

Proof. Since V^5 is dense in V^1, then for any $a \in V^1$ there is a sequence $\{b_m\} \subset V^5$ such that $\|b_m - a\|_{V^1} \to 0$ as $m \to \infty$. Let us put $\varepsilon_m = 1/\Big(m(1 + \|b_m\|^2_{V^4})\Big)$. Then, $\varepsilon_m \to 0$ as $m \to \infty$, and

$$\varepsilon_m \|b_m\|^2_{V^4} \leqslant 1. \tag{70}$$

By Theorem 10, for each $b_m \in V^5$, there exists a solution v_m of Equation (24) on $\mathbb{R}_+$ with $\varepsilon = \varepsilon_m$, and v_m satisfies the initial condition

$$v_m(0) = b_m.$$

By virtue of Theorem 10 taking into account inequality (70), the following estimates hold:

$$\|v_m(t)\|_{V^1} + \|v'_m(t)\|_{V^{-1}} \leqslant C_{18} + C_{19} e^{-\gamma t}\Big(C_6 \|b_m\|^2_{V^1} + 1\Big); \tag{71}$$
$$\varepsilon_m e^{-\gamma t}\|v'_m(t)\|_{V^5} \leqslant C_{12} + C_{13} e^{-\gamma t}\Big(C_6 \|b_m\|^2_{V^1} + 1\Big). \tag{72}$$

Similar to the proof of Theorem 10, we have that each of these inequalities for each m holds for all $t \in \mathbb{R}_+ \setminus Q_m$. Here, Q_m is some set of zero measure. Therefore, each of these inequalities holds for all m and for all $t \in \mathbb{R}_+ \setminus Q$, where $Q = \cup_m Q_m$ is a set of zero measure.

Let us show that the sequence $\{v_m\}$ is relatively compact in $C(\mathbb{R}_+, L_4(\Omega)^n)$. For any $T > 0$, due to estimate (71), the sequence $\{\Pi_T v_m\}$ is bounded in $L_\infty(0, T; V^1)$, and the sequence $\{\Pi_T v'_m\}$ is bounded in $L_\infty(0, T; V^{-1})$. That is, the sequence $\{\Pi_T v_m\}$ is bounded in $W_1[0, T]$. Analogously to Lemma 7, due to the compact embedding $W_1[0, T] \subset C([0, T], L_4(\Omega)^n)$, the sequence $\{\Pi_T v_m\}$ is relatively compact in $C([0, T], L_4(\Omega)^n)$. Then, by Lemma 1 by virtue of the arbitrariness of T, the sequence $\{v_m\}$ is relatively compact in $C(\mathbb{R}_+, L_4(\Omega)^n)$.

Since the sequence $\{v_m\}$ is relatively compact, it contains a subsequence $\{v_{m_k}\}$ which converges in $C(\mathbb{R}_+, L_4(\Omega)^n)$ to some function v_*. Let us show that v_* is the solution of (23), (13) on the semi-axis $\mathbb{R}_+$.

Let us show that $v_* \in W_1^{\text{loc}}(\mathbb{R}_+)$. For arbitrary $T > 0$, from estimate (71), the sequences $\{\Pi_T v_{m_k}\}$ and $\{\Pi_T v'_{m_k}\}$ are bounded in $L_\infty(0,T;V^1)$ and $L_\infty(0,T;V^{-1})$, respectively. Therefore, without loss of generality, the sequence $\{\Pi_T v_{m_k}\}$ converges $*$-weakly in $L_\infty(0,T;V^1)$ to v_*. Similarly, without loss of generality, the sequence $\{\Pi_T v'_{m_k}\}$ converges $*$-weakly in $L_\infty(0,T;V^{-1})$ to some function $u \in L_\infty(0,T;V^{-1})$. But in the sense of distributions on $(0,T)$ with values in V^{-1}, the sequence $\{\Pi_T v'_{m_k}\}$ converges to v'_*. Therefore, $u = \Pi_T v'_*$. Consequently, $\Pi_T v_* \in L_\infty(0,T;V^1)$, and $\Pi_T v'_* \in L_\infty(0,T;V^{-1})$. That is, $\Pi_T v_* \in W_1[0,T]$. Due to the arbitrariness of T, the function v_* belongs to $W_1^{\text{loc}}(\mathbb{R}_+)$.

Let us show that the function v_* is a solution to Equation (23) on $\mathbb{R}_+$, that is, the restriction $\Pi_T v_*$ on every interval $[0,T]$ $(T > 0)$ is a solution to Equation (23) on $[0,T]$.

Since the sequence $\{\hat{v}_{m_k}\}$ converges strongly to v_* in $C(\mathbb{R}_+, L_4(\Omega)^n)$, then for any $T > 0$, the sequence of restrictions $\{\Pi_T v_{m_k}\}$ converges strongly to $\Pi_T v_*$ in $C([0,T], L_4(\Omega)^n)$. Functions $\Pi_T v_{m_k}$ are solutions to Equation (24), that is,

$$(J + \varepsilon e^{-\gamma t} A^4 + \varkappa A)\Pi_T v'_{m_k} + \nu A \Pi_T v_{m_k} - B_1(\Pi_T v_{m_k}) \\ - \varkappa B_2(\Pi_T v_{m_k}) - \varkappa B_3(\Pi_T v_{m_k}) + N(v_{m_k}) = f. \quad (73)$$

From inequality (71), it follows that the sequence $\{\Pi_T v_{m_k}\}$ is bounded in $L_\infty(0,T;V^1)$, and the sequence $\{\Pi_T v'_{m_k}\}$ is bounded in $L_\infty(0,T;V^{-1})$. Due to (72), the sequence $\varepsilon_{m_k} v'_{m_k}$ is bounded in $L_\infty(0,T;V^5)$, and due to our choice, $\varepsilon_{m_k} \to 0$. Therefore, by Lemma 7, passing to the limit in (73) as $m_k \to \infty$, we obtain

$$(J + \varkappa A)\Pi_T v'_* + \nu A \Pi_T v_* - B_1(\Pi_T v_*) - \varkappa B_2(\Pi_T v_*) - \varkappa B_3(\Pi_T v_*) + N(v_*) = f.$$

Due to the arbitrariness of T, we obtain that v_* is a solution to problem (23), (13) on the semi-axis $\mathbb{R}_+$.

Let us check that v_* satisfies initial condition (13). The convergence in $C(\mathbb{R}_+, L_4(\Omega)^n)$ implies pointwise convergence. Consequently,

$$b_{m_k} = v_{m_k}(0) \to v_*(0) \quad \text{strongly in } L_4(\Omega)^n.$$

Due to the choice of the sequence $\{b_m\}$, there is the strong convergence $b_{m_k} \to a$ in V^1. Due to the uniqueness of the limit, $v_*(0) = a$. Namely, v_* satisfies initial condition (13).

Let us prove inequality (69). As already mentioned, inequality (71) holds for all m_k and for all t belonging to some subset $\mathbb{R}_+$ of full measure. Take some such t. From (71), it follows that the sequences $\{v_{m_k}(t)\}$ and $\{v'_{m_k}(t)\}$ are bounded in V^1 and V^{-1}, respectively. Consequently, each of them contains subsequences $v_l(t)$ and $v'_l(t)$, which weakly converge to $v_*(t)$ in V^1 and to $v'_*(t)$ in V^{-1}, respectively. Therefore,

$$\|v_*(t)\|_{V^1} \leqslant \varliminf_{l \to \infty} \|v_l(t)\|_{V^1} \leqslant C_{18} + C_{19} e^{-\gamma t}\Big(C_6 \|a\|_{V^1}^2 + 1\Big);$$
$$\|v'_*(t)\|_{V^{-1}} \leqslant \varliminf_{l \to \infty} \|v'_l(t)\|_{V^{-1}} \leqslant C_{18} + C_{19} e^{-\gamma t}\Big(C_6 \|a\|_{V^1}^2 + 1\Big).$$

Thus, for almost all $t \in \mathbb{R}_+$, the following estimates hold:

$$\|v_*(t)\|_{V^1} \leqslant C_{18} + C_{19} e^{-\gamma t}\Big(C_6 \|a\|_{V^1}^2 + 1\Big);$$
$$\|v'_*(t)\|_{V^{-1}} \leqslant C_{18} + C_{19} e^{-\gamma t}\Big(C_6 \|a\|_{V^1}^2 + 1\Big).$$

Adding these estimates, we obtain estimate (69). □

8. Trajectory Space and Attractors

Choose $E = V^1$ and $E_0 = V^{-1}$ as the two Banach spaces needed to define the trajectory space.

Let us define the trajectory space $\mathcal{H}^+$ for Equation (23) as follows. $\mathcal{H}^+$ consists of all solutions of (23) on $\mathbb{R}_+$, essentially bounded as functions with values in V^1 and satisfying for almost all $t > 0$ the estimate

$$\|v(t)\|_{V^1} + \|v'(t)\|_{V^{-1}} \leqslant C_{20}(1 + e^{-\gamma t}\|v\|^2_{L_\infty(\mathbb{R}_+,V^1)}). \tag{74}$$

Let us show the inclusion $\mathcal{H}^+ \subset C(\mathbb{R}_+; V^{-1}) \cap L_\infty(\mathbb{R}_+; V^1)$. The inclusion of $\mathcal{H}^+ \subset L_\infty(\mathbb{R}_+; V^1)$ follows directly from the definition of the trajectory space. Let v be a trajectory from $\mathcal{H}^+$. Then, from inequality (74), for an arbitrary $T > 0$, we obtain that $\Pi_T v' \in L_\infty(0, T; V^{-1})$. Therefore, $\Pi_T v$ belongs to the space $C([0, T], V^{-1})$ as an integral with a variable upper limit. Due to the arbitrariness of T, the function $v \in C(\mathbb{R}_+; V^{-1})$, as it is required.

Let us show that the space $\mathcal{H}^+$ is not empty. The following theorem holds.

Theorem 12. *Let the coefficients $\nu, \alpha_i, \beta_i, i = \overline{1, L}$ satisfy conditions (34). Then, for each $a \in V^1$, there exists a trajectory $v \in \mathcal{H}^+$, such that $v(0) = a$.*

Proof. By Theorem 11, there exists a solution $v \in W_1^{loc}(\mathbb{R}_+)$ to problem (23), (13) on $\mathbb{R}_+$. Let us show that v is a trajectory. For this, let us show that v satisfies estimate (74). Since for v inequality (69) holds, it suffices to show that

$$\|v(0)\|_{V^1} \leqslant \|v\|_{L_\infty(\mathbb{R}_+;V^1)}. \tag{75}$$

By estimate (69), v belongs to $L_\infty(\mathbb{R}_+; V^1)$, and v' belongs to $L_\infty(\mathbb{R}_+; V^{-1})$. Therefore, similarly to the proof of this fact for trajectories, we obtain that $v \in C(\mathbb{R}_+; V^{-1})$. Thus, $v \in C(\mathbb{R}_+; V^{-1}) \cap L_\infty(\mathbb{R}_+; V^1)$. By Theorem 1, it holds that $v \in C_w(\mathbb{R}_+; V^1)$. Therefore, for any $t \in \mathbb{R}_+$, the value $v(t) \in V^1$ is well defined. From this fact and from the definition of the norm in $L_\infty(\mathbb{R}_+, V^1)$, we obtain the required inequality (75). □

The main result of this paper is contained in the following two theorems on the existence of a minimal trajectory and a global attractor.

Theorem 13. *Let the coefficients $\nu, \alpha_i, \beta_i, i = \overline{1, L}$ satisfy conditions (34). Then, there exists a minimal trajectory attractor $\mathcal{U}$ of the trajectory space $\mathcal{H}^+$.*

Proof. By Theorem 2, it is sufficient to establish the existence of a trajectory semi-attractor.

Consider the set

$$P = \Big\{v \in C(\mathbb{R}_+; V^{-1}) \cap L_\infty(\mathbb{R}_+; V^1) \colon v' \in L_\infty(\mathbb{R}_+, V^1),$$
$$\|v(t)\|_{V^2} + \|v'(t)\|_{V^{-1}} \leqslant 2C_{20} \text{ for allmost all } t \in \mathbb{R}_+\Big\}.$$

From the definition of P, it immediately follows that P is bounded in $L_\infty(\mathbb{R}_+; V^1)$. Further, $T(h)P \subset P, h \geqslant 0$. Consequently, P is translation-invariant.

Let us show that P is relatively compact in $C(\mathbb{R}_+; V^{-1})$. From the definition of P, for any $T > 0$, the set $\Pi_T P$ is bounded in $L_\infty(0, T; V^1)$, and the set $\{v' \colon v \in \Pi_T P\}$ is bounded in $L_\infty(0, T; V^{-1})$. By Theorem 4, the set $\Pi_T P$ is relatively compact in $C([0, T], V^{-1})$. Due to the arbitrariness of T, by Lemma 1, the set P is relatively compact in $C(\mathbb{R}_+; V^{-1})$.

Let us show that P is an absorbing set for $\mathcal{H}^+$. Let B be an arbitrary subset of $\mathcal{H}^+$ bounded in $L_\infty(\mathbb{R}_+; V^1)$. Namely, let there exist a constant $R > 0$ such that for all $v \in B$ the inequality $\|v\|_{L_\infty(\mathbb{R}_+;V^1)} \leqslant R$ holds.

Let us choose $h_0 \geqslant 0$ such that $R^2 e^{-\gamma h_0} \leqslant 1$. Let v be an arbitrary function from B. Since v satisfies (74), then for $h \geqslant h_0$ we have

$$\|T(h)v(t)\|_{V^1} + \|T(h)v'(t)\|_{V^{-1}} = \|v(t+h)\|_{V^1} + \|v'(t+h)\|_{V^{-1}} \\ \leqslant C_{20}(1 + e^{-\gamma(t+h)}R^2) \leqslant C_{20}(1 + e^{-\gamma h_0}R^2) \leqslant 2C_{20}.$$

Thus, $T(h)v \in P$.

Since the function v is arbitrary, for all $h \geqslant h_0$, the inclusion $T(h)B \subset P$ holds. Therefore, P is an absorbing set.

Thus, the conditions of Lemma 2 are satisfied. Therefore, $\overline{P}$ is a trajectory semi-attractor. Then, by Theorem 2, there exists a trajectory attractor of the trajectory space $\mathcal{H}^+$. □

Theorem 14. *Let the coefficients $\nu, \alpha_i, \beta_i, i = \overline{1, L}$ satisfy the conditions in (34). Then, there exists a global attractor $\mathcal{A}$ of the trajectory space $\mathcal{H}^+$.*

Proof. The statement follows directly from Theorems 13 and 3. □

Author Contributions: Investigation, M.T. and A.U.; Writing—original draft, M.T. and A.U.; Writing—review and editing, M.T. and A.U. All authors have read and agreed to the published version of the manuscript.

Funding: This study was supported by the Russian Science Foundation, grant No. 23-21-00091.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Ladyzhenskaya, O.A. A dynamical system generated by the Navier-Stokes equations. *J. Sov. Math.* **1975**, *3*, 458–479. [CrossRef]
2. Ladyzhenskaya, O.A. On the determination of minimal global attractors for the Navier–Stokes and other partial differential equations. *Russ. Math. Surv.* **1987**, *42*, 27–73. [CrossRef]
3. Temam, R. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed.; Springer: New York, NY, USA, 1997; Volume 68. [CrossRef]
4. Seregin, G.A. On a dynamical system generated by the two-dimensional equations of the motion of a Bingham fluid. *J. Math. Sci.* **1994**, *70*, 1806–1816. [CrossRef]
5. Chepyzhov, V.V.; Vishik, M.I. Evolution equations and their trajectory attractors. *J. Math. Pures Appl.* **1997**, *76*, 913–964. [CrossRef]
6. Chepyzhov, V.V.; Vishik, M.I. *Attractors for Equations of Mathematical Physics*; American Mathematical Society: Providence, RI, USA, 2002; Volume 49.
7. Sell, G.R.; You, Y. *Dynamics of Evolutionary Equations*; Springer: New York, NY, USA, 2002; Volume 143.
8. Zvyagin, V.G.; Vorotnikov, D.A. *Topological Approximation Methods for Evolutionary Problems of Nonlinear Hydrodynamics*; Walter de Gruyter: Berlin, Germany, 2008; Volume 12. [CrossRef]
9. Vorotnikov, D.A.; Zvyagin, V.G. Trajectory and global attractors of the boundary value problem for autonomous motion equations of viscoelastic medium. *J. Math. Fluid Mech.* **2008**, *10*, 19–44. [CrossRef]
10. Pavlovsky, V.A. On theoretical description of weak aqueous solutions of polymers. *Dokl. Akad. Nauk SSSR* **1971**, *200*, 809–812. (In Russian)
11. Amfilokhiev, V.B.; Pavlovsky, V.A. Experimental data on laminar-turbulent transition for flows of polymer solutions in pipes. *Tr. Leningr. Korablestr. Inst.* **1975**, *104*, 3–5. (In Russian)
12. Amfilokhiev, V.B.; Voitkunskii, Y.I.; Mazaeva, N.P.; Khodornovskii, Y.S. Flows of Polymer Solutions in the Case of Convective Accelerations. *Tr. Leningr. Korablestr. Inst.* **1975**, *96*, 3–9. (In Russian)
13. Vinogradov, G.V.; Malkin, A.Y. *Rheology of Polymers: Viscoelasticity and Flow of Polymers*; Springer: Berlin/Heidelberg, Germany, 1980.
14. Oskolkov, A.P. Initial-boundary value problems for equations of motion of Kelvin-Voight fluids and Oldroyd fluids. *Proc. Steklov Inst. Math.* **1989**, *179*, 137–182.
15. Zvyagin, V.G.; Turbin, M.V. The study of initial-boundary value problems for mathematical models of the motion of Kelvin–Voigt fluids. *J. Math. Sci.* **2010**, *168*, 157–308. [CrossRef]
16. Mohan, M.T. On the three dimensional Kelvin-Voigt fluids: Global solvability, exponential stability and exact controllability of Galerkin approximations. *Evol. Equ. Control Theory* **2020**, *9*, 301–339. [CrossRef]
17. Di Plinio, F.; Giorgini, A.; Pata, V.; Temam, R. Navier-Stokes-Voigt equations with memory in 3D lacking instantaneous kinematic viscosity. *J. Nonlinear Sci.* **2018**, *28*, 653–686. [CrossRef]

18. Mohan, M.T. Global attractors, exponential attractors and determining modes for the three dimensional Kelvin-Voigt fluids with "fading memory". *Evol. Equ. Control Theory* **2022**, *11*, 125–167. [CrossRef]
19. Amrouche, C.; Berselli, L.C.; Lewandowski, R.; Nguyen, D.D. Turbulent flows as generalized Kelvin-Voigt materials: Modeling and analysis. *Nonlinear Anal.* **2020**, *196*, 111790. [CrossRef]
20. Berselli, L.C.; Kim, T.Y.; Rebholz, L.G. Analysis of a reduced-order approximate deconvolution model and its interpretation as a Navier-Stokes-Voigt regularization. *Discret. Contin. Dyn. Syst. B* **2016**, *21*, 1027–1050. [CrossRef]
21. Kalantarov, V.K.; Levant, B.; Titi, E.S. Gevrey regularity for the attractor of the 3D Navier-Stokes-Voight equations. *J. Nonlinear Sci.* **2009**, *19*, 133–152. [CrossRef]
22. Kalantarov, V.K.; Titi, E.S. Global attractors and determining modes for the 3D Navier-Stokes-Voight equations. *Chin. Ann. Math. Ser. B* **2009**, *30*, 697–714. [CrossRef]
23. Kalantarov, V.K.; Titi, E.S. Global stabilization of the Navier-Stokes-Voight and the damped nonlinear wave equations by finite number of feedback controllers. *Discret. Contin. Dyn. Syst. B* **2018**, *23*, 1325–1345. [CrossRef]
24. Turbin, M.; Ustiuzhaninova, A. Pullback attractors for weak solution to modified Kelvin-Voigt model. *Evol. Equ. Control Theory* **2022**, *11*, 2055–2072. [CrossRef]
25. Ustiuzhaninova, A.S.; Turbin, M.V. Trajectory and Global Attractors for a Modified Kelvin–Voigt Model. *J. Appl. Ind. Math.* **2021**, *15*, 158–168. [CrossRef]
26. Antontsev, S.N.; de Oliveira, H.B.; Khompysh, K. Generalized Kelvin-Voigt equations for nonhomogeneous and incompressible fluids. *Commun. Math. Sci.* **2019**, *17*, 1915–1948. [CrossRef]
27. Antontsev, S.N.; De Oliveira, H.B.; Khompysh, K. The classical Kelvin-Voigt problem for incompressible fluids with unknown non-constant density: Existence, uniqueness and regularity. *Nonlinearity* **2021**, *34*, 3083–3111. [CrossRef]
28. Zvyagin, V.G.; Turbin, M.V. The optimal feedback control problem for Voigt model with variable density. *Russ. Math.* **2020**, *64*, 80–84. [CrossRef]
29. Zvyagin, V.; Turbin, M. Optimal feedback control problem for inhomogeneous Voigt fluid motion model. *J. Fixed Point Theory Appl.* **2021**, *23*, 4. [CrossRef]
30. Turbin, M.V. Research of a mathematical model of low-concentrated aqueous polymer solutions. *Abstr. Appl. Anal.* **2006**, *2006*, 012497. [CrossRef]
31. Turbin, M.; Ustiuzhaninova, A. Existence of weak solution to initial-boundary value problem for finite order Kelvin-Voigt fluid motion model. *Boletín Soc. Mat. Mex.* **2023**, *29*, 54. [CrossRef]
32. DiPerna, R.J.; Lions, P.L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **1989**, *98*, 511–547. [CrossRef]
33. Zvyagin, V.G.; Kondrat'ev, S.K. Attractors of equations of non-Newtonian fluid dynamics. *Russ. Math. Surv.* **2014**, *69*, 845–913. [CrossRef]
34. Gajewski, H.; Gröger, K.; Zacharias, K. Nichtlineare operatorgleichungen und operatordifferentialgleichungen. *Math. Nachrichten* **1975**, *67*, 337–341. [CrossRef]
35. Temam, R. *Navier-Stokes Equations: Theory and Numerical Analysis*; AMS Chelsea: Providence, RI, USA, 2001.
36. Ladyzhenskaya, O.A. *The Mathematical Theory of Viscous Incompressible Flow*; Gordon and Breach Science Publishers: New York, NY, USA, 1969; Volume 2.
37. Solonnikov, V.A. Estimates of Green's tensors for some boundary-value problems. *Dokl. Akad. Nauk SSSR* **1960**, *130*, 988–991. (In Russian)
38. Vorovich, I.I.; Yudovich, V.I. Steady flow of a viscous incompressible fluid. *Mat. Sb.* **1961**, *53*, 393–428. (In Russian)
39. Fursikov, A.V. *Optimal Control of Distributed Systems. Theory and Applications*; American Mathematical Society: Providence, RI, USA, 2000; Volume 187.
40. Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.* **1986**, *146*, 65–96. [CrossRef]
41. Orlov, V.P.; Sobolevskii, P.E. On mathematical models of a viscoelasticity with a memory. *Differ. Integral Equ.* **1991**, *4*, 103–115. [CrossRef]
42. Orlov, V.P.; Parshin, M.I. On a problem in the dynamics of a thermoviscoelastic medium with memory. *Comput. Math. Math. Phys.* **2015**, *55*, 650–665. [CrossRef]