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Pricing of Al-Urbun and a Class of Al-Istijrar Islamic Contracts under the Black–Scholes Framework

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Abstract: Islamic financial contracts necessarily need to abide by Shariah principles. As such, some contracts have been introduced for risk-hedging real transactions that differ from those seen in conventional financial markets. In this paper, we examine two such products, Al-Urbun and Al-Istijrar, and determine fair prices for both the Al-Urbun and a class of Al-Istijrar under the Black–Scholes framework.

Keywords: Al-Urbun contract; Al-Istijrar contract; Black–Scholes equation; Islamic finance

MSC: 35G05; 91G50; 91-10

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1. Introduction

One of the most popular financial products available to investors is the European option contract, which has seen impressive growth in trade over the last 50 years. A European call option is a contract between two parties: the buyer (or holder) and the seller (or writer) that gives the holder the right, but not the obligation to buy an underlying asset from the writer at a later date at a price agreed upon at the opening of the contract. For that option, the buyer pays the writer an amount, known as the premium, at the opening of the contract.

The problems with the use of options in Islamic Finance, as detailed in [1], include the prohibition of gambling (Maysir), unnecessary risk taking (Gharar) and transactions based on ignorance (Jahl), trading in permitted (Halal) products only, presence of mutual consent for entering into and canceling contracts and the nature of the underlying asset (mal). In particular, the International Fiqh Academy of Organization of Islamic Cooperation [2] released a resolution with regard to the permissibility of option contracts. As currently applied in the global financial markets, option contracts are a new type of contracts that do not fall under any one of the Shariah nominate contracts. Since the object of the contract is neither a sum of money nor a utility or a financial right which may be waived then the contract is not permissible, according to Shariah. Since these contracts are initially not permissible, neither is their trading.

With regular instruments, mainly derivatives, hedging and speculation cannot be separated. However, while the Accounting and Auditing Organization for Islamic Financial Institutions (AAOIFI, 2008) [3,4] (in standard 1, trading in currencies), has indicated that options are forbidden in Shariah, it has instead approved the use of Urbun. Bai' al-urbun or Urbun, is a partial payment of the price of an asset, which enables the buyer to buy an asset for a specific price at a specific future date, but does not oblige the buyer to do so. If the buyer does choose to buy the asset, then the Urbun becomes part of the purchase price; but if the buyer chooses not to purchase the asset, the seller keeps the Urbun as compensation for their efforts. This is in contrast to a European call option, where the writer keeps the premium regardless of whether the holder exercises or not. For further details, we refer the reader to [5].

Most papers that analyze these products do so from a conceptual point of view (see, e.g., [6–8]) and very few papers are found in the literature that focus on the modeling and pricing of Urbun.

In [9], the authors studied the possibility of modeling Bai' al-urbun using the Binomial Model. This model, widely used for pricing options, is adapted to Shariah rules and conditions for pricing the Urbun deposit. They consider an American-style option and compare their results for the Urbun with those for American call options.

In [10], the authors price Urbun, $C(S,t)$, using the Black–Scholes partial differential Equation (PDE) as for the European call option, but with the final condition $C(S,T) = |S-K|$. This corresponds to pricing the sum of a European put option plus a European call option. Also, unfortunately, their solution does not satisfy their specified boundary condition $C(0,t) = 0$.

In [11], the authors' objective was to model Islamic options based on the Urbun principles and show how it was distinct from traditional options. With Black–Scholes as its benchmark, the authors use an artificial neural network to price the Islamic option. Their result showed that while artificial neural networks were able to price the Islamic options, there was a concerning lack of information provided by the model.

Other Islamic alternative products to conventional derivatives exist that necessarily must avoid conditions of zero-sum and also conditions of uncertainty. One of these that will be considered here is Al-Istijrar. Istijrar was recently introduced by the Muslim Commercial Bank in Pakistan and is being used for commodity financing. Furthermore, Istijrar financing has promising prospects in the Islamic banking industry in Bangladesh (see [12]). Istijrar is concerned with trade but does not include any likelihood of risk-free return by either party. The transaction includes the sale of a real commodity, purchasing and ownership. In this contract, the buyer and seller cannot speculate on the price movements. This constraint on speculation differentiates Istijrar from conventional financial products and makes it comply with the norms of Islamic ethics [13]. This product, which is detailed in Section 3, however, is very complex and involves averages and embedded options. Such a contract may allow either party to reset the ultimate buying price when various trigger points are reached in terms of the high or low values of the asset. Whether such actions are taken may depend on one party's perception of the other party's strategy. This then becomes a problem in game theory. In this paper, we consider a particular case of the product in which the values at the boundaries are set.

Not much can be found in the literature on the mathematical pricing of the Al-Istijrar contracts. However, in [13], Obaidullah noted that the contract with the buyer has the nature of an up-and-in call barrier option and the contract with the seller has the nature of a down-and-in put option contract. (He also noted that the options had the nature of average rate (Asian) options). As the value of these options to both parties must be equal, he arrives at the condition for equality using the standard valuation approach suggested by [14], which assumes that the underlying price follows a multiplicative binomial process.

The Black–Scholes framework has also been used to price products that lead to nonlinear cases, e.g., when incorporating the effects of market illiquidity (see, e.g., [15–17]).

In Sections 2 and 3 of this paper, we focus on the Al-Urbun and Al-Istijrar contracts, respectively. We provide more detail on the contracts and examine their pricing from a mathematical perspective under the Black–Scholes framework. We conclude in Section 4.

2. Al-Urbun

As stated in the Introduction, Bai' al-urbun is a partial payment of an asset's price that allows the buyer to buy the asset at a particular price at a particular date in the future. If the buyer chooses to buy the asset, the Urbun is included in the purchase price; if the buyer does not buy the asset, the seller keeps the money as compensation.

The following example illustrates the Urbun contract:

A buyer pays a deposit of 500 SR with a purchase (exercise) price of 5000 SR for 100 stocks. (i.e., 50 SR per stock with a deposit of 5 SR). The time to expiration of the contract is 3 months. At the exercise date, in 3 months, if the market price of the stock is more than 50 SR per stock, the buyer will pay 4500 SR for the remainder of the purchase price. He could then sell the stocks at the market price. The profit for the buyer is the market price less the exercise price. The seller must sell the stock at the predetermined exercise price. If the market price is above 45 SR and below 50 SR, say 47 SR, the holder will still buy the stock from the seller for 50 SR because the loss from the stock $(47 - 50)$ SR plus retrieval of the Urbun, 5 SR, (the payoff) is still positive and is better than a payoff of 0. Note however, that the final profit is still negative $(47 - 50 + 5 - 5 = -3)$ SR as the Urbun 5 SR was initially paid. Again, it is still better than losing the entire Urbun. If the price in the market is lower than 45 SR, the buyer will forfeit the purchase and the seller will keep the Urbun.

Tables 1 and 2 detail the payoffs and profits from European call options and the Urbun for different final stock prices.

Table 1. Payoffs and Profits at Expiry from a European Call Option with premium C.

	Payoff		**Profit**	
	Holder	**Writer**	**Holder**	**Writer**
$S > K$	$S - K$	$K - S$	$S - K - C$	$K - S + C$
$S < K$	0	0	$-C$	C

Table 2. Payoffs and Profits at Expiry from an Urbun Option with deposit a.

	Payoff		**Profit**	
	Holder	**Writer**	**Holder**	**Writer**
$S > K$	$S - K + a\ (> 0)$	$K - a - S\ (< 0)$	$S - K\ (> 0)$	$K - S\ (< 0)$
$K - a < S < K$	$S - K + a\ (> 0)$	$K - a - S\ (< 0)$	$S - K\ (< 0)$	$K - S\ (> 0)$
$S < K - a$	0	0	$-a$	a

Notice that with the Urbun contract when $K - a < S < K$, the holder still exercises and makes a loss of $S - K$ but that is less than the loss he would have if he did not exercise, i.e., $-a$ as $-a < S - K$.

Pricing of Al-Urbun

We consider the option to buy stock with price S at time $t = T$ for the exercise price of K. We assume that the underlying asset price is governed by

$$dS = \mu S dt + \sigma S dZ, \quad (1)$$

where μ and σ are, respectively, the constant expected return and volatility of the stock and dZ is an increment in a Wiener process Z.

We wish to determine a fair price for the Urbun, or deposit, a.

Result 1. *The fair price of the Urbun contract $V(S,t)$ for the option to purchase a stock S at price K at time $t = T$ for an initial deposit a, is*

$$V(S,t) = C_{BS}(S,t;K - a,T), \quad (2)$$

where

$$a = C_{BS}(S,0;K - a,T), \quad (3)$$

and where $C_{BS}(S,t;X,T) = SN(d_1) - Xe^{-r(T-t)}N(d_2)$, $d_{1,2} = \dfrac{(\ln(\frac{S}{X}) + (r \pm \frac{\sigma^2}{2})(T-t))}{\sigma\sqrt{T-t}}$ *is the Black–Scholes value of a European call option with exercise price* X *and expiry* T. r *is the constant risk-free interest rate, and* $N(.)$ *denotes the cumulative distribution function of the standard normal distribution (see [18]).*

Proof. The proof follows simply from that of the European call option. □

Setting up a portfolio of one short Urbun contract and $\frac{\partial V}{\partial S}$ long shares, i.e.,

$$\Pi = -V(S,t) + \frac{\partial V}{\partial S}S$$

and considering the change in the portfolio in a time dt (see [19]) demonstrates that the portfolio must be risk-free during the time dt. This then leads to the pricing PDE (as for the European call option)

$$V_t + \frac{\sigma^2 S^2}{2}V_{SS} + rSV_S - rV = 0, \tag{4a}$$

which for the Urbun contract we solve subject to

$$V(S,T) = \max(S - (K-a), 0), \tag{4b}$$

$$V(0,t) = 0, \tag{4c}$$

$$\lim_{S\to\infty} V(S,t) \sim S. \tag{4d}$$

Further, as we have an extra unknown, namely a, we have an extra condition

$$V(S,0) = a. \tag{4e}$$

The unique solution to (4a)–(4d) is (2). Substituting (4e) into (2) we obtain (3).

Equation (3) gives an implicit solution for the Urbun deposit a.

With the Urbun contract, when one exercises at $t = T$, they essentially pay $K - a$, the remainder of the exercise price, as a was paid as a deposit at the opening of the contract. The RHS of (3) represents the Black–Scholes value of a European call option with exercise price $K - a$. This gives the fair price of the option which is the amount paid (the premium or in this case the deposit) at the opening of the contract, i.e., 'a' (the LHS).

Result 2. *The Urbun is an increasing function of* S *and when* $S = K$ *then* $a = K$.

Proof. Consider $a = a(S)$. From (2) we have

$$a = SN(d_1) - (K-a)e^{-rT}N(d_2). \tag{5}$$

We show that $a'(S) > 0$.

Differentiating (5) with respect to S we have

$$a'(S) = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} + a'(S)e^{-rT}N(d_2) - (K-a)e^{-rT}N'(d_2)\frac{\partial d_2}{\partial S}. \tag{6}$$

Using $\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T}} + \frac{a'(S)}{(K-a)\sigma\sqrt{T}}$ and then making $a'(S)$ the subject, we obtain

$$a'(S) = \frac{N(d_1)}{1 - e^{-rT}N(d_2)} > 0. \tag{7}$$

□

Here we have used the relationship $SN'(d_1) - (K-a)e^{-rT}N'(d_2) = 0$ which is easily proven.

Further we have that when $S = K$ in (2), then $a = K$ satisfies (2) as $N(d_1) = 1$ and $K - a = 0$.

Note: From Result 1, this means that when $S = K$, the fair value of the Urbun is the entire exercise price, and when $S > K$, there is no fair value of the Urbun, i.e., no solution to (2). In the current setting, the stock price is expected to grow at a positive growth rate so if initially $S > K$ then it would be expected that the option will be exercised and there would be no benefit to the seller. Hence if $S > K$, and the holder can secure an Urbun, then he is in the enviable position of having an advantage because the Urbun is actually worth more than K.

Result 3. *American-style Urbun contracts can imply arbitrage opportunities. For example, if $S > K$ at the opening of the Urbun option contract, the Urbun could be exercised straightaway—the holder pays a, then exercises and receives $S - K + a$, thus making $S - K > 0$ immediately.*

Figure 1 and Table 3 illustrate a comparison of Urbun prices with the European call option prices with the parameters $K = 100, \sigma = 0.25, r = 0.05, T = 1$. As expected, the Urbun prices are always greater than the corresponding European call option prices and the differences increase with S.

Table 3. Comparison of Urbun and European call option prices with $K = 100, \sigma = 0.25, r = 0.05$, $T = 1$.

S	Urbun	European Call Option
50	0.0274	0.0274
60	0.2460	0.2402
70	1.1810	1.0775
80	4.0269	3.1415
90	12.3141	6.8698
95	24.6992	9.3950

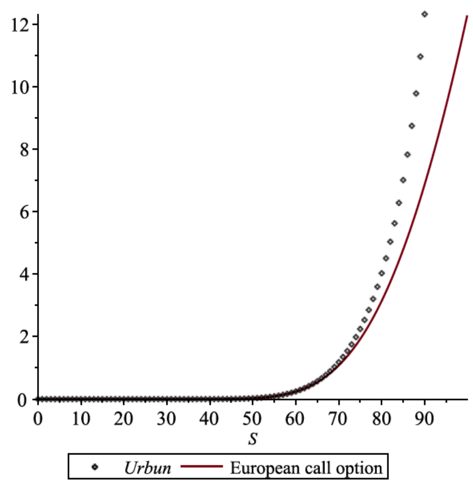

Figure 1. Comparison between Urbun and European call option with $K = 100, \sigma = 0.25, r = 0.05$, $T = 1$.

3. Al-Istijrar

Al-istijrar, or Istijrar, also involves trade between two parties, a buyer who might be a company looking to finance the purchase of an underlying asset, and a bank or other financial institution. The sale price of the traded commodity is calculated as the average of the market price over the financing period. The contract has embedded options for both parties that are triggered if the market price reaches an upper or lower bound during the financing period. The option gives the right to a party to set the sale price at some predetermined level. At the opening of the contract, there is one master agreement where all terms and conditions are finalized.

As stated in [13], ‘Istijrar does not allow any room for controversy, or difference of opinion as the options are embedded in the contract and are not tradable in themselves. The contract does not allow for any kind of speculative gain as it is backed by real purchase, ownership and sale of a commodity’.

A typical contract might be as follows: A company that seeks finance for a short term to buy a commodity approaches a bank. The bank buys the commodity at the current price, say S_0, and sells it to the company, to pay for it at a mutually agreed upon date, say 9 months. The sale price is unlike a usual trade and is determined at the end of the financing period, i.e., in 9 months, as the average of a series of asset prices during the 9 months. Both the company and the bank agree on a publically available, irrefutable source of price information and also on the sampling interval for observing asset prices on which the average price is calculated. The Istijrar contract has embedded options for both parties to fix the sale price at different predetermined levels. These options are activated if the asset price increases or decreases too much and crosses either a pre-specified upper or lower bound. The buyer company can choose to fix the sale price, payable on the due date, if S reaches S_u, an upper bound during the 9-month financing period. The bank, on the other hand, can choose to fix the sale price, payable on the due date, if S reaches S_l, a lower bound.

Note that if the buyer company expects the price will continue to rise after S_u, (so the average of the price during the entire period would be driven up) then they would exercise their option at S_u. However, if they expect that the increasing price trend price would stop and that asset prices will fall, then they would be better off not exercising.

If the price falls below S_l and the bank expects the price decline to continue (so the average of the prices during the entire period would be driven down) then they would exercise the option to fix the payment. However, if the bank expects the price decline to stop and thinks the prices will increase, then it might be better not to exercise the option.

This hedging instrument allows for the sharing of risk in which the settlement price benefits both parties. It is not a zero-sum game, as both parties benefit from the contract.

Pricing of a Particular Case of an Al-Istijrar Contract

The Al-Istijrar contract has features in common with Asian and barrier options, so the pricing of these contracts will in some ways be similar to them.

Averaging of the underlying asset in the contracts can be either arithmetic or geometric. In this paper, we use arithmetic averaging, which is the sum of all the constituent asset prices, equally weighted, and then divided by the total number of prices used. If we consider closely spaced prices over a finite time then the sums, we calculate when averages become integrals of the functions of the asset over the averaging period, leading to a continuously sampled average. In other words, if the sampling takes place so frequently then it may be best to view them as being continuously sampled rather than a string of discrete samples.

The continuously sampled arithmetic average is defined as

$$\frac{1}{t}\int_0^t S(t')dt', \tag{8}$$

which is

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}S(t_i)\right]. \tag{9}$$

Pricing of the Istijrar is complex and challenging. Here we make reasonable assumptions about the boundary conditions to make the mathematical model tractable and manageable. This helps us to gain insight and understanding of the underlying principles and processes that govern the contract.

The class of A-Istijrar we consider here is designed in such a way that the 'options' at the boundaries would always be exercised as it is for the relevant participant's benefit. Hence we assume that if the stock price reaches S_u that the buyer client would exercise and set the value on an agreed upon pre-determined value. Further, we assume that if it decreases and reaches S_l then the bank will exercise and the value there be set at a pre-determined value.

To value a contract contingent on an arithmetic average we define (see [19])

$$I = \int_0^t S(t')dt' \tag{10}$$

as a new independent variable, called a state variable. As the history of the asset price is independent of the current price, we can treat S, I, t as independent variables—different realizations of the random walk lead to different values of I. Hence the value of the contract is not just a function of S and t but also of I.

We again assume that the real process followed by the asset is given by (1). The pricing equation for the value of the Istijrar contract, $V(S, I, t)$, is then necessarily the same as for Asian options contingent on arithmetic averaging (see [14]), namely,

$$V_t + SV_I + \frac{\sigma^2 S^2}{2}V_{SS} + rSV_S - rV = 0. \tag{11}$$

The final condition for the Istijrar contract is $V(S, I, T) = I/T$.

We take the boundary conditions to be

$$V(S_u, I, t) = \frac{e^{-r(T-t)}}{T}[I + S_u^*(T-t)] + k_1, \tag{12}$$

$$V(S_l, I, t) = \frac{e^{-r(T-t)}}{T}[I + S_l^*(T-t)] + k_2, \tag{13}$$

$$\lim_{I\to\infty} V(S, I, t) \to \infty, \tag{14}$$

where k_1, k_2 are set constants and S_u^* and S_l^* are constant estimates of the average value of S over $[t, T]$ given $S = S_u$ or $S = S_l$. So at the upper(lower) boundary $S = S_u$ $(S = S_l)$ the buyer (bank) sets the value as the discounted estimated average at time T of the asset prices, plus a set constant. Here we are using the Mean Value Theorem for integrals to estimate $\int_t^T S(t')dt'$ (see, e.g., [20]). Note that at $t = T$ the value is $I/T + k_1$ at the upper boundary, so we expect the client would choose $k_1 < 0$. Similarly, at $t = T$ the value is $I/T + k_2$ at the lower boundary, so we expect the bank would choose $k_2 > 0$.

We let $V = \exp(-r(T-t))W(S, I, t)$ so that the new problem is

$$W_t + SW_I + \frac{\sigma^2 S^2}{2}W_{SS} + rSW_S = 0, \tag{15}$$

subject to $W(S, I, T) = I/T$, and

$$\begin{aligned} W(S_u, I, t) &= \frac{1}{T}[I + S_u^*(T-t)] + k_1 e^{r(T-t)}, && (16)\\ W(S_l, I, t) &= \frac{1}{T}[I + S_l^*(T-t)] + k_2 e^{r(T-t)}, && (17)\\ \lim_{I\to\infty} W(S, I, t) &\to \infty. && (18) \end{aligned}$$

Equation (15) has a classical symmetry $S^* = S, t^* = t, W^* = W + \epsilon, I^* = I + T\epsilon$ (see [21]). This leads to the functional form of the similarity reduction given by $W(S, I, t) = \frac{I}{T} + f(S, t)$ (with this form for W, the PDE and given conditions reduce to a problem for $f(S, t)$ with only two independent variables). Substituting this into the above problem we obtain

$$f_t + \frac{\sigma^2 S^2}{2} f_{SS} + rS f_S = -S/T, \tag{19}$$

subject to $f(S, T) = 0$ and

$$\begin{aligned} f(S_u, t) &= \frac{S_u^*(T-t)}{T} + k_1 e^{r(T-t)}, && (20)\\ f(S_l, t) &= \frac{S_l^*(T-t)}{T} + k_2 e^{r(T-t)}. && (21) \end{aligned}$$

Next, we let $\tau = T - t$ and $S = e^x$ so our problem reduces to

$$f_\tau - \frac{\sigma^2}{2} f_{xx} + f_x\left[-r + \frac{\sigma^2}{2}\right] = \frac{e^x}{T}, \tag{22}$$

subject to

$$\begin{aligned} f(x, 0) &= 0, && (23)\\ f(\ln S_u, \tau) &= \frac{S_u^* \tau}{T} + k_1 e^{r\tau}, && (24)\\ f(\ln S_l, \tau) &= \frac{S_l^* \tau}{T} + k_2 e^{r\tau}. && (25) \end{aligned}$$

Now with $f = e^{\alpha x + \beta\tau} u(x, \tau)$ where $\alpha = \frac{\frac{\sigma^2}{2} - r}{\sigma^2}, \beta = -\frac{\alpha^2\sigma^2}{2}$, our problem simplifies to

$$u_\tau - \frac{\sigma^2}{2} u_{xx} = \frac{1}{T} e^x e^{-\alpha x - \beta\tau}, \tag{26}$$

subject to

$$\begin{aligned} u(x, 0) &= 0, && (27)\\ u(\ln S_u, \tau) &= S_u^{-\alpha} e^{-\beta\tau}\left[k_1 e^{r\tau} + \frac{S_u^* \tau}{T}\right], && (28)\\ u(\ln S_l, \tau) &= S_l^{-\alpha} e^{-\beta\tau}\left[k_2 e^{r\tau} + \frac{S_l^* \tau}{T}\right]. && (29) \end{aligned}$$

Finally, to shift our domain to start at 0, we let $z = x - \ln S_l$ so our problem is

$$u_\tau = \frac{\sigma^2}{2} u_{zz} + \frac{1}{T} S_l^{1-\alpha} e^{z(1-\alpha)} e^{-\beta\tau}, \tag{30}$$

subject to

$$u(z,0) = 0, \tag{31}$$

$$u(0,\tau) = S_l^{-\alpha} e^{-\beta\tau}\left[k_2 e^{r\tau} + \frac{S_l^* \tau}{T}\right], \tag{32}$$

$$u(L,\tau) = S_u^{-\alpha} e^{-\beta\tau}\left[k_1 e^{r\tau} + \frac{S_u^* \tau}{T}\right], \tag{33}$$

where $L = \ln S_u - \ln S_l$.

The solution to the problem [22] is

$$u(z,\tau) = \int_0^\tau \int_0^L \frac{1}{T} S_l^{1-\alpha} e^{\xi(1-\alpha)} e^{-\beta t'} G(z,\xi,\tau - t') d\xi dt' \tag{34}$$

$$+ \frac{\sigma^2}{2} \int_0^\tau \frac{e^{-\beta t'}}{S_l^\alpha}\left[k_2 e^{rt'} + S_l^* \frac{t'}{T}\right] H_1(z,\tau - t') dt' \tag{35}$$

$$- \frac{\sigma^2}{2} \int_0^\tau \frac{e^{-\beta t'}}{S_u^\alpha}\left[k_1 e^{rt'} + S_u^* \frac{t'}{T}\right] H_2(z,\tau - t') dt', \tag{36}$$

where

$$G(z,\xi,\tau) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi\xi}{L}\right) \exp\left(-\frac{\sigma^2}{2}\frac{n^2\pi^2\tau}{L^2}\right) \tag{37}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \sum_{n=-\infty}^{\infty}\left[\exp\left(-\frac{(z-\xi+2nL)^2}{2\sigma^2\tau}\right) - \exp\left(-\frac{(z+\xi+2nL)^2}{2\sigma^2\tau}\right)\right]. \tag{38}$$

The first form of the series above for G converges rapidly for large τ while the second form converges rapidly for small τ.

Then,

$$H_1(z,\tau) = \frac{\partial}{\partial\xi} G(z,\xi,\tau)|_{\xi=0}, \tag{39}$$

$$H_2(z,\tau) = \frac{\partial}{\partial\xi} G(z,\xi,\tau)|_{\xi=L}. \tag{40}$$

Undoing the change of variables we obtain

$$V(S,I,t) = e^{-r(T-t)}\left[\frac{I}{T} + S^\alpha e^{\beta(T-t)} u(\ln S - lnS_l, T-t)\right]. \tag{41}$$

Figure 2 details the Istijrar prices with parameters $S_u = 50, S_l = 5, r = 0.05, \sigma = 0.2,$ $T = 0.25, S_l^* = \frac{4}{3}S_l, S_u^* = \frac{3}{4}S_u, k_1 = -2, k_2 = 2$. Notice how the values generally increase with S and at a particular S value the Istijrar values are close to S. However close to the lower boundary the Istijrar values are greater than S while close to the upper boundary, they are a fair bit smaller than S. We used $n = 5$ in (37) as adding more terms made no difference to the solution to 4 decimal places.

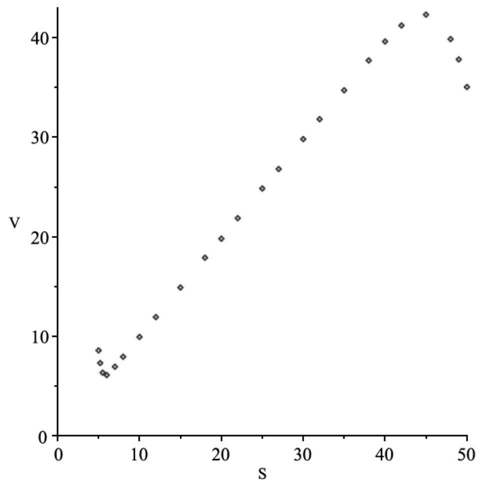

Figure 2. Istijrar prices with $S_u = 50, S_l = 5, r = 0.05, \sigma = 0.2, T = 0.25, S_l^* = \frac{4}{3}S_l, S_u^* = \frac{3}{4}S_u$, $k_1 = -2, k_2 = 2$.

4. Conclusions

The Islamic finance industry is a rapidly growing part of the world's financial sector. However, even with the introduction of financial instruments such as the Urbun and Istijrar, there do not appear to be many published studies that focus on the mathematical modeling and pricing of these instruments. This paper addresses this issue. In the Black–Scholes framework, we have determined an implicit solution for the fair price of the Urbun and shown that under this framework, if the current stock price is greater than the exercise price, then no fair value can be found for its price. Otherwise, the price can be found using an application of the Black–Scholes value for the European call option. We have also considered a form of Istijrar whereby at specified upper (lower) bounds reached by the stock price during the life of the contract, the value is set at pre-determined values that would benefit the buyer (seller). By assuming reasonable forms for the boundary conditions, we were able to formulate an explicit solution from which we can gain insight and understanding of the underlying principles of the Istijrar contract. Given that currently, there are no other explicit solutions for the Istijrar, we believe that our solution can offer important perspicacity to the contract.

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