mathematics

Article

A New Notion of Fuzzy Function Ideal Convergence

Dimitrios Georgiou *,† and Georgios Prinos †

Department of Mathematics, University of Patras, 265 00 Patras, Greece; gprinos161168@yahoo.gr
* Correspondence: georgiou@math.upatras.gr
† These authors contributed equally to this work.

Abstract: P.M. Pu and Y.M. Liu extended Moore-Smith's convergence of nets to fuzzy topology and Y.M. Liu provided analogous results to J. Kelley's classical characterization theorem of net convergence by introducing the notion of fuzzy convergence classes. In a previous paper, the authors of this study provided modified versions of this characterization by using an alternative notion of convergence of fuzzy nets, introduced by B.M.U. Afsan, named fuzzy net ideal convergence. Our main scope here is to generalize and simplify the preceding results. Specifically, we insert the concept of a fuzzy function ideal convergence class, $\mathcal{L}$, on a non-empty set, X, consisting of triads $(f, e, \mathcal{I})$, where f is a function from a non-empty set, D, to the set $\mathrm{FP}(X)$ of fuzzy points in X, which we call fuzzy function, $e \in \mathrm{FP}(X)$, and $\mathcal{I}$ is a proper ideal on D, and we provide necessary and sufficient conditions to establish the existence of a unique fuzzy topology, δ, on X, such that $(f, e, \mathcal{I}) \in \mathcal{L}$ iff f $\mathcal{I}$-converges to e, relative to the fuzzy topology δ.

Keywords: fuzzy set; fuzzy topology; fuzzy function ideal convergence; fuzzy function ideal convergence class

MSC: 54A20

check for updates

Citation: Georgiou, D.; Prinos, G. A New Notion of Fuzzy Function Ideal Convergence. *Mathematics* **2024**, *12*, 260. https://doi.org/10.3390/math12020260

Academic Editors: Antonin Dvorak, Manuel Sanchis and Salvador Romaguera

Received: 10 December 2023
Revised: 4 January 2024
Accepted: 9 January 2024
Published: 12 January 2024

1. Introduction

The main concept of this paper is the convergence of a fuzzy function, with respect to an ideal, in fuzzy topological spaces. In ordinary topological spaces, the notion of convergence of a function, with respect to an ideal, under the name function $\mathcal{I}$-convergence (see [1–3]), is the dual notion of convergence of a function, with respect to a filter, which H. Cartan introduced in [4] (see also [5] (p. 71, Definition 1)).

The concept of convergence of a function, with respect to a filter, naturally generalizes classical net and filter convergence in topological spaces (see [2,3]). It is worth noting that the concept of convergence of a sequence, with respect to a filter or ultrafilter, has been reintroduced and studied by many authors (see [6–9]). By using ideals instead of filters, the notion of convergence, with respect to a filter, can be equivalently reformulated (see, e.g., frequently quoted works [10–12]). The reason for this modified version of convergence was an effort to generalize previous results on statistical convergence (an extension of the classical notion of convergence of sequences relative to the asymptotic density, e.g., [13–15]).

B.K. Lahiri and P. Das, in [16,17], investigated the notion of the ideal convergence of sequences and nets in topological spaces. In the context of net ideal convergence, the authors of [18,19] provided a modified version of J. Kelley's classical theorem [20] (p. 74, Theorem 9) for convergence classes. More precisely, in [18] they considered a non-empty set, X, and a class, $\mathcal{C}$, consisting of triples of the form $((s_d)_{d \in D}, x, \mathcal{I})$, where $(s_d)_{d \in D}$ is a net with a domain on the directed set, D, and values on X, $\mathcal{I}$ is a D-admissible ideal on D and $x \in X$ and provided a set of axioms on the class, $\mathcal{C}$, which are necessary and sufficient to ensure the existence of a unique topology, τ, on X, such that $((s_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}$ iff $(s_d)_{d \in D}$ $\mathcal{I}$-converges to x, relative to the topology, τ. Subsequently, in [19], they provided similar results by considering arbitrary ideals. In continuation, in [21] they extended and simplified the results of the last two papers by considering functions instead of nets and a smaller set of axioms, to be fulfilled by class $\mathcal{C}$, in order for the last to be topological.

The introduction of the fundamental notion of a fuzzy set in 1965, by L. Zadeh [22], provided the natural background for generalizing many of the concepts of general topology to the fuzzy setting. Following the generalization of Moore-Smith convergence of nets (see [23–25]) to fuzzy topological spaces, which was provided by P.M. Pu and Y.M. Liu in [26,27], a characterization that correlates fuzzy topologies with fuzzy net convergence classes was introduced by Y.M. Liu in [28], which succeeded in generalizing J. Kelley's theorem.

The concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, and $\mathcal{I}$-Cauchyness for sequences of fuzzy numbers were defined and studied in [29]. The notion of $\mathcal{I}$-convergence of fuzzy nets in fuzzy topological spaces was provided by B.M.U. Afsan in [30]. Moreover, the authors of [31], by using the previous notion, obtained a modification of the Y.M. Liu characterization theorem on fuzzy convergence classes. Particularly, they introduced the concept of a fuzzy net ideal convergence class, $\mathcal{H}$, on a non-empty set, X, consisting of triples of the form $((s_d)_{d\in D}, e, \mathcal{I})$, where $(s_d)_{d\in D}$ is a fuzzy net in X, $\mathcal{I}$ is an ideal on D, and $e \in \mathrm{FP}(X)$, and provided a set of axioms on the class $\mathcal{H}$ to obtain the following result: there exists a unique fuzzy topology, $\Psi(\mathcal{H})$, on X, such that $((s_d)_{d\in D}, e, \mathcal{I}) \in \mathcal{H}$ iff $(s_d)_{d\in D}$ $\mathcal{I}$-converges to e, with respect to $\Psi(\mathcal{H})$.

In this paper, we extend and simplify the results of the latter work by considering fuzzy functions instead of fuzzy nets and a concise set of axioms. We should note that previous ideas are reorganized efficiently and full proofs of the most important points are provided, rather than pointing out the necessary adaptations. In addition, this work extends the results of [21] to fuzzy topological spaces.

The rest of this paper is divided into three sections as follows. In Section 2, we provide the preliminaries that will be used later. In Section 3, we present some special properties concerning the convergence of a fuzzy function, with respect to an ideal, on fuzzy topological spaces. Finally, in Section 4, we consider the notion of a fuzzy function ideal convergence class on a non-empty set, X, and prove analogous to the results of [19] for the more general case of fuzzy function ideal convergence.

2. Preliminaries

In this section, we review basic concepts that will be used in the following sections, and we refer the reader to [22,26,28,32] for more details.

Let X be a non-empty set. We will use the symbols I and I^X to represent the unit closed interval $[0,1]$ and the set of all functions with domain X and range I, respectively. A function $A : X \to I$ is called a *fuzzy set* in X (due to Zadeh [22]), i.e., a fuzzy set in X is an element of I^X. For every $x \in X$, $A(x)$ is called the *grade of membership* of x in A and the set $\{x \in X : A(x) > 0\}$ is called the *support* of A. If A takes only the values 0 and 1, then A is called a *crisp set* in X. Particularly, we will use the notation $\mathbf{1}$ for the crisp set that always takes the value 1 on X and $\mathbf{0}$ for the crisp set that always takes the value 0 on X. In addition, if A, B are fuzzy sets in X, we say that A is *contained* in B, which we will denote by $A \leqslant B$, whenever $A(x) \leq B(x)$, for every $x \in X$.

Let $\mathcal{A} = \{A_\lambda | \lambda \in \Lambda\}$ be a family of fuzzy sets in X, with Λ being the indexed set. The *union* $\vee\mathcal{A}$ and the *intersection* $\wedge\mathcal{A}$ of the family are the fuzzy sets defined, respectively, by the following rules:

$$(\vee\mathcal{A})(x) = \sup\{A_\lambda(x) : \lambda \in \Lambda\},\ x \in X$$

$$(\wedge\mathcal{A})(x) = \inf\{A_\lambda(x) : \lambda \in \Lambda\},\ x \in X.$$

If A is a fuzzy set, the *complement* A' of A is a fuzzy set, defined by the formula:

$$A'(x) = 1 - A(x),\ x \in X.$$

The following De Morgan's laws also hold:

$$(\vee\{A_\lambda : \lambda \in \Lambda\})' = \wedge\{A'_\lambda : \lambda \in \Lambda\}$$

$$(\wedge\{A_\lambda : \lambda \in \Lambda\})' = \vee\{A'_\lambda : \lambda \in \Lambda\}.$$

A family, δ, of fuzzy sets in X is called a *fuzzy topology* for X (due to Chang [32]) if

(1) $\mathbf{0}, \mathbf{1} \in \delta$,
(2) $A \wedge B \in \delta$, whenever $A, B \in \delta$, and
(3) $\vee\{A_\lambda : \lambda \in \Lambda\} \in \delta$, whenever $A_\lambda \in \delta$, for every $\lambda \in \Lambda$.

Moreover, the pair (X, δ) is called a *fuzzy topological space*, or *fts* for short. Every member of δ is called a δ*-open* (or simply *open*) fuzzy set. The complement of a δ-open fuzzy set is called a δ*-closed* (or simply *closed*) fuzzy set. Let δ_1 and δ_2 be two fuzzy topologies for X. We say that δ_2 is *finer* than δ_1 and δ_1 is *coarser* than δ_2 if the inclusion relation $\delta_1 \subseteq \delta_2$ holds.

In this paper, we adopted the notions of fuzzy point and Q-neighborhood from [26]. A fuzzy set in X is called a *fuzzy point* if its support is a singleton $\{x\}$, for some $x \in X$. If its value at x is $\lambda \in (0, 1]$, we will denote the fuzzy point by x_λ. The set of all fuzzy points in X will be denoted by $\mathrm{FP}(X)$. The fuzzy point x_λ is said to be *contained* in a fuzzy set, A, or to *belong* to A, denoted by $x_\lambda \mathrel{\boldsymbol{\in}} A$ if $\lambda \leq A(x)$. An arbitrary fuzzy set, A, is the union of all the fuzzy points that belong to A.

A fuzzy point x_λ is said to be *quasi-coincident* with a fuzzy set A, which we will denote by $x_\lambda \,\mathrm{q}\, A$ if $\lambda > A'(x)$, or equivalently $\lambda + A(x) > 1$. If A and B are fuzzy sets in X, we will say that A is *quasi-coincident* with B, and we will denote this by $A \,\mathrm{q}\, B$, if there exists $x \in X$ such that $A(x) > B'(x)$, or equivalently $A(x) + B(x) > 1$. In this case, we also say that A and B are *quasi-coincident (with each other) at* x. If A and B are quasi-coincident at x, both $A(x)$ and $B(x)$ are not zero. If A is not quasi-coincident with B, then we will denote this by $A \,\overline{\mathrm{q}}\, B$.

Let (X, δ) be an fts. A fuzzy set, A, in X is said to be a *Q-neighborhood of the fuzzy point* x_λ, if there exists $B \in \delta$ such that $x_\lambda \,\mathrm{q}\, B \leqslant A$. A Q-neighborhood A of a fuzzy point is said to be open if $A \in \delta$. Generally, a Q-neighborhood of a fuzzy point does not necessarily contain the fuzzy point itself.

Definition 1 ([26])**.** *Let (X, δ) be an fts and A be a fuzzy set in X. The intersection of all the δ-closed fuzzy sets containing A is called the (fuzzy) closure of A, denoted by $\overline{A}$, or by $\mathrm{cl}_\delta A$. Obviously, $\overline{A}$ is the smallest δ-closed fuzzy set containing A and $\overline{(\overline{A})} = \overline{A}$.*

Definition 2 ([26])**.** *A map, $\sigma : I^X \to I^X$, is called a fuzzy closure operator on X if f satisfies the following Kuratowski closure axioms:*

(FCO1) $\sigma(\mathbf{0}) = \mathbf{0}$,
(FCO2) $A \leqslant \sigma(A)$,
(FCO3) $\sigma(\sigma(A)) = \sigma(A)$, *and*
(FCO4) $\sigma(A \vee B) = \sigma(A) \vee \sigma(B)$.

If X is an fts, then the map $\rho : I^X \to I^X$ with $\rho(A) = \overline{A}$ is a fuzzy closure operator on X, and conversely, every fuzzy closure operator on X determines a fuzzy topology for X. For this, we have the following:

Proposition 1 ([26])**.** *If σ is a fuzzy closure operator on X and $\kappa = \{A \in I^X : \sigma(A) = A\}$, then the family $\delta = \{A' : A \in \kappa\}$ is a fuzzy topology for X, and for every $B \in I^X$, $\mathrm{cl}_\delta B = \sigma(B)$. In this case, δ is said to be the fuzzy topology associated with the fuzzy closure operator, σ.*

A partially preordered set $(D, \geqslant)$ (simply denoted as D) is called *directed* if every two elements of D have an upper bound in D. If $\{(E_d, \geqslant_d)\}_{d \in D}$ is a family of directed sets, the cartesian product $\prod_{d \in D} E_d$ of the family is directed by $\geqslant$, where $f \geqslant g$ if $f(d) \geqslant_d g(d)$, for all $d \in D$.

A *fuzzy net* in X is an arbitrary function, $s : D \to \mathrm{FP}(X)$, where D is directed. If we set $s(d) = s_d$, for all $d \in D$, then the fuzzy net s will be denoted by $(s_d)_{d \in D}$.

A fuzzy net, $t = (t_\lambda)_{\lambda \in \Lambda}$, in X is said to be a *fuzzy semisubnet* of the fuzzy net $s = (s_d)_{d \in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$, i.e., $t_\lambda = s_{\varphi(\lambda)}$ for every $\lambda \in \Lambda$. We write $(t_\lambda)^{\varphi}_{\lambda \in \Lambda}$ to indicate the fact that φ is the function mentioned above.

A fuzzy net $t = (t_\lambda)_{\lambda \in \Lambda}$ in X is said to be a *fuzzy subnet* of the fuzzy net $s = (s_d)_{d \in D}$ in X if t is a fuzzy semisubnet of s and for every $d \in D$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant d$ whenever $\lambda \in \Lambda$ with $\lambda \geqslant \lambda_0$.

Let A be a fuzzy set in X. A fuzzy net $s = (s_d)_{d \in D}$ in X is said to be

(1) *quasi-coincident* with A if, for each $d \in D$, s_d is quasi-coincident with A,
(2) *eventually quasi-coincident* with A if there is an element $d_0 \in D$, such that if $d \in D$ and $d \geqslant d_0$, then s_d is quasi-coincident with A,
(3) *frequently quasi-coincident* with A if for each $d \in D$ there is $d' \in D$ such that $d' \geqslant d$ and $s_{d'}$ is quasi-coincident with A, and
(4) *in* A if for each $d \in D$, $s_d \in A$.

We say that a fuzzy net $s = (s_d)_{d \in D}$, in an fts (X, δ), *converges* to a fuzzy point e in X, relative to δ, if s is eventually quasi-coincident with each Q-neighborhood of e. In this case, we write $\lim_{d \in D} s_d = e$.

Proposition 2 ([26] (Theorem 11.1))**.** *In an fts (X, δ), a fuzzy point $e \in \overline{A}$ iff there is a fuzzy net $s = (s_d)_{d \in D}$ in A such that s converges to e.*

Suppose that D is a directed set and for each $d \in D$ there is a directed set E^d and a fuzzy net $s^d = (s^d(n))_{n \in E^d}$. Then, under product ordering, we have a directed set $F = D \times \prod_{d \in D} E_d$ and a fuzzy net s defined by

$$s(d, f) = s^d(f(d)),\ d \in D,\ f \in \prod_{d \in D} E_d.$$

The fuzzy net s is called an *induced net (associated with D and each s^d)*.

In what follows, let X be a non-empty set and let $\mathcal{G}$ be a class consisting of pairs (s, e), where $s = (s_d)_{d \in D}$ is a fuzzy net in X and e is a fuzzy point in X.

Definition 3 ([28])**.** *We say that $\mathcal{G}$ is a fuzzy convergence class for X if it satisfies the conditions listed below. For convenience, we say that s converges $(\mathcal{G})$ to e or that $\lim_{d \in D} s_d \equiv e(\mathcal{G})$ if $(s, e) \in \mathcal{G}$.*

(G1) If s is such that $s_d = e$, for each $d \in D$, then s converges $(\mathcal{G})$ to e.
(G2) If s converges $(\mathcal{G})$ to e, then so does each fuzzy subnet of s.
(G3) If s does not converge $(\mathcal{G})$ to e, then there exists a fuzzy subnet t of s, no fuzzy subnet of which converges $(\mathcal{G})$ to e.
(G4) We consider the following:
  (1) D is a directed set.
  (2) E_d is a directed set, for each $d \in D$.
  (3) $s^d = (s^d(n))_{n \in E_d}$ is a fuzzy net in X, converging $(\mathcal{G})$ to s_d for each $d \in D$, and the fuzzy net $(s_d)_{d \in D}$, thus obtained, converges $(\mathcal{G})$ to e.
  Then, the induced net (associated with D and each s^d) converges $(\mathcal{G})$ to e.
(G5) For each point $x \in X$ and real directed set $D \subseteq (0, 1]$, if $r \leqslant \sup D$, then the fuzzy net $(x_d)_{d \in D}$ converges $(\mathcal{G})$ to x_r.

Theorem 1 ([28])**.** *Let (X, δ) be a fuzzy topological space. Then, the class of pairs $\{(s, e) :$ the fuzzy net s converges to $e\}$ is a fuzzy convergence class, denoted by $\phi(\delta)$.*

Proposition 3 ([28])**.** *Let Ω be a family of fuzzy points in X and $A = \vee \Omega$. Let the class of pairs, $\mathcal{G}$, satisfy the conditions* (G4) *and* (G5)*. If a fuzzy net s in A converges $(\mathcal{G})$ to e, then there exists a fuzzy net $\bar{s}$ that consists of fuzzy points in Ω and converges $(\mathcal{G})$ to e.*

Theorem 2 ([28] (Theorem 2))**.** *(fuzzy convergence classes theorem). We consider a map,* $\mathrm{c} : I^X \to I^X$*, induced as follows: for each* $A \in I^X$*, we define*

$$\mathcal{G}(A) = \{e : \text{ for some fuzzy net } s \text{ in } A,\ (s,e) \in \mathcal{G}\}$$

$$\mathrm{c}(A) = \vee\mathcal{G}(A).$$

If $\mathcal{G}$ *is a fuzzy convergence class for X, then the following holds:*

(1) The correspondence $A \mapsto \mathrm{c}(A)$ *is a fuzzy closure operator and the fuzzy topology thus obtained will be denoted by* $\psi(\mathcal{G})$*,*
(2) $\phi(\psi(\mathcal{G})) = \mathcal{G}$*, and*
(3) $\psi(\phi(\delta)) = \delta$*, for a fuzzy topology* δ *on X.*

Therefore, there exists a bijective map between the set of all fuzzy topologies δ *for X and the set of all fuzzy convergence classes* $\mathcal{G}$ *for X. Moreover, this map is order-reversing, i.e., if* $\delta_1 \supseteq \delta_2$*, then* $\phi(\delta_1) \subseteq \phi(\delta_2)$*.*

Let D be a non-empty set. A family, $\mathcal{I}$, of subsets of D is called *ideal* on D if $\mathcal{I}$ has the following properties:

(1) $\varnothing \in \mathcal{I}$,
(2) if $\mathrm{A} \in \mathcal{I}$ and $\mathrm{B} \subseteq \mathrm{A}$, then $\mathrm{B} \in \mathcal{I}$, and
(3) if $\mathrm{A}, \mathrm{B} \in \mathcal{I}$, then $\mathrm{A} \cup \mathrm{B} \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$. In some cases, we will also use the notation $\mathcal{I}_D$ for the above ideal.

Definition 4 ([17,30])**.** *Let D be a directed set and* $M_d = \{d' \in D : d' \geqslant d\}$ *for all* $d \in D$*. A proper ideal* $\mathcal{I}$ *on D is called* admissible *if* $D \setminus M_d \in \mathcal{I}$ *for all* $d \in D$*. Moreover,*

$$\mathcal{I}_0(D) = \{A \subseteq D : A \subseteq D \setminus M_d \text{ for some } d \in D\}$$

is a proper ideal on D.

Proposition 4 ([18])**.** *We suppose the following:*

(1) D is a non-empty set.
(2) $\mathcal{I}_D$ *is a proper ideal on D.*
(3) E_d *is a non-empty set, for each* $d \in D$*.*
(4) $\mathcal{I}_{E_d}$ *is a proper ideal on* E_d*, for each* $d \in D$*.*
(5) $\mathcal{I}_D \times \mathcal{I}_{\prod\limits_{d\in D} E_d}$ *is the family of all subsets of* $D \times \prod\limits_{d\in D} E_d$ *for which* $A \in \mathcal{I}_D \times \mathcal{I}_{\prod\limits_{d\in D} E_d}$ *iff there exists* $A_D \in \mathcal{I}_D$*, such that*

$$\{f(d) : (d,f) \in A\} \in \mathcal{I}_{E_d}, \text{ for each } d \in D \setminus A_D.$$

Then, the family $\mathcal{I}_D \times \mathcal{I}_{\prod\limits_{d\in D} E_d}$ *is a proper ideal on* $D \times \prod\limits_{d\in D} E_d$*.*

Let D and Λ be non-empty sets and suppose that $\varphi : \Lambda \to D$ is a function. Then, for every ideal $\mathcal{I}$ on D, the family $\{\mathrm{A} \subseteq \Lambda : \varphi(\mathrm{A}) \in \mathcal{I}\}$ is an ideal on Λ, which will be denoted by $\mathcal{I}_\Lambda(\varphi)$.

In the remainder of the current section, we review the concept of convergence of fuzzy nets, via ideals, in an fts. We note that although in [30] (Section 3) the ideals are supposed to be admissible, and therefore proper, the proofs hold for arbitrary proper ideals.

Definition 5 ([30])**.** *Let* (X,δ) *be an fts and* $\mathcal{I}$ *an ideal of a directed set D. We say that a fuzzy net* $(s_d)_{d\in D}$ $\mathcal{I}$-converges *to a fuzzy point e in X, relative to* δ*, if for every open Q-neighborhood U of e* *we have* $\{d \in D : s_d \,\overline{\mathrm{q}}\, U\} \in \mathcal{I}$*. In this case, we write* $\mathcal{I} - \lim_{d\in D} s_d = e$ *and we say that* e *is the* $\mathcal{I}$-limit *of the fuzzy net* $(s_d)_{d\in D}$.

Proposition 5 ([17,30])**.** *Let* (X, δ) *be an fts and* e *be a fuzzy point in* X*. A fuzzy net* $(s_d)_{d\in D}$ *in* X *converges to* e *iff the fuzzy net* s $\mathcal{I}_0(D)$ *converges to* e*.*

Proposition 6 ([30] (Theorem 3.5))**.** *Let* (X, δ) *be an fts and* A *be a fuzzy set in* X*. If there is a fuzzy net* $(s_d)_{d\in D}$ *in* A *that* $\mathcal{I}$*-converges to the fuzzy point* e *in* X*, where* $\mathcal{I}$ *is a proper ideal on* D*, then* $e \in \mathrm{cl}_\delta(A)$*.*

In addition, the converse of Proposition 6 also holds, if we take into account Propositions 2 and 5.

Finally, the authors of [19] obtained the following results for the ideal convergence of fuzzy nets.

Definition 6. *Let* X *be a non-empty set and let* $\mathcal{H}$ *be a class consisting of triads* $(s, e, \mathcal{I})$*, where* $s = (s_d)_{d\in D}$ *is a fuzzy net in* X*,* e *is a fuzzy point in* X*, and* $\mathcal{I}$ *is an ideal of* D*. We say that* $\mathcal{H}$ *is a fuzzy ideal convergence class for* X *if it satisfies the conditions listed below. For convenience, we say that* s $\mathcal{I}$*-converges* $(\mathcal{H})$ *to* e *or that* $\mathcal{I} - \lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$ *if* $(s, e, \mathcal{I}) \in \mathcal{H}$*.*

- *(C′1)* *If* $(s_d)_{d\in D}$ *is a fuzzy net such that* $s_d = e$ *for every* $d \in D$ *and* $\mathcal{I}$ *is an ideal of* D*, then* $\mathcal{I} - \lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$*.*
- *(C′2)* *If* $\mathcal{I} - \lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$*, where* $\mathcal{I}$ *is an ideal of* D*, then for every fuzzy semisubnet* $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ *of the fuzzy net* $(s_d)_{d\in D}$ *we have* $\mathcal{I}_\Lambda(\varphi) - \lim\limits_{\lambda\in\Lambda} t_\lambda \equiv e(\mathcal{H})$*.*
- *(C′3)* *If* $\mathcal{I} - \lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$*, where* $\mathcal{I}$ *is a proper ideal of* D*, then there exists a fuzzy semisubnet* $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ *of the fuzzy net* $(s_d)_{d\in D}$ *such that* $\mathcal{I}_0(\Lambda) - \lim\limits_{\lambda\in\Lambda} t_\lambda \equiv e(\mathcal{H})$*.*
- *(C′4)* *Let* D *be a directed set and* $\mathcal{I}_D$ *a proper ideal of* D*. If the fuzzy net* $(s_d)_{d\in D}$ *does not* $\mathcal{I}_D$*-converge* $(\mathcal{H})$ *to* e*, then there exists a fuzzy semisubnet* $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ *of the fuzzy net* $(s_d)_{d\in D}$ *such that:*
  - *(1)* *No fuzzy semisubnet* $(r_k)^{f}_{k\in K}$ *of* $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ $\mathcal{I}_K$*-converges* $(\mathcal{H})$ *to* e*, for every proper ideal* $\mathcal{I}_K$ *of* K*.*
  - *(2)* $\mathcal{I}_\Lambda(\varphi)$ *is a proper ideal of* Λ*.*
- *(C′5)* *We consider the following:*
  - *(1)* D *is a directed set.*
  - *(2)* $\mathcal{I}_0(D)$ *is a proper ideal of* D*.*
  - *(3)* E_d *is a directed set for each* $d \in D$*.*
  - *(4)* $\mathcal{I}_0(E_d)$ *is a proper ideal of* E_d *for each* $d \in D$*.*
  - *(5)* $(s(d, e))_{e\in E_d}$ *is a fuzzy net in* X *for each* $d \in D$*.*
  - *(6)* $\mathcal{I}_0(D) - \lim\limits_{d\in D} t_d \equiv e(\mathcal{H})$*, where* $\mathcal{I}_0(E_d) - \lim\limits_{e\in E_d} s(d, e) \equiv t_d(\mathcal{H})$ *for every* $d \in D$*.*

  Then, the fuzzy net $r : D \times \prod\limits_{d\in D} E_d \to \mathrm{FP}(X)$*, where* $r(d, f) = s(d, f(d))$*, for every* $(d, f) \in D \times \prod\limits_{d\in D} E_d$*,* $\mathcal{I}_0\left(D \times \prod\limits_{d\in D} E_d\right)$*-converges* $(\mathcal{H})$ *to* e*.*
- *(C′6)* *For each point* $x \in X$ *and real directed set* $D \subseteq (0, 1]$*, if* $r \leqslant \sup D$ *then the fuzzy net* $(x_d)_{d\in D}$ $\mathcal{I}_0(D)$*-converges* $(\mathcal{H})$ *to* x_r*.*
- *(C′7)* *If* $(s_d)_{d\in D}$ *is a fuzzy net in* X*, then* $\mathcal{P}(D) - \lim\limits_{d\in D} s_d \equiv e(\mathcal{H})$ *for every fuzzy point* $e \in X$*.*

Remark 1. *Let* (X, δ) *be a fuzzy topological space. Then, the class* $\Phi(\delta)$ *consisting of triads* $((s_d)_{d\in D}, e, \mathcal{I})$*, where* $(s_d)_{d\in D}$ *is a fuzzy net in* X*,* e *is a fuzzy point in* X*,* $\mathcal{I}$ *is an ideal of* D*, and* $(s_d)_{d\in D}$ $\mathcal{I}$*-converges to* x*, relative to* δ*, is a fuzzy ideal convergence class since it satisfies all the* *conditions of Definition 6. We say that the fuzzy topology δ* generates *the fuzzy ideal convergence class $\Phi(\delta)$.*

Proposition 7. *Let Ω be a family of fuzzy points in X and $A = \vee\Omega$. Let the class of triads $\mathcal{H}$ satisfy the conditions* (C′3), (C′5), *and* (C′6). *If a fuzzy net $s = (s_d)_{d\in D}$ in A $\mathcal{I}$-converges $(\mathcal{H})$ to e, where $\mathcal{I}$ is a proper ideal of D, then there exists a fuzzy net $\bar{s} = (\bar{s}_k)_{k\in K}$ that consists of fuzzy points in Ω and $\mathcal{I}_0(K)$-converges $(\mathcal{H})$ to e.*

Moreover, the following theorem sets up a one-to-one correspondence between the fuzzy topologies for a non-empty set, X, and the fuzzy net ideal convergence classes on it.

Theorem 3 (fuzzy ideal convergence classes theorem)**.** *Let $\mathcal{H}$ be a fuzzy ideal convergence class for a non-empty set, X. We consider a map $\mathrm{cl} : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define $\mathrm{cl}(A) \in I^X$ to be such that a fuzzy point $e \in \mathrm{cl}(A)$ iff for a fuzzy net $(s_d)_{d\in D}$ in A and a proper ideal $\mathcal{I}$ of the directed set D, $(s_d)_{d\in D}$ $\mathcal{I}$-converges $(\mathcal{H})$ to e, i.e., $(s, e, \mathcal{I}) \in \mathcal{H}$. Then,* cl *is a fuzzy closure operator for a fuzzy topology denoted by $\Psi(\mathcal{H})$ on X and $((s_d)_{d\in D}, e, \mathcal{I}) \in \mathcal{H}$ iff $(s_d)_{d\in D}$ $\mathcal{I}$-converges to e, relative to $\Psi(\mathcal{H})$.*

Corollary 1. *Let $\mathcal{H}$ be a fuzzy ideal convergence class and δ be a fuzzy topology for a non-empty set, X. We have the following:*

(1) $\Phi(\Psi(\mathcal{H})) = \mathcal{H}$ *and*

(2) $\Psi(\Phi(\delta)) = \delta$.

Therefore, there exists a bijective map between the set of all fuzzy topologies δ for X and the set of all fuzzy ideal convergence classes $\mathcal{H}$ for X. Moreover, this map is order-reversing, i.e., if $\delta_1 \supseteq \delta_2$, then $\Phi(\delta_1) \subseteq \Phi(\delta_2)$.

3. The Notion of Convergence of a Fuzzy Function

In this section, we examine the characteristic properties of the notion of convergence of a fuzzy function, with respect to an ideal, on an fts.

Let X be a non-empty set. We will say that a function $f : D \to \mathrm{FP}(X)$ is a *fuzzy function* in X, and we will use the symbolization $f|D$. Consequently, a fuzzy net is a fuzzy function.

Definition 7 ([30])**.** *Let (X, δ) be an fts and $\mathcal{I}$ be a proper ideal on a non-empty set, D. We say that the fuzzy function $f|D$ $\mathcal{I}$-*converges *to a fuzzy point e in X, relative to δ if, for every open Q-neighborhood U of e, we have*

$$\{d \in D : f(d)\,\bar{\mathrm{q}}\,U\} \in \mathcal{I}.$$

*In this case, we write $\mathcal{I}$-*lim *$f = e$ and we say that e is the $\mathcal{I}$-*limit *of the fuzzy function f.*

The following example illustrates the concept of fuzzy function $\mathcal{I}$-convergence.

Example 1. *Let $x \in X$, $\delta = \{\mathbf{1}, \mathbf{0}, x_{1/2}\}$ and $e = x_{3/4}$, then the fuzzy open Q-neighborhoods of e in (X, δ) are $\mathbf{1}$ and $x_{1/2}$. Let the fuzzy function $f|\mathbb{N}$ in X, where*

$$f(n) = \begin{cases} x_{1/2-1/n+3}, & \text{if } n \text{ is even} \\ x_{1/2+1/n+1}, & \text{if } n \text{ is odd,} \end{cases}$$

and let the proper ideal $\mathcal{I} = \{A : A \subseteq 2\mathbb{N}\}$ on $\mathbb{N}$. Then, f $\mathcal{I}$-converges to e, since for every fuzzy open Q-neighborhood U of e we have $\{n \in \mathbb{N} : f(n)\,\bar{\mathrm{q}}\,U\} = \emptyset$ or $2\mathbb{N} \in \mathcal{I}$. However, f, as a fuzzy net, does not converge to e.

Let the fuzzy function $f|D$ and A be a fuzzy set in X. We will say that f is *in A* if, for each $d \in D$, $f(d) \in A$.

The proofs of the following propositions are similar to the proofs of Proposition 6, and Propositions 4.5 and 4.6 in [31].

In what follows, (X, δ) is an fts and e is a fuzzy point in X.

Proposition 8. *If there is a fuzzy function $f|D$ that $\mathcal{I}$-converges to e and f is in A, then $e \in \mathrm{cl}_\delta(A)$.*

Note that the converse of Proposition 8 also holds.

Proposition 9. *If $f|D$ is a constant fuzzy function with $f(d) = e$, for every $d \in D$, then f $\mathcal{I}$-converges to e, for every proper ideal $\mathcal{I}$ on D.*

A fuzzy function $g|\Lambda$ is said to be a *fuzzy subfunction* of the fuzzy function $f|D$ if there exists a function $\varphi : \Lambda \to D$ such that $g = f \circ \varphi$. Occasionally, we will write g^φ to indicate the fact that φ is the function mentioned above.

Proposition 10. *If $f|D$ $\mathcal{I}$-converges to e, then for every fuzzy subfunction $g^\varphi|\Lambda$ of f we have that g^φ $\mathcal{I}_\Lambda(\varphi)$-converges to e, whenever $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal on Λ.*

Proposition 11. *If $f|D$ does not $\mathcal{I}$-converge to e, where $\mathcal{I}$ is a proper ideal on D, then there exists a fuzzy subfunction $g^\varphi|\Lambda$ of f such that:*

(1) $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal on Λ.

(2) No fuzzy subfunction $h^\theta|K$ of g $\mathcal{J}$-converges to e, for every proper ideal $\mathcal{J}$ on K.

Proof. Since the fuzzy function $f|D$ does not $\mathcal{I}$-converge to e, there exists an open quasi-neighborhood, U, of e such that $\{d \in D : f(d)\,\overline{\mathrm{q}}\,U\} \notin \mathcal{I}$.

Let $\Lambda = \{d \in D : f(d)\,\overline{\mathrm{q}}\,U\}$, which is obviously a non-empty subset of D, and let $\varphi : \Lambda \to D$ be the inclusion map. Then, the fuzzy function $g = f \circ \varphi$ is a fuzzy subfunction of f and $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal on Λ, since $\varphi(\Lambda) = \Lambda \notin \mathcal{I}$.

We will prove that no fuzzy subfunction $h^\theta|K$ of g $\mathcal{J}$-converges to e, for every proper ideal $\mathcal{J}$ on K. Let $h^\theta|K$ be a fuzzy subfunction of g and $\mathcal{J}$ be a proper ideal on K. It will suffice to show that

$$\{k \in K : h(k)\,\overline{\mathrm{q}}\,U\} \notin \mathcal{J}.$$

Indeed, let $k \in K$. Then, $h(k) = g(\theta(k)) = f(\varphi(\theta(k))) = f(\theta(k))$. Since $\theta(k) \in \Lambda$, from the definition of Λ we have $f(\theta(k))\,\overline{\mathrm{q}}\,U$. Hence,

$$\{k \in K : h(k)\,\overline{\mathrm{q}}\,U\} = K.$$

Since $\mathcal{J}$ is a proper ideal on K, $\{k \in K : h(k)\,\overline{\mathrm{q}}\,U\} \notin \mathcal{J}$. □

Proposition 12. *We suppose the following:*

(1) $t|D$ is a fuzzy function that $\mathcal{I}_D$-converges to e.

(2) $s(d, \cdot)|E_d$ is a fuzzy function that $\mathcal{I}_{E_d}$-converges to $t(d)$, for every $d \in D$.

Let the fuzzy function

$$r|D \times \prod_{d \in D} E_d, \text{ where } r(d, f) = s(d, f(d)),$$

for every $(d, f) \in D \times \prod_{d \in D} E_d$. Then, r $\mathcal{I}_D \times \prod_{d \in D} \mathcal{I}_{E_d}$-converges to e (see Proposition 4).

Proof. Let U be an open quasi-neighborhood of e. We must prove that

$$A = \{(d, f) \in D \times \prod_{d \in D} E_d : r(d, f)\,\overline{\mathrm{q}}\,U\} \in \mathcal{I}_D \times \mathcal{I}_{\prod_{d \in D} E_d}.$$

It will suffice to show that there exists $A_D \in \mathcal{I}_D$ such that

$$\{f(d) : (d,f) \in A\} \in \mathcal{I}_{E_d}, \text{ for each } d \in D \setminus A_D.$$

Indeed, we set

$$A_D = \{d \in D : t(d)\,\overline{\mathrm{q}}\,U\}.$$

By condition (1), $A_D \in \mathcal{I}_D$. Let $d \in D \setminus A_D$. Then, $t(d)\,\mathrm{q}\,U$. Therefore, by condition (2),

$$\{\varepsilon \in E_d : s(d,\varepsilon)\,\overline{\mathrm{q}}\,U\} \in \mathcal{I}_{E_d}.$$

Since

$$\{f(d) \in E_d : s(d,f(d))\,\overline{\mathrm{q}}\,U\} \subseteq \{\varepsilon \in E_d : s(d,\varepsilon)\,\overline{\mathrm{q}}\,U\},$$

we have $\{f(d) \in E_d : s(d,f(d))\,\overline{\mathrm{q}}\,U\} \in \mathcal{I}_{E_d}$. However,

$$\{f(d) \in E_d : s(d,f(d))\,\overline{\mathrm{q}}\,U\} = \{f(d) : (d,f) \in A\}.$$

Hence, $\{f(d) : (d,f) \in A\} \in \mathcal{I}_{E_d}$. □

In addition, by taking into account Proposition 5 and condition (G5) in Definition 3, we have the following.

Proposition 13 ([31])**.** *For every point $x \in X$ and real directed set $D \subseteq (0,1]$, if $r \leq \sup D$, then the fuzzy function (net) $(x_d)_{d\in D}$ $\mathcal{I}_0(D)$-converges to x_r.*

4. Main Results

In this section, we give a modification of Theorems 2 and 3 for the fuzzy function ideal convergence. It is worth noting that, in this instance, we employ a restricted set of axioms that the fuzzy function convergence class must obey (see Definition 6).

Definition 8. *Let X be a non-empty set and let $\mathcal{L}$ be a class consisting of triads $(f,e,\mathcal{I})$, where $f|D$ is a fuzzy function in X, e is a fuzzy point in X, and $\mathcal{I}$ is a proper ideal on D. We say that $\mathcal{L}$ is a fuzzy function ideal convergence class for X if it satisfies the conditions listed below. We say that f $\mathcal{I}$-converges $(\mathcal{L})$ to e or that $\mathcal{I} - \lim f \equiv e(\mathcal{L})$ if $(f,e,\mathcal{I}) \in \mathcal{L}$.*

- *(L1) If $f|D$ is a constant fuzzy function with $f(d) = e$, for every $d \in D$, then f $\mathcal{I}$-converges $(\mathcal{L})$ to e, for every proper ideal $\mathcal{I}$ on D.*
- *(L2) If $f|D$ $\mathcal{I}$-converges $(\mathcal{L})$ to e, then for every fuzzy subfunction $g^{\varphi}|\Lambda$ of f we have that g^{φ} $\mathcal{I}_{\Lambda}(\varphi)$-converges $(\mathcal{L})$ to e, whenever $\mathcal{I}_{\Lambda}(\varphi)$ is a proper ideal on Λ.*
- *(L3) If the fuzzy function $f|D$ does not $\mathcal{I}$-converge $(\mathcal{L})$ to e, where $\mathcal{I}$ is a proper ideal on D, then there exists a fuzzy subfunction $g^{\varphi}|\Lambda$ of f such that:*
  - *(1) $\mathcal{I}_{\Lambda}(\varphi)$ is a proper ideal on Λ.*
  - *(2) No fuzzy subfunction $h^{\theta}|K$ of g $\mathcal{J}$-converges $(\mathcal{L})$ to e, for every proper ideal $\mathcal{J}$ on K.*
- *(L4) We suppose the following:*
  - *(1) $t|D$ is a fuzzy function that $\mathcal{I}_D$-converges $(\mathcal{L})$ to e.*
  - *(2) $s(d,\cdot)|E_d$ is a fuzzy function that $\mathcal{I}_{E_d}$-converges $(\mathcal{L})$ to $t(d)$, for every $d \in D$.*

  Let the fuzzy function

  $$r|D \times \prod_{d\in D} E_d, \text{ where } r(d,f) = s(d,f(d)),$$

  for every $(d,f) \in D \times \prod_{d\in D} E_d$. Then, r $\mathcal{I}_D \times \prod_{d\in D} \mathcal{I}_{E_d}$-converges $(\mathcal{L})$ to e (see Proposition 4).
- *(L5) For every point $x \in X$ and real directed set $D \subseteq (0,1]$, if $r \leq \sup D$, then the fuzzy function (net) $(x_d)_{d\in D}$ $\mathcal{I}_0(D)$-converges $(\mathcal{L})$ to x_r.*

Remark 2. *Let (X,δ) be an fts. Then, the class $\Xi(\delta)$ consisting of triads $(f,e,\mathcal{I})$, where $f|D$ is a fuzzy function on X and e is a fuzzy point in X such that f $\mathcal{I}$-converges to e, relative to δ, is a* *fuzzy function ideal convergence class since it satisfies all the conditions of Definition 6. We say that the fuzzy topology δ* generates *the fuzzy function ideal convergence class $\Xi(\delta)$.*

Proposition 14. *Let Ω be a family of fuzzy points in X and $A = \vee\Omega$. Let the class of triads $\mathcal{L}$ satisfy the conditions (L4) and (L5). If a fuzzy function $f|D$ $\mathcal{I}$-converges $(\mathcal{L})$ to e and f is in A, then there exists a fuzzy function $\overline{f}|K$ such that $\overline{f}(K)$ consists of fuzzy points in Ω and $\overline{f}$ $\mathcal{J}$-converges $(\mathcal{L})$ to e.*

Proof. Suppose that the fuzzy function $f|D$ $\mathcal{I}$-converges $(\mathcal{L})$ to e and f is in A. Then, we continue as in the proof of [28] (Proposition 1). For each $d \in D$, let y and r be the support point and the membership grade, respectively, of $f(d)$, i.e., $f(d) = y_r$. Since $y_r \in A$, we can consider a family of fuzzy points $\{y_{r_n}\} \subseteq \Omega$, such that $y_r \leqslant \vee\{y_{r_n}\}$. If we denote E^d to be the set of reals, r_n, we obtain the fuzzy function (net) $t^d = (y_{r_n})_{r_n \in E^d}$. Since $r \leq \sup E^d$, by condition (L5) we have that t^d $\mathcal{I}_0(E^d)$-converges $(\mathcal{L})$ to $f(d)$. Now condition (L4) applies and we obtain the desired fuzzy function. $\square$

The following theorem sets up a one-to-one correspondence between the fuzzy topologies for a non-empty set, X, and the fuzzy function ideal convergence classes on it.

Theorem 4. *(fuzzy function ideal convergence class theorem) Let $\mathcal{L}$ be a fuzzy function ideal convergence class for a non-empty set, X. We consider a map $\mathrm{cl} : I^X \to I^X$ induced as follows: for each $A \in I^X$, we define*

(i) *$\mathcal{C}(A)$ to be the set of all fuzzy points e in X for which there exists a fuzzy function $f|D$, in A, such that $(f, e, \mathcal{I}) \in \mathcal{L}$, and*

(ii) $\mathrm{cl}(A) = \vee\mathcal{C}(A)$.

Then, cl *is a fuzzy closure operator for a fuzzy topology denoted by $\Theta(\mathcal{L})$ on X and $(f, e, \mathcal{I}) \in \mathcal{L}$ iff f $\mathcal{I}$-converges to e, relative to $\Theta(\mathcal{L})$.*

Proof. Firstly, we prove that a fuzzy point $e \in \mathrm{cl}(A)$ iff for some fuzzy function $f|D$, in A, f $\mathcal{I}$-converges $(\mathcal{L})$ to e, i.e., $(f, e, \mathcal{I}) \in \mathcal{L}$. It is enough to prove that for each fuzzy point $e \in \mathrm{cl}(A)$ there exists a fuzzy function $f|D$, in A, such that f $\mathcal{I}$-converges $(\mathcal{L})$ to e. Indeed, let $e \in \mathrm{cl}(A)$ and denote the support point and the membership grade of e by x and $\lambda \in (0, 1]$, respectively, i.e., $e = x_\lambda$. Let R be the set of all $r \in (0, 1]$ for which there exists a fuzzy function $s^r|M^r$, in A, such that $(s^r, x_r, \mathcal{I}_{M^r}) \in \mathcal{L}$. Clearly, $R \neq \emptyset$ and $\sup R \geq \lambda$. Therefore, from (L5) the fuzzy function (net) $(x_r)_{r \in R}$ $\mathcal{I}_0(R)$-converges $(\mathcal{L})$ to e. Now from the definition of R there exists a fuzzy function $s^r|M^r$, in A, such that s^r $\mathcal{I}_{M^r}$-converges $(\mathcal{L})$ to x_r for each $r \in R$. It follows from (L4) that there exists a fuzzy function, in A, such that $\mathcal{I}_0(R) \times \prod_{r \in R} \mathcal{I}_{M^r}$-converges $(\mathcal{L})$ to e.

Next, we prove that cl is a fuzzy closure operator on X.

(FCO1) Is clear.

(FCO2) Let $A \in I^X$ and $e \in A$. We consider the fuzzy function $f|D$ in A, where $f(d) = e$, for every $d \in D$. By condition (L1) of Definition 6, we have that f $\mathcal{I}$-converges $(\mathcal{L})$ to e for every proper ideal $\mathcal{I}$ on D. Therefore, $e \in \mathrm{cl}(A)$.

(FCO3) Let $A, B \in I^X$. Then, $\mathrm{cl}(A) \leqslant \mathrm{cl}(A \vee B)$ and $\mathrm{cl}(B) \leqslant \mathrm{cl}(A \vee B)$. Therefore, $\mathrm{cl}(A) \vee \mathrm{cl}(B) \leqslant \mathrm{cl}(A \vee B)$. We prove that $\mathrm{cl}(A \vee B) \leqslant \mathrm{cl}(A) \vee \mathrm{cl}(B)$. Let $e \in \mathrm{cl}(A \vee B)$. Then, there exists a fuzzy function $f|D$ in $A \vee B$ such that f $\mathcal{I}$-converges $(\mathcal{L})$ to e. Denote

$$D_A = \{d \in D : f(d) \in A\} \text{ and } D_B = \{d \in D : f(d) \in B\}.$$

Then, we have $D_A \notin \mathcal{I}$ or $D_B \notin \mathcal{I}$, otherwise $D_A \cup D_B = D \in \mathcal{I}$, which is a contradiction. Without loss of generality, assume that $D_A \notin \mathcal{I}$. Let the inclusion map $\varphi_A : D_A \to D$ and the fuzzy function $g|D_A$ such that $g = f \circ \varphi$. Then, g is a fuzzy subfunction of f with g

in A. Since f $\mathcal{I}$-converges $(\mathcal{L})$ to e, by condition (L2) of Definition 6 we have that g $\mathcal{I}_{D_A}(\varphi_A)$-converges $(\mathcal{L})$ to e since the ideal $\mathcal{I}_{D_A}(\varphi_A)$ on D_A is proper because $\varphi_A(D_A) = D_A \notin \mathcal{I}$. Thus, $e \in \mathrm{cl}(A)$ and therefore $e \in \mathrm{cl}(A) \vee \mathrm{cl}(B)$.

(FCO4) We prove that $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. We have $A \leqslant \mathrm{cl}(A)$ and so $\mathrm{cl}(A) \leqslant \mathrm{cl}(\mathrm{cl}(A))$. We prove that $\mathrm{cl}(\mathrm{cl}(A)) \leqslant \mathrm{cl}(A)$. Let $e \in \mathrm{cl}(\mathrm{cl}(A))$. Then, there exists a fuzzy function $t|D$, in $\mathrm{cl}(A)$, such that t $\mathcal{I}_D$-converges $(\mathcal{L})$ to e. Therefore, for every $d \in D$ there exist a fuzzy function $s(d,\cdot)|E_d$, in A, such that $s(d,\cdot)$ $\mathcal{I}_{E_d}$-converges $(\mathcal{L})$ to $t(d)$. By condition (L4) of Definition 6 there exists a fuzzy function, in A, such that $\mathcal{I}_D \times \prod\limits_{d\in D} \mathcal{I}_{E_d}$-converges $(\mathcal{L})$ to e ($\mathcal{I}_D \times \prod\limits_{d\in D} \mathcal{I}_{E_d}$ is proper). Hence, $e \in \mathrm{cl}(A)$.

Therefore, cl determines a fuzzy topology $\Theta(\mathcal{L})$ on X.

We prove that if the fuzzy function $f|D$ $\mathcal{I}$-converges $(\mathcal{L})$ to the fuzzy point e in X, then f $\mathcal{I}$-converges to e, with respect to $\Theta(\mathcal{L})$. Suppose that f $\mathcal{I}$-converges $(\mathcal{L})$ to e and does not $\mathcal{I}$-converge to e, with respect to $\Theta(\mathcal{L})$. By Proposition 11 and its proof, there exist an open Q-neighborhood U of e and a fuzzy subfunction $g^{\varphi}|\Lambda$ of f such that:

1. $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal on Λ, and
2. $g(\lambda) \in U'$, for every $\lambda \in \Lambda$.

Since f $\mathcal{I}$-converges $(\mathcal{L})$ to e, by condition (L2) of Definition 6, g $\mathcal{I}_\Lambda(\varphi)$-converges $(\mathcal{L})$ to e. Therefore, $e \in \mathrm{cl}(U') = U'$. This contradicts the fact that e is quasi-coincident with U.

We prove that if the fuzzy function $f|D$ $\mathcal{I}$-converges to the fuzzy point e in X, with respect to $\Theta(\mathcal{L})$, then f $\mathcal{I}$-converges $(\mathcal{L})$ to e. Suppose that f $\mathcal{I}$-converges to e, with respect to $\Theta(\mathcal{L})$, and does not $\mathcal{I}$-converge $(\mathcal{L})$ to e. By condition (L3) of Definition 6, there exists a fuzzy subfunction $t^{\varphi}|\Lambda$ of f such that:

1. $\mathcal{I}_\Lambda(\varphi)$ is a proper ideal on Λ, and
2. no fuzzy subfunction $r^{\psi}|K$, of t, $\mathcal{I}_K$-converges $(\mathcal{L})$ to e, for every proper ideal $\mathcal{I}_K$ on K.

By Proposition 10, t $\mathcal{I}_\Lambda(\varphi)$-converges to e, with respect to $\Theta(\mathcal{L})$. Set $A = \vee\{t(\lambda) : \lambda \in \Lambda\}$. The fuzzy function t is in A, so by Proposition 6, $e \in \mathrm{cl}(A)$. By the property of $\mathrm{cl}(A)$, there exists a fuzzy function $w|N$, in A, such that w $\mathcal{I}_N$-converges $(\mathcal{L})$ to e. By Proposition 14, there exists a fuzzy function $\overline{w}|K$ such that $\overline{w}(K)$ consists of fuzzy points in the set $\{t(\lambda) : \lambda \in \Lambda\}$ (so $\overline{w}$ is a fuzzy subfunction of t), that $\mathcal{I}_K$-converges $(\mathcal{L})$ to e, which is a contradiction. $\square$

Additionally, we have the following, analogous to Corollary 1, result.

Corollary 2. *Let $\mathcal{L}$ be a fuzzy function ideal convergence class and δ be a fuzzy topology for a non-empty set, X. We have the following:*

(1) $\Xi(\Theta(\mathcal{L})) = \mathcal{L}$, *and*
(2) $\Theta(\Xi(\delta)) = \delta$.

Therefore, there exists a bijective map between fuzzy topologies δ for X and the fuzzy function ideal convergence classes $\mathcal{L}$ for X. Moreover, this map is order-reversing, i.e., if $\delta_1 \supseteq \delta_2$, then $\Xi(\delta_1) \subseteq \Xi(\delta_2)$.

Finally, in the following example, we see a class, under the morphology of the classes of Definition 8, which is not a fuzzy function ideal convergence class.

Example 2. *Let X be a non-empty set and let the class*

$$\mathcal{L} = \{(f|D, e, \mathcal{I}) : \{d \in D : f(d) \neq e\} \in \mathcal{I},\ \textit{for some } e \in \mathrm{FP}(X)\},$$

where $f|D$ is a fuzzy function in X and $\mathcal{I}$ is a proper ideal on D. Then, $\mathcal{L}$ is not a fuzzy function ideal convergence class for X. Indeed, let $x \in X$ and let the fuzzy function $f(n) = x_{1-1/n+2}$, $n \in \mathbb{N}$, then f $\mathcal{I}_0(\mathbb{N})$-converges to x_1, relative to every fuzzy topology δ of X; however, $(f, x_1, \mathcal{I}_0(\mathbb{N})) \notin \mathcal{L}$, since $\{n \in \mathbb{N} : f(n) \neq x_1\} = \mathbb{N}$.

5. Discussion

It is commonly accepted that the notion of convergence is a fundamental topic in topology; moreover, topology and net convergence are characterized by each other, by using the so-called notion of net convergence class in [20] (the analogous characterization for fuzzy topology is given in [28]). In this paper, we insert the notion of fuzzy function ideal convergence as a natural generalization of both fuzzy net convergence in [26] and fuzzy net ideal convergence in [30,31] and provide a characterization of fuzzy topology via the notion of fuzzy function ideal convergence class. Specifically, we examine the necessary and sufficient conditions that a fuzzy function ideal convergence class $\mathcal{L}$, on a non-empty set X, should fulfill to determine a unique fuzzy topology δ on X such that $\mathcal{I}$-convergence ($\mathcal{L}$) coincides with $\mathcal{I}$-convergence, with respect to δ. All the results obtained here are parallel to and extend those given in [18,19,21], for the ordinary topology, while simultaneously simplify the exposition and the underlying theory. In order to increase the utility of the present work, future research options may include the extension of the lattice background, L, to completely distributive lattices with an ordering–reversing involution as a tool to investigate the notion of L-fuzzy function ideal convergence and its applications in the more general context of L-fuzzy topological spaces.

Author Contributions: Conceptualization, D.G. and G.P.; methodology, D.G. and G.P.; investigation, D.G. and G.P.; writing—original draft preparation, D.G. and G.P.; writing—review and editing, D.G. and G.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Acknowledgments: The authors are grateful to the reviewers for reading the paper and making constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Das, P.; Sengupta, S.; Glab, S.; Bienias, M. Certain aspects of ideal convergence in topological spaces. *Topol. Appl.* **2019**, 107005. [CrossRef]
2. Das, P.; Sleziak, M.; Toma, V. $\mathcal{I}^{\mathcal{K}}$-Cauchy functions. *Topol. Appl.* **2014**, *173*, 9–27. [CrossRef]
3. Máčaj, M.; Sleziak, M. $\mathcal{I}^{\mathcal{K}}$-convergence. *Real Anal. Exch.* **2010**, *36*, 177–194. [CrossRef]
4. Cartan, H. Filtres et ultrafiltres. *C. R. Acad. Sci. Paris* **1937**, *205*, 777–779.
5. Bourbaki, N. Chapters I–IV. In *Elements of Mathematics—General Topology*; Springer: Berlin/Heidelberg, Germany, 1989.
6. Bernstein, A.R. A new kind of compactness for topological spaces. *Fund. Math.* **1970**, *66*, 185–193. [CrossRef]
7. Katětov, M. Products of filters. *Comment. Math. Univ. Carolinae* **1968**, *9*, 173–189.
8. Hrbacek, K.; T. Jech, T. *Introduction to Set Theory*, 3rd ed.; Revised and Expanded; CRC Press: Boca Raton, FL, USA, 2017.
9. Robinson, A. On generalized limits and linear functionals. *Pacif. J. Math.* **1964**, *14*, 269–283. [CrossRef]
10. Kostyrko, P.; Máčaj, M.; Šalát, T. Statistical Convergence and $\mathcal{I}$-Convergence. 1999. Available online: http://thales.doa.fmph.uniba.sk/\macaj/ICON.pdf (accessed on 18 November 2023).
11. Kostyrko, P.; Šalát, Wilczyński, W. $\mathcal{I}$-convergence. *Real Anal. Exch.* **2000**, *26*, 669–686. [CrossRef]
12. Nuray, F.; Ruckle, W.H. Generalized statistical convergence and convergence free spaces. *J. Math. Anal. Appl.* **2000**, *245*, 513–527. [CrossRef]
13. Caserta, A.; Di Maio, G.; Kočinac, L.D.R. Statistical convergence in function spaces. *Abstr. Appl. Anal.* **2011**, 420419. [CrossRef]
14. Di Maio, G.; Kočinac, L.D.R. Statistical convergence in topology. *Topol. Appl.* **2008**, *156*, 28–45. [CrossRef]
15. Šalát, T. On statistically convergent sequences of real numbers. *Math. Slovaca* **1980**, *30*, 139–150.
16. Lahiri, B.K.; Das, P. $\mathcal{I}$ and $\mathcal{I}^*$-convergence in topological spaces. *Math. Bohem.* **2005**, *130*, 153–160. [CrossRef]
17. Lahiri, B.K.; Das, P. $\mathcal{I}$ and $\mathcal{I}^*$-convergence of nets. *Real Anal. Exch.* **2008**, *33*, 431–442. [CrossRef]
18. Georgiou, D.N.; Iliadis, S.D.; Megaritis, A.C.; Prinos, G.A. Ideal-convergence classes. *Topol. Appl.* **2017**, *222*, 217–226. [CrossRef]
19. Georgiou, D.N.; Megaritis, A.C.; Prinos, G.A. A study on convergence and ideal convergence classes. *Topol. Appl.* **2018**, *241*, 38–49. [CrossRef]
20. Kelley, J.L. *General Topology*; Van Nostrand: New York, NY, USA, 1955.
21. Georgiou, D.N.; Prinos, G.A. On the notion of convergence of a function along an ideal. *Topol. Proc.* 2023, *accepted*.
22. Zadeh, L.A. Fuzzy sets. *Inform. Control* **1965**, *8*, 338–353. [CrossRef]
23. Birkhoff, G. Moore-Smith convergence in general topology. *Ann. Math.* **1937**, *38*, 39–56. [CrossRef]

24. Moore, E.H. Definition of limit in general integral analysis. *Proc. Nat. Acad. Sci. USA* **1915**, *1*, 628. [CrossRef]
25. Moore, E.H.; Smith, H.L. A general theory of limits. *Am. J. Math.* **1922**, *44*, 102–121. [CrossRef]
26. Pu, P.M.; Liu, Y.M. Fuzzy topology I. Neighborhood structure of a fuzzy point and Moore-Smith convergence. *J. Math. Anal. Appl.* **1980**, *76*, 571–599.
27. Pu, P.M.; Liu, Y.M. Fuzzy topology II. Product and quotient spaces. *J. Math. Anal. Appl.* **1980**, *77*, 20–37.
28. Liu, Y.M. On fuzzy convergence classes. *Fuzzy Sets Syst.* **1989**, *30*, 47–51. [CrossRef]
29. Kumar, V.; Kumar, K. On the ideal convergence of sequences of fuzzy numbers. *Inf. Sci.* **2008**, *178*, 4670–4678. [CrossRef]
30. Afsan, B.M.U. Ideal convergence of nets in fuzzy topological spaces. *Ann. Fuzzy Math. Inform.* **2014**, *8*, 635–643.
31. Georgiou, D.N.; Megaritis, A.C.; Prinos, G.A. New characterizations of fuzzy topology. *Mat. Vesn.* **2023**, *75*, 118–133.
32. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, *24*, 182–190. [CrossRef]