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Subordinations Results on a q-Derivative Differential Operator

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Abstract: In this research paper, we utilize the q-derivative concept to formulate specific differential and integral operators denoted as $\mathcal{R}_q^{n,m,\lambda}$, $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$. These operators are introduced with the aim of generalizing the class of Ruscheweyh operators within the set of univalent functions. We extract certain properties and characteristics of the set of differential subordinations employing specific techniques. By utilizing the newly defined operators, this paper goes on to establish subclasses of analytic functions defined on an open unit disc. Additionally, we delve into the convexity properties of the two recently introduced q-integral operators, $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$. Special cases of the primary findings are also discussed.

Keywords: analytic functions; q-derivative; subordinations; q-convex functions; Ruscheweyh derivative

MSC: 30C45

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1. Introduction

In recent times, the q-analysis has garnered substantial attention from mathematicians, particularly in the realm of function theory, as evidenced in the comprehensive research available in [1]. The expansion of operator theory in this context has served as inspiration for numerous researchers, leading to the publication of various articles. The q-calculus provides valuable tools extensively employed to investigate diverse classes of analytic functions. Several geometric aspects, including coefficient estimates, convexity, close to convexity, distortion bounds and radii of starlikeness, have been explored within these proposed classes of functions.

Srivastava recently published a survey and expository review paper [2], offering valuable insights for researchers and scholars delving into the subject matter. The survey extensively examines the mathematical descriptions and applications of fractional q-derivative operators and fractional q-calculus within the realm of geometric function theory. The investigation delves into the intricacies of how these fractional operators and calculus concepts are employed in describing mathematical functions and their geometric properties. The survey also explores the practical applications and implications of fractional q-derivative operators within the broader context of geometric function theory. Overall, it provides a thorough exploration of the theoretical foundations and practical uses of these mathematical tools in the specified mathematical domain. Additionally, Srivastava and collaborators [3] specifically examined certain classes of q-starlike functions associated with conic regions.

The utilization of q-calculus in geometric function theory traces back to 1990, when Ismail and colleagues, referenced by [4], first applied q-calculus. They employed the q-derivative operator D_q to investigate an extension of the class of starlike functions within the open unit disk. Another significant contribution was made by Purohit and Raina, as cited in [5], where they introduced a generalized q-Taylor's formula in fractional q-calculus. In a different context, Mohammed and Darus, in their work denoted by [6], directed their focus towards the approximation and geometric properties of q-operators within specific subclasses of analytic functions situated in compact disks. This research showcased the versatility of q-calculus in addressing geometric aspects and the approximation within a specific set of analytic functions. Collectively, these studies highlight the diverse applications of q-calculus in exploring various facets of geometric function theory. Kanas and Raducanu [7] applied fractional q-calculus operators to examine specific function classes using the concept of the conic domain. Bounds for q-convex functions and q-starlike with respect to symmetric points were studied by Ramachandran et al. [8] using fractional q-calculus operators. In the study conducted by Srivastava and colleagues [9], they developed comprehensive findings concerning the partial sums of meromorphically starlike functions. These functions were defined within a specific class of q-derivative operators.The research aimed to provide broader insights into the characteristics and properties of these meromorphically starlike functions, leveraging the framework of a designated class of q-derivative operators. In the research conducted by Ibrahim and collaborators, referenced by [10], they introduced a novel q-differential operator within the open unit disk. This operator played a crucial role in characterizing the analytic geometric representation of solutions to the well-known Beltrami differential equation within a complex domain. The study aimed to contribute to the understanding of solutions to the Beltrami equation in a complex setting, utilizing the introduced q-differential operator in the context of the open unit disk. In the work conducted by Nezir and co-authors, and referenced by [11], they introduced particular subclasses of analytic and univalent functions within the open unit disk. These subclasses were defined based on the q-derivative, and the study involved an examination of conditions that analytic and univalent functions must satisfy to belong to these specific classes. The research aimed to provide a deeper understanding of the properties and characteristics of analytic and univalent functions in the context of the introduced q-derivative, shedding light on the conditions governing membership in the defined subclasses. Analytic functions in q-analogue associated with the cardioid domain and limacon domain are examined with respect to various properties by Ul-Haq et al. [12]. In the research conducted by Deniz et al. [13], they delved into the exploration of j-neighborhoods associated with various subclasses of convex and starlike functions, defined based on the q-Ruscheweyh derivative operator. In the research conducted by Khan and colleagues, denoted by [14], they explored diverse subclasses of analytic functions, q-starlike functions, and symmetric q-starlike functions. This exploration was carried out through the application of q-analogue values of integral and derivative operators. This study aimed to investigate and characterize the properties and behaviors of these specific subclasses of analytic functions under the influence of q-calculus, employing integral and derivative operators with q-analogue values. The research contributed to advancing the understanding of analytic functions within the framework of q-calculus and the application of relevant operators.

These discoveries, among numerous others, underscore the pressing need for significant progress in q-calculus and fractional q-calculus within the framework of geometric function theory in complex analysis. Various researchers have played a crucial role in advancing this theory by introducing specific classes through the application of q-calculus. The contributions of these researchers have collectively expanded the scope and understanding of geometric function theory, paving the way for further exploration and developments in the realm of complex analysis. The recognition of q-calculus as a valuable tool in defining classes and understanding geometric properties emphasizes its importance in the ongoing evolution of geometric function theory. To access more recent contributions on this subject, interested individuals can refer to the provided references [15–20]. All of these sources are likely to contain the latest research findings and advancements in the field, offering a comprehensive overview of the current state of knowledge regarding q-calculus and its applications within geometric function theory.

In this research paper, the central focus lies in the application of the concept of the q-derivative to derive specific differential and integral operators, denoted as $\mathcal{R}_q^{n,m,\lambda}$, $F_q^{n,m,\lambda}$

and $G_q^{n,m,\lambda}$. These operators are introduced with the aim of generalizing the class of Ruscheweyh operators within the set of univalent functions. This paper proceeds to establish various properties and characteristics related to the set of differential subordinations. The derivation of these properties involves employing specific techniques tailored to the q-derivative, leading to the attainment of interesting results in the realm of differential subordination. By utilizing the newly defined operators, this paper goes on to establish subclasses of analytic functions defined on an open unit disc. Furthermore, the research delves into the convexity properties of the two recently introduced q-integral operators. These operators are defined within specific classes of analytic functions, and their properties are examined in the context of the newly introduced q-differential operator.

In this context, we revisit fundamental concepts from the Geometric Function Theory literature, which are essential for ensuring clarity and comprehension of the forthcoming analysis.

2. Main Results

In the customary notation, $\mathcal{H}(U)$ represent the set of analytic functions in the open unit disk. Consider the subclass $\mathcal{A}$ of $\mathcal{H}(U)$, consisting of analytic functions f defined on the open unit disk $U = \{z \in \mathbb{C} \mid |z| < 1\}$. Members of this subclass are subject to normalization conditions, specifically $f(0) = 0 = f^{'}(0) - 1$. In simpler terms, functions f belonging to $\mathcal{A}$ can be expressed in the form of a power series:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,\ z \in U. \tag{1}$$

We revisit certain notations and concepts of q-calculus employed in this paper. The theoretical underpinning of this framework rests upon the incorporation of q-analogues into traditional formulas and functions. This foundation is established by acknowledging and utilizing the concept that involves expressing traditional mathematical structures in terms of q-analogues. The integration of q-analogues into established mathematical frameworks forms the basis for developing a comprehensive theory that extends and adapts classical formulas and functions in the realm of q-calculus, built upon the recognition that

$$\lim_{q\to 1} \frac{1-q^{\alpha}}{1-q} = \alpha,\ q \in (0,1),\ \alpha \in \mathbb{N}, \tag{2}$$

hence, the quantity $\frac{1-q^{\alpha}}{1-q}$ is occasionally referred to as the basic number $[\alpha]_q$. The q-factorial, denoted as $[\alpha]_q!$, is a mathematical concept related to q-calculus that is specified by the following formula:

$$[\alpha]_q! = \begin{cases} [\alpha]_q \cdot [\alpha-1]_q \cdots [1]_q, \text{ for } \alpha = 1,2,\ldots; \\ 1, \text{ for } \alpha = 0. \end{cases} \tag{3}$$

The q-factorial is a special function that arises in various areas of mathematics, including combinatorics, number theory, and quantum algebra.

It is important to note that when q approaches 1, the q-factorial converges to the classical factorial function. In this sense, the q-factorial provides a q-analogue or a deformation of the standard factorial.

The q-derivative of a function $f(z)$ with respect to the variable z is determined by the following definition:

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z},\ q \in (0,1), z \in U,\ z \neq 0 \tag{4}$$

and $D_q(f(0)) = f^{'}(0)$, where D_q denotes the q consequently, we infer that

$$D_q(f(z)) = 1 + \sum_{j=2}^{\infty} [j]_q a_j z^{j-1},\ q \in (0,1),\ z \in U, z \neq 0. \tag{5}$$

Hence, for a function $f(z) = z^k$, the q-derivative is expressed as

$$D_q\left(z^k\right) = \frac{\left(q^k - 1\right)z^{k-1}}{q-1} \cdot z^{k-1} = [k]_q z^{k-1}, \tag{6}$$

then $\lim_{q\to 1} D_q(f(z)) = \lim_{q\to 1}[k]_q z^{k-1} = kz^{k-1} = f'(z)$, where $f'(z)$ is the ordinary derivative.

Given the assumption of the definition of the q-derivatives operator, for f and g belonging to set $\mathcal{A}$, the following rules apply:

$$mD_q((f(z)) \pm ng(z)) = mD_q f(z) \pm nD_q g(z), \text{ for } m, n \in \mathbb{C}\ ,$$

$$D_q(f(z)g(z)) = g(z)D_q f(z) + f(qz)D_q g(z),$$

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(z)g(qz)}, \text{ with } g(z)g(qz) \neq 0.$$

Furthermore, the q-integral of a function $f(x)$ over a subset of $\mathbb{C}$ is determined by

$$\int_0^z f(t)d_q t = z(1-q)\sum_{k=0}^{\infty} q^k f\left(zq^k\right). \tag{7}$$

Principle of Subordination (see [21]): If f and g are analytic functions in the domain U, we express that f is subordinate to g, denoted as $f \prec g$, when there exists a Schwarz function w that is analytic in U, satisfying $w(0) = 0$ and $|w(z)| < 1$. This function w should be such that $f(z)$ equals $g(w(z))$, for all z in U. Specifically, when the function g is univalent in U, the mentioned subordination is equivalent to $f(0)$ being equal to $g(0)$ and the image of f over U being a subset of the image of g over U.

A function $f \in \mathcal{A}$ is said to be starlike of order α, $0 \le \alpha < 1$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in U.$$

The collection of all of these functions is represented by $\mathcal{S}^*(\alpha)$.

A function f belonging to the set $\mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha)$ of convex functions of order α, where $0 \le \alpha < 1$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ z \in U.$$

Particularly, the classes $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ are, respectively, the well-known classes of starlike and convex functions in U.

The q-analogues to the functions classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are provided in the following manner.

A function f belonging to the set $\mathcal{A}$ is said to be in the class $\mathcal{S}_q^*(\alpha)$ of starlike functions with respect to q-differentiation of order α, where $0 \le \alpha < 1$, if it meets the requirements

$$Re\left\{\frac{zD_q(f(z))}{f(z)}\right\} > \alpha,\ z \in U.$$

A function f belonging to the set $\mathcal{A}$ is said to be in the class $\mathcal{C}_q(\alpha)$ of convex functions with respect to q-differentiation of order α, for $-1 \le \alpha < 1$, if it meets the conditions

$$Re\left\{1 + \frac{zD_q^2(f(z))}{D_q(f(z))}\right\} > \alpha,\ z \in U.$$

The classes $\mathcal{S}_q^*(0) = \mathcal{S}_q^*$ and $\mathcal{C}_q(\alpha) = \mathcal{C}_q$ represent the classes of starlike and convex functions with respect to q-differentiation.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{US}_q(\alpha, k)$ of k-uniformly starlike functions with respect to q-differentiation of order α, for $0 \leq \alpha < 1$, if it meets the conditions

$$Re\left\{\frac{zD_q(f(z))}{f(z)} - \alpha\right\} > k\left|\frac{zD_q(f(z))}{f(z)} - 1\right|, \ z \in U.$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UC}_q(\alpha, k)$ of k-uniformly- convex functions with respect to q-differentiation of order α, for $-1 \leq \alpha < 1$, if it meets the conditions

$$Re\left\{1 + \frac{zD_q^2(f(z))}{D_q(f(z))} - \alpha\right\} > k\left|\frac{zD_q^2(f(z))}{D_q(f(z))}\right|, \ z \in U.$$

Definition 1 ([22])**.** *Consider an function f belonging to the set $\mathcal{A}$. Let $\mathcal{R}_q^n$ denote the q-analogue of the Ruscheweyh operator, defined as follows:*

$$\mathcal{R}_q^n f(z) = z + \sum_{j=2}^{\infty} \frac{[j+n-1]_q!}{[n]_q![j-1]_q!} a_j z^j, \tag{8}$$

where $[\alpha]_q$ and $[\alpha]_q!$ are specified within (2) and (3).

Remark 1. *It can be inferred that when $q \to 1$ in the preceding definition, we acquire*

$$\begin{aligned} \lim_{q\to 1} \mathcal{R}_q^n f(z) = z + \lim_{q\to 1}\left[\sum_{j=2}^{\infty} \tfrac{[j+n-1]_q!}{[n]_q![j-1]_q!} a_j z^j\right] = \\ = z + \sum_{j=2}^{\infty} \tfrac{(j+n-1)!}{n!\cdot(j-1)!} a_j z^j = \mathcal{R}^n f(z), \end{aligned} \tag{9}$$

where $\mathcal{R}^n f(z)$ is a Ruscheweyh differential operator defined in [23] and examined by various researchers; see [24–26].

We hereby introduce a new q-operator, denoted as $\mathcal{R}_q^{n,m,\lambda}$, with the following definition:

$$\begin{aligned} \mathcal{R}_q^{n,0,\lambda} f(z) &= \mathcal{R}_q^n f(z), \\ \mathcal{R}_q^{n,1,\lambda} f(z) &= (1-\lambda)\mathcal{R}_q^n f(z) + \lambda z D_q\left(\mathcal{R}_q^n f(z)\right), \\ &\ldots \\ \mathcal{R}_q^{n,m,\lambda} f(z) &= \mathcal{R}_q^{n,1,\lambda}\left(\mathcal{R}_q^{n,m-1,\lambda} f(z)\right), \end{aligned} \tag{10}$$

for $n, m \in \mathbb{N}, 0 < q < 1, \lambda \geq 0, z \in U$.

Assuming $f \in \mathcal{A}$ is represented by (1), we can derive the following from (10)

$$\mathcal{R}_q^{n,m,\lambda} f(z) = z + \sum_{j=2}^{\infty}\left(1 - \lambda + [j]_q\lambda\right)^m \frac{[j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 z^j, \tag{11}$$

for $n, m \in \mathbb{N}, 0 < q < 1, \lambda \geq 0, z \in U$.

Proposition 1. *For $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$, $z \in U$, the operator $\mathcal{R}_q^{n,m,\lambda}$ satisfies the following identity:*

$$q^n z\left(D_q\left(\mathcal{R}_q^{n,m,\lambda} f(z)\right)\right) = [n+1]_q \mathcal{R}_q^{n+1,m,\lambda} f(z) - [n]_q \mathcal{R}_q^{n,m,\lambda} f(z), \tag{12}$$

Proof. Considering that $[n+1]_q = [n]_q + q^n$, we acquire

$$
\begin{aligned}
&[n+1]_q \mathcal{R}_q^{n+1,m,\lambda} f(z) - [n]_q \mathcal{R}_q^{n,m,\lambda} f(z) = \\
&\Big([n+1]_q - [n]_q\Big) z + \sum_{j=2}^{\infty} \Big(1-\lambda+[j]_q\lambda\Big)^m \tfrac{[j+n-1]_q!}{[n]_q![j-1]_q!} \Big([j+n]_q - [n]_q\Big) a_j^2 z^j = \\
&q^n z + \sum_{j=2}^{\infty} \Big(1-\lambda+[j]_q\lambda\Big)^m \tfrac{[j+n-1]_q!}{[n]_q![j-1]_q!} \Big(\tfrac{q^{j+n}-1}{q-1} - \tfrac{q^n-1}{q-1}\Big) a_j^2 z^j = \\
&q^n z + \sum_{j=2}^{\infty} \Big(1-\lambda+[j]_q\lambda\Big)^m \tfrac{[j+n-1]_q!}{[n]_q![j-1]_q!} q^n [j]_q a_j^2 z^j = \\
&q^n z \left(1 + \sum_{j=2}^{\infty} \Big(1-\lambda+[j]_q\lambda\Big)^m \tfrac{[j+n-1]_q!}{[n]_q![j-1]_q!} [j]_q a_j^2 z^{j-1}\right) = \\
&q^n z \left(\tfrac{(q-1)z}{(q-1)z} + \sum_{j=2}^{\infty} \Big(1-\lambda+[j]_q\lambda\Big)^m \tfrac{[j+n-1]_q!}{[n]_q![j-1]_q!} \tfrac{q^j-1}{q-1} a_j^2 z^{j-1}\right) = \\
&q^n z \left(\tfrac{qz+\sum_{j=2}^{\infty}\left(1-\lambda+[j]_q\lambda\right)^m \frac{[j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 q^j z^j}{(q-1)z} - \tfrac{z+\sum_{j=2}^{\infty}\left(1-\lambda+[j]_q\lambda\right)^m \frac{[j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 z^j}{(q-1)z}\right) = \\
&q^n z \tfrac{\mathcal{R}_q^{n,m,\lambda} f(qz) - \mathcal{R}_q^{n,m,\lambda} f(z)}{(q-1)z} = q^n z \Big(D_q\Big(\mathcal{R}_q^{n,m,\lambda} f(z)\Big)\Big).
\end{aligned}
$$

The demonstration for (12) is finished. □

Proposition 2. *For natural numbers n and m, with $0 < q < 1$ and $z \in U$, the operator $\mathcal{R}_q^{n,m,1}$ obeys the following equality:*

$$
z\Big(D_q\Big(\mathcal{R}_q^{n,m,1} f(z)\Big)\Big) = \mathcal{R}_q^{n,m+1,1} f(z). \tag{13}
$$

Proof. We can derive from (11)

$$
\begin{aligned}
D_q\Big(\mathcal{R}_q^{n,m,1} f(z)\Big) &= \frac{\mathcal{R}_q^{n,m,1} f(qz) - \mathcal{R}_q^{n,m,1} f(z)}{qz - z} = \\
&\frac{1}{z(q-1)}\left(qz - z + \sum_{j=2}^{\infty} \frac{[j]_q^m [j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 z^j \Big(q^j - 1\Big)\right) = \\
1 + \sum_{j=2}^{\infty} \frac{[j]_q^m [j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 z^{j-1} \frac{q^j-1}{q-1} &= 1 + \sum_{j=2}^{\infty} \frac{[j]_q^{m+1} [j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 z^{j-1}.
\end{aligned}
$$

Thus, the subsequent identity is valid for the operator $\mathcal{R}_q^{n,m,1}$

$$
z\Big(D_q\Big(\mathcal{R}_q^{n,m,1} f(z)\Big)\Big) = z + \sum_{j=2}^{\infty} \frac{[j]_q^{m+1} [j+n-1]_q!}{[n]_q![j-1]_q!} a_j^2 z^j = \mathcal{R}_q^{n,m+1,1} f(z).
$$

The proof is completed. □

From the definition, it is evident that by setting specific parameters, the operator $\mathcal{R}_q^{n,m,\lambda}$ transforms into well-known operators. Particularly, for $q \to 1$, the q -operator $\mathcal{R}_q^{n,m,\lambda}$ becomes the generalised Darus and Al- Shaqsi derivative operator [27]; for the case of $m = 0$, the q-operator $\mathcal{R}_q^{n,m,\lambda}$ turn into q-analogue of the Ruscheweyh operator introduced in [22]. Additionally, for $q \to 1$, the q-operator $\mathcal{R}_q^{n,m,\lambda}$ convert into the q-analogue of the Ruscheweyh operator given by (8). In the particular case where $\lambda = 0$, $n = 0$ and q approaches 1, the q-operator $\mathcal{R}_q^{n,m,\lambda}$ takes a special form, and it is asserted that in this limit, it coincides with an operator introduced by Al-Oboudi [28].

Prior to presenting our findings, we present the generalized lemmas introduced in [29,30], utilizing q-derivative.

Lemma 1 ([22]). *Suppose the function v is analytic, convex, and univalent in the domain U, with $v(0) = 1$. Let $g(z) = 1 + b_1 z + b_2 z^2 + \ldots$ be an analytic function in U. If*

$$g(z) + \frac{1}{a} z D_q(g(z)) \prec v(z),\ z \in U,\ a \in \mathbb{C} \backslash \{0\}, \tag{14}$$

then

$$g(z) \prec \frac{a}{z^a} \int_0^z t^{a-1} v(t) d_q t, \quad for Rea \geq 0.$$

Proof. Assume that the function v is analytic, convex and univalent in U and g is analytic in U.

Let $q \to 1$ in (14).

We acquire

$$g(z) + \frac{1}{a} z g'(z) \prec v(z),\ z \in U,\ a \in \mathbb{C} \backslash \{0\}.$$

Subsequently, employing the lemma in [29], we obtain

$$g(z) \prec h(z),$$

where $h(z) = \frac{a}{z^a} \int_0^z t^{a-1} v(t) dt, z \in U$. □

Lemma 2 ([22]). *Consider v be a convex function in U and let $h(z) = v(z) + \alpha z D_q(v(z))$, for $z \in U$ and $\alpha > 0$. If $g(z) = 1 + b_1 z + b_2 z^2 + \ldots$ is analytic in U and*

$$g(z) + \alpha z D_q(g(z)) \prec h(z), \text{ for } z \in U,$$

then

$$g(z) \prec v(z),\ z \in U$$

and this result is sharp.

Proof. The method of proving this is akin to the approach used in proving the Lemma 1. □

Lemma 3 ([22]). *Let v be an univalent function in the unit disk U and let θ and ϕ be analytic functions in a domain D containing $v(U)$ with $\phi(\omega) \neq 0$, when $\omega \in v(U)$. Consider $Q(z) = zD_q(v(z))\phi(v(z))$ and $h(z) = \theta(v(z)) + Q(z)$. Suppose that*

1. *Q is starlike univalent in U;*
2. *$Re\left(\frac{zD_q(h(z))}{Q(z)}\right) > 0$, for $z \in U$.*

If p is an analytic function in U, with $p(0) = v(0)$, $p(U) \subseteq D$ and

$$zD_q(p(z))\phi(p(z)) + \theta(p(z)) \prec zD_q(v(z))\phi(v(z)) + \theta(v(z)) = h(z),$$

then $p \prec v$ and v is the best dominant.

Proof. The method of proving this is akin to the approach used in proving the Lemma 1. □

Utilizing the new q-operator, $\mathcal{R}_q^{n,m,\lambda}$, we apply the techniques of the theory of differential subordination to undertake an investigation, leading to the discovery of intriguing new differential subordination relationships and the identification of the best dominant.

We are set to demonstrate the initial outcome.

Theorem 1. *Let $n, m \in \mathbb{N}$, $0 < q < 1$, $\gamma > 0$, and $-1 \leq M \leq N < 1$. If $f \in \mathcal{A}$ satisfies the subsequent subordination condition*

$$(1-\gamma)\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} + \gamma \frac{\mathcal{R}_q^{n,m+1,1} f(z)}{z} \prec \frac{1+Mz}{1+Nz}, \tag{15}$$

then

$$Re\left\{ \left(\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} \right)^{\frac{1}{s}} \right\} \geq \left(\frac{1}{q\gamma} \int_0^1 u^{\frac{1}{q\gamma}-1} \frac{1-Mu}{1-Nu} du \right)^{\frac{1}{s}}, s \geq 1. \tag{16}$$

and this result is sharp.

Proof. Allow

$$h(z) = \frac{\mathcal{R}_q^{n,m,1} f(z)}{z} = 1 + c_1 z + c_2 z^2 + \ldots, \quad z \in U. \tag{17}$$

The function $h(z)$ is analytic in U, for $f \in \mathcal{A}$. Through employing the logarithmic q-differentiation, we obtain

$$D_q(h(z)) = D_q\left(\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} \right) = \frac{z D_q\left(\mathcal{R}_q^{n,m,1} f(z) \right) - \mathcal{R}_q^{n,m,1} f(z)}{q z^2} = \frac{\mathcal{R}_q^{n,m+1,1} f(z) - \mathcal{R}_q^{n,m,1} f(z)}{q z^2}.$$

So,

$$\frac{z D_q(h(z))}{h(z)} = \frac{\mathcal{R}_q^{n,m+1,1} f(z) - \mathcal{R}_q^{n,m,1} f(z)}{q \mathcal{R}_q^{n,m,1} f(z)} = \frac{1}{q}\left(\frac{\mathcal{R}_q^{n,m+1,1} f(z)}{\mathcal{R}_q^{n,m,1} f(z)} - 1 \right)$$

We derive

$$1 + \frac{q z D_q(h(z))}{h(z)} = \frac{\mathcal{R}_q^{n,m+1,\lambda} f(z)}{\mathcal{R}_q^{n,m,1} f(z)}.$$

So,

$$1 + \frac{q z D_q(h(z))}{h(z)} = \frac{\mathcal{R}_q^{n,m+1,\lambda} f(z)}{z h(z)}.$$

Multiplying the result by $h(z)$, we obtain

$$\frac{\mathcal{R}_q^{n,m+1,1} f(z)}{z} = h(z) + q z D_q(h(z)).$$

Therefore, we have

$$(1-\gamma)\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} + \gamma \frac{\mathcal{R}_q^{n,m+1,1} f(z)}{z} = (1-\gamma)h(z) + \gamma\big(h(z) + q z D_q(h(z))\big) = h(z) + \gamma q z D_q(h(z)).$$

The expression for differential subordination (15) can be stated as

$$h(z) + \gamma q z D_q(h(z)) \prec \frac{1+Mz}{1+Nz}.$$

Utilizing Lemma 1, we deduce

$$h(z) \prec \frac{1}{q\gamma} z^{-\frac{1}{q\gamma}} \int_0^z t^{\frac{1}{q\gamma}-1} \frac{1+Mt}{1+Nt} dt.$$

By employing the concept of subordination, we obtain

$$\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} = \frac{1}{q\gamma} \int_0^1 u^{\frac{1}{q\gamma}-1} \frac{1+Muw(z)}{1+Nuw(z)} du.$$

Considering the range $-1 \leq M \leq N < 1$, we acquire

$$Re\left(\frac{\mathcal{R}_q^{n,m,1} f(z)}{z}\right) > \left(\frac{1}{q\gamma} \int_0^1 u^{\frac{1}{q\gamma}-1} \frac{1-Mu}{1-Nu} du\right). \tag{18}$$

Employing the inequality $Re\left(w^{\frac{1}{s}}\right) \geq (Rew)^{\frac{1}{s}}$, for $s \geq 1$ and $Rew > 0$, the inequality (16) is a direct consequence of (18).

To establish the sharpness of (16), we define the function f in set $\mathcal{A}$ as:

$$\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} = \frac{1}{q\gamma} \int_0^1 u^{\frac{1}{q\gamma}-1} \frac{1-Mu}{1-Nu} du$$

We acquire

$$(1-\gamma)\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} + \gamma \frac{\mathcal{R}_q^{n,m+1,1} f(z)}{z} = \frac{1+Mz}{1+Nz}$$

and

$$\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} \to \frac{1}{q\gamma} \int_0^1 u^{\frac{1}{q\gamma}-1} \frac{1-Mu}{1-Nu} du, \text{ as } z \to -1.$$

The proof of the theorem is now concluded. □

Corollary 1. *Let $n, m \in \mathbb{N}$, $0 < q < 1$, $\gamma > 0$, and $0 \leq \alpha < 1$. If $f \in \mathcal{A}$ satisfies the subsequent subordination condition*

$$(1-\gamma)\frac{\mathcal{R}_q^{n,m,1} f(z)}{z} + \gamma \frac{\mathcal{R}_q^{n,m+1,1} f(z)}{z} \prec \frac{(2\alpha-1)z+1}{z+1}, \tag{19}$$

then

$$Re\left\{\left(\frac{\mathcal{R}_q^{n,m,1} f(z)}{z}\right)^{\frac{1}{s}}\right\} \geq \left((2\alpha-1) + \frac{2(1-\alpha)}{q\gamma} \int_0^1 u^{\frac{1}{q\gamma}-1} \frac{1}{u+1} du\right)^{\frac{1}{s}}, s \geq 1. \tag{20}$$

Proof. Applying identical steps as in the proof of Theorem 1 for $h(z) = \frac{\mathcal{R}_q^{n,m,1} f(z)}{z}$, the differential subordination (19) transitions to:

$$h(z) + \gamma q z D_q(h(z)) \prec \frac{(2\alpha-1)z+1}{z+1}.$$

Hence

$$Re\left\{\left(\frac{\mathcal{R}_q^{n,m,1}f(z)}{z}\right)^{\frac{1}{s}}\right\} > \left(\frac{1}{q\gamma}\int_0^1 u^{\frac{1}{q\gamma}-1}\frac{(2\alpha-1)u+1}{u+1}du\right)^{\frac{1}{s}} =$$

$$\left(\frac{1}{q\gamma}\int_0^1 u^{\frac{1}{q\gamma}-1}\left((2\alpha-1)-\frac{2(\alpha-1)}{u+1}\right)du\right)^{\frac{1}{s}} =$$

$$\left((2\alpha-1)+\frac{2(1-\alpha)}{q\gamma}\int_0^1 u^{\frac{1}{q\gamma}-1}\frac{1}{u+1}du\right)^{\frac{1}{s}},$$

and the statement of Corollary 1 is valid. □

Example 1. *For the function* $f(z) = z + z^2$, $n = 1$, $m = 1$, $\lambda = 1$, $\gamma = 2$, $\alpha = \frac{1}{4}$, $s = 1$, *we have* $(1-\gamma)\frac{\mathcal{R}_q^{n,m,1}f(z)}{z} + \gamma\frac{\mathcal{R}_q^{n,m+1,1}f(z)}{z} = -\frac{\mathcal{R}_q^{1,1,1}f(z)}{z} + 2\frac{\mathcal{R}_q^{1,2,1}f(z)}{z} = -\frac{z+[2]_q^2z^2}{z} + 2\frac{z+[2]_q^3z^2}{z} = 1 + z\left(2q^3+5q^2+4q+1\right)$. *Utilizing the Corollary 1, we acquire* $1+\left(2q^3+5q^2+4q+1\right)z \prec \frac{2-z}{2(z+1)}$, *for* $z \in U$, *leading to*

$$Re\left\{z\left(q^2+2q+1\right)+1\right\} \geq -\frac{1}{2}+\frac{3}{4q}\int_0^1 u^{\frac{1}{2q}-1}\frac{1}{u+1}du, \text{ for } z \in U.$$

Theorem 2. *Assume* $n, m \in \mathbb{N}$, $0 < q < 1$ *and* $0 \leq p < 1$. *Moreover, consider the parameter* $\alpha \in \mathbb{C}\backslash\{0\}$ *such that* $\left|\frac{2\alpha-2\alpha p-q}{q}\right| \leq 1$ *or* $\left|\frac{2\alpha-\alpha p+q}{q}\right| \leq 1$. *If the function* $f \in \mathcal{A}$ *fulfills the subsequent inequality:*

$$Re\left(\frac{\mathcal{R}_q^{n,m+1,1}f(z)}{\mathcal{R}_q^{n,m,1}f(z)}\right) > p, \text{ for } z \in U, \tag{21}$$

then

$$\left(\frac{\mathcal{R}_q^{n,m,1}f(z)}{z}\right)^{\alpha} \prec (z-1)^{\frac{2\alpha p-2\alpha}{q}}, \text{ for } z \in U,$$

and $(z-1)^{\frac{2\alpha p-2\alpha}{q}}$ *is the best dominant.*

Proof. Allow

$$h(z) = \left(\frac{\mathcal{R}_q^{n,m,1}f(z)}{z}\right)^{\alpha}, \quad z \in U. \tag{22}$$

By employing a logarithmic q-differentiation, we derive

$$D_q(h(z)) = D_q\left(\frac{\mathcal{R}_q^{n,m,1}f(z)}{z}\right)^{\alpha} = \alpha\left(\frac{\mathcal{R}_q^{n,m,1}f(z)}{z}\right)^{\alpha-1}\frac{\mathcal{R}_q^{n,m+1,1}f(z)-\mathcal{R}_q^{n,m,1}f(z)}{qz^2}.$$

So

$$\frac{D_q(h(z))}{h(z)} = \frac{\alpha}{q}\frac{\mathcal{R}_q^{n,m+1,1}f(z)}{\mathcal{R}_q^{n,m,1}f(z)} - \frac{\alpha}{q}.$$

We obtain

$$\frac{\mathcal{R}_q^{n,m+1,1}f(z)}{\mathcal{R}_q^{n,m,1}f(z)} = \frac{q}{\alpha}\frac{D_q(h(z))}{h(z)} + 1.$$

From (21), we deduce

$$\frac{\mathcal{R}_q^{n,m+1,1} f(z)}{\mathcal{R}_q^{n,m,1} f(z)} \prec \frac{z(1-2\alpha)+1}{1-z}.$$

By defining

$$\theta(\omega) := 1 \text{ and } \phi(\omega) := \frac{q}{\alpha\omega},\ v(z) = (z-1)^{\frac{2\alpha p - 2\alpha}{q}}$$

it can be readily confirmed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$, $\omega \in \mathbb{C}\backslash\{0\}$. Moreover, by allowing $Q(z) = zD_q(v(z))\phi(v(z)) = \frac{z(2-2p)}{1-z}$, we determine that $Q(z)$ is starlike and univalent in U. Allow $g(z) = \theta(v(z)) + Q(z) = \frac{z(1-2p)+1}{1-z}$. Hence, the criteria of Lemma 3 are satisfied, leading to

$$g(z) \prec v(z), \text{ i.e., } \left(\frac{\mathcal{R}_q^{n,m,1} f(z)}{z}\right)^{\alpha} \prec v(z),\ z \in U,$$

and v is the best dominant. □

Theorem 3. *Assume that g is an analytic and convex function within the domain U and let h be defined by $h(z) = u(z) + \frac{\alpha q^{n+1}}{[n+1]_q} zD_q(u(z))$, for $z \in U$, $\alpha \in \mathbb{C}\backslash\{0\}$. If $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$, and $f \in \mathcal{A}$ satisfies*

$$\begin{aligned}\alpha\Bigg[\left(1+\frac{q^{n+1}}{[n+1]_q}\right)\frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + q\left(1-\frac{q^n}{[n+1]_q}\right)\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\Bigg]+ \\ +\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)}\left(1-\alpha q\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right) \prec h(z),\ z \in U,\end{aligned} \tag{23}$$

then

$$\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec u(z),\ z \in U \tag{24}$$

and this result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)},\ z \in U. \tag{25}$$

The function $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is analytic in U. By applying logarithmic q-differentiation to both sides of the Equation (25), we derive

$$\frac{D_q(p(z))}{p(z)} = \frac{\mathcal{R}_q^{n,m,\lambda} f(z) D_q\left(\mathcal{R}_q^{n+1,m,\lambda} f(z)\right) - \mathcal{R}_q^{n+1,m,\lambda} f(z) D_q\left(\mathcal{R}_q^{n,m,\lambda} f(z)\right)}{\mathcal{R}_q^{n,m,\lambda} f(qz)\mathcal{R}_q^{n,m,\lambda} f(z)}. \tag{26}$$

By multiplying the outcome with z and making use of the identity (12), we obtain

$$\begin{aligned}z\frac{D_q(p(z))}{p(z)} = \frac{[n+2]_q}{q^{n+1}}\frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n+1,m,\lambda} f(z)\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{[n+1]_q}{q^{n+1}}\frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \\ -\frac{[n+1]_q}{q^n}\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + \frac{[n]_q}{q^n}\frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}.\end{aligned} \tag{27}$$

Multiplying the result by $p(z)$, we obtain

$$
\begin{aligned}
zD_q(p(z)) = \frac{[n+2]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{[n+1]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \\
- \frac{q[n+1]_q}{q^{n+1}} \frac{\left(\mathcal{R}_q^{n+1,m,\lambda} f(z)\right)^2}{\mathcal{R}_q^{n,m,\lambda} f(z) \mathcal{R}_q^{n,m,\lambda} f(qz)} + \frac{q[n]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}.
\end{aligned}
\tag{28}
$$

Considering that $[n+2]_q = [n+1]_q + q^{n+1}$, $[n]_q = [n+1]_q - q^n$ and by adding $p(z)$, we deduce

$$
\begin{aligned}
p(z) + \alpha \frac{q^{n+1}}{[n+1]_q} zD_q(p(z)) = \alpha \left(1 + \frac{q^{n+1}}{[n+1]_q}\right) \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + \\
+ \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \left(1 - \alpha q \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right) + \alpha q \left(1 - \frac{q^n}{[n+1]_q}\right) \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \alpha \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}.
\end{aligned}
\tag{29}
$$

So,

$$
\begin{aligned}
p(z) + \alpha \frac{q^{n+1}}{[n+1]_q} zD_q(p(z)) = \\
\alpha \left[\left(1 + \frac{q^{n+1}}{[n+1]_q}\right) \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + q \left(1 - \frac{q^n}{[n+1]_q}\right) \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} \right] + \\
+ \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \left(1 - \alpha q \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right).
\end{aligned}
$$

Subsequently, (23) transforms into

$$
p(z) + \alpha \frac{q^{n+1}}{[n+1]_q} zD_q(p(z)) \prec h(z) = u(z) + \alpha \frac{q^{n+1}}{[n+1]_q} zD_q(u(z)),
$$

for $z \in U$. Utilizing Lemma 2, we obtain

$$
p(z) \prec u(z) \text{ i.e., } \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec u(z),
$$

for $z \in U$, and this result is sharp. □

Theorem 4. *Suppose the function h is analytic, convex and univalent in U with $h(0) = 1$. If $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$ and $f \in \mathcal{A}$ satisfy,*

$$
\begin{aligned}
\frac{[\gamma]_q}{q^\gamma} \left[\left(1 + \frac{q^{n+1}}{[n+1]_q}\right) \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + q \left(1 - \frac{q^n}{[n+1]_q}\right) \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} \right] + \\
+ \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \left(1 - \frac{[\gamma]_q}{q^{\gamma-1}} \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right) \prec h(z),\ z \in U,
\end{aligned}
\tag{30}
$$

then

$$
\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec u(z),\ z \in U,
$$

where $u(z) = \frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}} z^{-\frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}}} \int_0^z h(t) t^{\frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}} - 1} d_q t$. *The function u is the best dominant.*

Proof. Consider $p(z) = \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)}$ which is analytic in U. Following the analogous steps outlined in Theorem 3, in the view of (29), the relation (30) transforms into

$$
p(z) + \frac{[\gamma]_q q^{n+1-\gamma}}{[n+1]_q} zD_q(p(z)) \prec h(z),\ z \in U.
$$

In the light of Lemma 1, we find

$$p(z) = \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec u(z),\ z \in U,$$

where

$$u(z) = \frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}} z^{-\frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}}} \int_0^z h(t) t^{\frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}}-1} d_q t,$$

and the function u is the best dominant. □

Corollary 2. *Consider the function $h(z) = \frac{1+(2\beta-1)z}{1+z}$, where $0 \le \beta < 1$, which is convex in U. If $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \ge 0$ and $f \in \mathcal{A}$ satisfies the differential subordination*

$$\begin{gathered}\frac{[\gamma]_q}{q^\gamma}\left[\left(1+\frac{q^{n+1}}{[n+1]_q}\right)\frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + q\left(1-\frac{q^n}{[n+1]_q}\right)\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right]+ \\ +\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)}\left(1-\alpha q\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right) \prec \frac{1+(2\beta-1)z}{1+z}, \\ z \in U,\ \alpha \in \mathbb{C}\backslash\{0\},\end{gathered} \tag{31}$$

then

$$\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec u(z),\ z \in U,$$

in which u is defined as $u(z) = (2\beta-1) + \frac{2(1-\beta)[n+1]_q}{[\gamma]_q q^{n+1-\gamma}} z^{-\frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}}} \int_0^z \frac{t^{\frac{[n+1]_q}{[\gamma]_q q^{n+1-\gamma}}-1}}{1+t} d_q t$. The function u is the best dominant.

Proof. Obviously, the function h is analytic, convex and univalent in U with $h(0) = 1$. The proof closely resembles the proof of Theorem 4. □

Theorem 5. *Let $\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \in \mathcal{H}(U)$, $z \in U$, $f \in \mathcal{A}$, $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \ge 0$, and let the function $v(z)$ be both convex and univalent in U, with $v(0) = 1$. Suppose that*

$$Re\left\{\frac{1}{v(qz)}\left[v(z) + z\left(\frac{qv(z)D_q^2(v(z))}{D_q(v(z))} - D_q(v(z))\right)\right]\right\} > 0,\ z \in U, \tag{32}$$

and for $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$,

$$\begin{gathered}\psi_q^n(\alpha,\beta;z) = \alpha + \frac{[n+2]_q}{q\beta[n+1]_q}\frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n+1,m,\lambda} f(z)}\frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \\ -\frac{1}{\beta}\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + \left(\frac{[n]_q}{\beta[n+1]_q} - \frac{1}{\beta q}\right)\frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)},\ z \in U.\end{gathered} \tag{33}$$

If v fulfills the subsequent subordination

$$\psi_q^n(\alpha,\beta;z) \prec \alpha + \frac{q^n z D_q(v(z))}{\beta[n+1]_q v(z)}, \tag{34}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$, then

$$\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec v(z),\ z \in U$$

and v is the best dominant.

Proof. Define the function p as follows:

$$p(z) := \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)},\ z \in U,\ z \neq 0, f \in \mathcal{A}.$$

The function p is analytic in U, with $p(0) = 1$. Through logarithmic q-differentiation with respect to z on both sides of this function, multiplying the result by z, and leveraging the identity (12), we acquire:

and

$$\begin{aligned}\frac{zD_q(p(z))}{p(z)} = \frac{[n+2]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n+1,m,\lambda} f(z)} - \frac{[n+1]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \\ - \frac{[n+1]_q}{q^n} \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + \frac{[n]_q}{q^n} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}.\end{aligned} \tag{35}$$

By defining

$$\theta(\omega) := \alpha \text{ and } \phi(\omega) := \frac{q^n}{\beta[n+1]_q \omega},\ \alpha, \beta \in \mathbb{C}, \beta \neq 0,$$

it can be readily confirmed that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$, $\omega \in \mathbb{C}\backslash\{0\}$. Moreover, by allowing $Q(z) = zD_q(v(z))\phi(v(z)) = \frac{q^n zD_q(v(z))}{\beta[n+1]_q v(z)}$, we determine that $Q(z)$ is starlike and univalent in U. Allow $h(z) = \theta(v(z)) + Q(z) = \alpha + \frac{q^n zD_q(v(z))}{\beta[n+1]_q v(z)}$. Upon differentiating the function h with respect to z and conducting the calculations, we obtain:

$$\begin{aligned}\frac{zD_q(h(z))}{Q(z)} &= \frac{v(z)}{v(qz)} - z\frac{D_q(v(z))}{v(qz)} + \frac{qzv(z)D_q^2(v(z))}{v(qz)D_q(v(z))} = \\ &\frac{1}{v(qz)}\left[v(z) + z\left(\frac{qv(z)D_q^2(v(z))}{D_q(v(z))} - D_q(v(z))\right)\right].\end{aligned}$$

Therefore, we obtain

$$Re\left(\frac{zD_q(h(z))}{Q(z)}\right) = Re\left\{\frac{1}{v(qz)}\left[v(z) + z\left(\frac{qv(z)D_q^2(v(z))}{D_q(v(z))} - D_q(v(z))\right)\right]\right\} > 0.$$

By employing (35), we derive

$$\begin{aligned}\alpha + \frac{q^n zD_q(p(z))}{\beta[n+1]_q p(z)} = \alpha + \frac{q^n}{\beta[n+1]_q}\left\{\frac{[n+2]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n+1,m,\lambda} f(z)} - \right. \\ \left. \frac{[n+1]_q}{q^{n+1}} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \frac{[n+1]_q}{q^n} \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + \frac{[n]_q}{q^n} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}\right\} = \\ \alpha + \frac{[n+2]_q}{q\beta[n+1]_q} \frac{\mathcal{R}_q^{n+2,m,\lambda} f(z)}{\mathcal{R}_q^{n+1,m,\lambda} f(z)} \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} - \\ \frac{1}{\beta} \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)} + \left(\frac{[n]_q}{\beta[n+1]_q} - \frac{1}{\beta q}\right) \frac{\mathcal{R}_q^{n,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(qz)}.\end{aligned}$$

Utilizing (34), we find

$$\alpha + \frac{q^n zD_q(p(z))}{\beta[n+1]_q p(z)} \prec \alpha + \frac{q^n zD_q(v(z))}{\beta[n+1]_q v(z)}.$$

Hence, the criteria of Lemma 3 are satisfied, leading to

$$p(z) \prec v(z),\ \text{i.e.,}\ \frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec v(z),\ z \in U,$$

and v is the best dominant. □

Corollary 3. *Allowing* $v(z) = \frac{1}{(1-z)^{2\beta \frac{[n+1]_q}{q^n}}}$, $z \in U, \beta \in \mathbb{C}, \beta \neq 0, n, m \in \mathbb{N}, 0 < q < 1, \lambda \geq 0$ *and suppose that* (32) *is valid. If* $f \in \mathcal{A}$ *and*

$$\psi_q^n(\alpha, \beta; z) \prec \alpha + \frac{2z}{1-z},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, *where* $\psi_q^n(\alpha, \beta; z)$ *is defined in* (32), *then*

$$\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} \prec \frac{1}{(1-z)^{2\beta \frac{[n+1]_q}{q^n}}}$$

and $\frac{1}{(1-z)^{2\beta \frac{[n+1]_q}{q^n}}}$ *is the best dominant.*

Proof. The corollary is derived by applying Theorem 5 to $q(z) = \frac{1}{(1-z)^{2\beta \frac{[n+1]_q}{q^n}}}$. □

In the following, employing the newly defined q-operator and drawing inspiration from operators introduced in references [31,32], the paper introduces two novel integral operators, along with some new classes of analytic functions defined through these operators. Breaz and Breaz [31], as well as Breaz, Owa, and Breaz, [32] initiated and explored the subsequent integral operators

$$F_{\gamma_1,\dots,\gamma_l}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \dots \left(\frac{f_l(t)}{t}\right)^{\gamma_l} dt = \int_0^z \prod_{i=1}^{l} \left(\frac{f_i(t)}{t}\right)^{\gamma_i} dt,$$
$$G_{\gamma_1,\dots,\gamma_l}(z) = \int_0^z \left(f_1'(t)\right)^{\gamma_1} \dots \left(f_l'(t)\right)^{\gamma_l} dt = \int_0^z \prod_{i=1}^{l} \left(f_i'(t)\right)^{\gamma_i} dt,$$

where $f_i \in \mathcal{A}$, $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$, $i \in \{1, 2, \dots, l\}$, $l \in \mathbb{N}$.

Now, we present two novel q-integral operators in the following manner.

Definition 2. *Let* $f_i \in \mathcal{A}$, $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$, $i \in \{1, 2, \dots, l\}$, $l \in \mathbb{N}$, $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$. *Then,* $F_q^{n,m,\lambda}(z) : \mathcal{A} \to \mathcal{A}$ *is characterized by*

$$F_q^{n,m,\lambda}(z) = F_{q,\gamma_1,\dots,\gamma_l}^{n,m,\lambda}(f_1, \dots, f_l) = \int_0^z \prod_{i=1}^{l} \left(\frac{\mathcal{R}_q^{n,m,\lambda} f_i(t)}{t}\right)^{\gamma_i} d_q t \tag{36}$$

and $G_q^{n,m,\lambda}(z) : \mathcal{A} \to \mathcal{A}$ *is expressed as*

$$G_q^{n,m,\lambda}(z) = G_{q,\gamma_1,\dots,\gamma_l}^{n,m,\lambda}(f_1, \dots, f_l) = \int_0^z \prod_{i=1}^{l} \left(D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(t)\right)\right)^{\gamma_i} d_q t, \tag{37}$$

where $\mathcal{R}_q^{n,m,\lambda}$ *is defined by* (11).

Remark 2. *As* $q \to 1$ *and* $n = 0$, *we arrive at the two integral operators investigated by Breaz et al. in [31,32].*

Subsequently, we examine the q-integral operators $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$ defined by (36) and (37). Specifically, we investigate the convexity properties of the operators $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$.

By employing the operator $\mathcal{R}_q^{n,m,\lambda} f(z)$ defined by (11) and applying q-differentiation, we define two new subclasses of analytic functions in the following approach.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{US}_q^n(\alpha, k)$, if and only if

$$Re\left\{ \frac{[n+1]_q}{q^n} \left(\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} - 1 \right) + 1 - \alpha \right\} \geq \\ \geq k \left| \frac{[n+1]_q}{q^n} \left(\frac{\mathcal{R}_q^{n+1,m,\lambda} f(z)}{\mathcal{R}_q^{n,m,\lambda} f(z)} - 1 \right) \right|, \tag{38}$$

for $-1 \leq \alpha < 1, k \geq 0, n, m \in \mathbb{N}, 0 < q < 1, \lambda \geq 0$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UC}_q^n(\alpha, k)$, if and only if

$$Re\left\{ 1 + \frac{zD_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)} - \alpha \right\} \geq k \left| \frac{zD_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)} \right| \tag{39}$$

for $-1 \leq \alpha < 1, k \geq 0, n, m \in \mathbb{N}, 0 < q < 1, \lambda \geq 0$.

Theorem 6. *Let $n, m \in \mathbb{N}$, $0 < q < 1$, $\lambda \geq 0$, $\gamma = (\gamma_1, \ldots, \gamma_l) \in \mathbb{R}_+^l$, $-1 \leq \alpha_i < 1$, $k_i > 0$ and $f_i \in \mathcal{US}_q^n(\alpha_i, k_i)$, for all $i \in \{1, 2, \ldots, l\}$, $l \in \mathbb{N}$. If*

$$0 \leq 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1) < 1, \tag{40}$$

then, the q-integral operator $F_q^{n,m,\lambda}(z)$, defined by (36), exhibits convexity with respect to q-differentiation of order λ, with $\lambda = 1 + \sum\limits_{i=1}^{l} \gamma_i(\alpha_i - 1)$.

Proof. Looking at (36), it is evident that $F_q^{n,m,\lambda}(z)$ belongs to the class $\mathcal{A}$. It is straightforward to confirm that

$$D_q\left(F_q^{n,m,\lambda}(z)\right) = \prod_{i=1}^{l} \left(\frac{\mathcal{R}_q^{n,m,\lambda} f_i(z)}{z} \right)^{\gamma_i}, \; z \in U. \tag{41}$$

This equality suggests that

$$\ln D_q\left(F_q^{n,m,\lambda}(z)\right) = \gamma_1 \ln \frac{\mathcal{R}_q^{n,m,\lambda} f_1(z)}{z} + \ldots + \gamma_l \ln \frac{\mathcal{R}_q^{n,m,\lambda} f_l(z)}{z},$$

or, in other words

$$\ln D_q\left(F_q^{n,m,\lambda}(z)\right) = \gamma_1 \left[\ln \mathcal{R}_q^{n,m,\lambda} f_1(z) - \ln z \right] + \ldots + \gamma_l \left[\ln \mathcal{R}_q^{n,m,\lambda} f_l(z) - \ln z \right].$$

By q-differentiating both sides of the aforementioned equality, we obtain

$$\frac{D_q^2\left(F_q^{n,m,\lambda}(z)\right)}{D_q\left(F_q^{n,m,\lambda}(z)\right)} = \sum_{i=1}^{l} \gamma_i \left(\frac{D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{\mathcal{R}_q^{n,m,\lambda} f_i(z)} - \frac{1}{z} \right). \tag{42}$$

Hence,

$$1 + \frac{zD_q^2\left(F_q^{n,m,\lambda}(z)\right)}{D_q\left(F_q^{n,m,\lambda}(z)\right)} = \sum_{i=1}^{l} \gamma_i \frac{zD_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{\mathcal{R}_q^{n,m,\lambda} f_i(z)} - \sum_{i=1}^{l} \gamma_i + 1.$$

This relationship is tantamount to

$$Re\left\{1+\frac{zD_q^2\left(F_q^{n,m,\lambda}(z)\right)}{D_q\left(F_q^{n,m,\lambda}(z)\right)}\right\}=\sum_{i=1}^{l}\gamma_i Re\left\{\frac{zD_q\left(\mathcal{R}_q^{n,m,\lambda}f_i(z)\right)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}\right\}-\sum_{i=1}^{l}\gamma_i+1. \tag{43}$$

Utilizing (12), we obtain for $i\in\{1,2,\ldots,l\}$

$$\begin{aligned}\frac{zD_q\left(\mathcal{R}_q^{n,m,\lambda}f_i(z)\right)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}&=\frac{[n+1]_q\mathcal{R}_q^{n+1,m,\lambda}f_i(z)-[n]_q\mathcal{R}_q^{n,m,\lambda}f_i(z)}{q^n\mathcal{R}_q^{n,m,\lambda}f_i(z)}=\\&=\frac{1}{q^n}\left[[n+1]_q\frac{\mathcal{R}_q^{n+1,m,\lambda}f_i(z)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}-[n]_q\right].\end{aligned}$$

Taking into account that $[n+1]_q=[n]_q+q^n$, we obtain

$$\begin{aligned}\frac{zD_q\left(\mathcal{R}_q^{n,m,\lambda}f_i(z)\right)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}&=\frac{[n+1]_q}{q^n}\frac{\mathcal{R}_q^{n+1,m,\lambda}f_i(z)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}+\frac{q^n-[n+1]_q}{q^n}=\\&=\frac{[n+1]_q}{q^n}\left(\frac{\mathcal{R}_q^{n+1,m,\lambda}f_i(z)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}-1\right)+1.\end{aligned} \tag{44}$$

The relation (43) is equivalent to

$$\begin{aligned}Re\left\{1+\frac{zD_q^2\left(F_q^{n,m,\lambda}(z)\right)}{D_q\left(F_q^{n,m,\lambda}(z)\right)}\right\}&=\sum_{i=1}^{l}\gamma_i Re\left\{\frac{[n+1]_q}{q^n}\left(\frac{\mathcal{R}_q^{n+1,m,\lambda}f_i(z)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}-1\right)+1-\alpha_i\right\}+\\&+\sum_{i=1}^{l}\gamma_i(\alpha_i-1)+1.\end{aligned}$$

As $f_i\in\mathcal{US}_q^n(\alpha,k)$, for all $i\in\{1,2,\ldots,l\}$, by making use of (38), we obtain

$$\begin{aligned}Re\left\{1+\frac{zD_q^2\left(F_q^{n,m,\lambda}(z)\right)}{D_q\left(F_q^{n,m,\lambda}(z)\right)}\right\}&\geq 1+\sum_{i=1}^{l}\gamma_i(\alpha_i-1)+\\&+\sum_{i=1}^{l}\gamma_i k_i\left|\frac{[n+1]_q}{q^n}\left(\frac{\mathcal{R}_q^{n+1,m,\lambda}f_i(z)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}-1\right)\right|.\end{aligned}$$

Since $\sum_{i=1}^{l}\gamma_i k_i\left|\frac{[n+1]_q}{q^n}\left(\frac{\mathcal{R}_q^{n+1,m,\lambda}f_i(z)}{\mathcal{R}_q^{n,m,\lambda}f_i(z)}-1\right)\right|>0$, for all $i\in\{1,2,\ldots,l\}$, we conclude

$$Re\left\{1+\frac{zD_q^2\left(F_q^{n,m,\lambda}(z)\right)}{D_q\left(F_q^{n,m,\lambda}(z)\right)}\right\}>1+\sum_{i=1}^{l}\gamma_i(\alpha_i-1).$$

Therefore, the integral operator $F_q^{n,m,\lambda}(z)$ is a convex of order λ, where $\lambda=1+\sum_{i=1}^{l}\gamma_i(\alpha_i-1)$. □

Now, we establish the convexity result concerning to the q-differentiation of the operator $G_q^n(z)$.

Theorem 7. *Let $n,m\in\mathbb{N}$, $0<q<1$, $\lambda\geq 0$, $\gamma=(\gamma_1,\ldots,\gamma_l)\in\mathbb{R}_+^l$, $-1\leq\alpha_i<1$, $k_i>0$ and $f_i\in\mathcal{UC}_q^n(\alpha_i,k_i)$, for all $i\in\{1,2,\ldots,l\}$, $l\in\mathbb{N}$. If*

$$0\leq 1+\sum_{i=1}^{l}\gamma_i(\alpha_i-1)<1, \tag{45}$$

then, the q-integral operator $G_q^{n,m,\lambda}(z)$ defined by (37) is convex with respect to the q-differentiation of order λ, with $\lambda = 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1)$.

Proof. From (37), it is evident that $G_q^{n,m,\lambda}(z) \in \mathcal{A}$. It is straightforward to confirm that

$$D_q\left(G_q^{n,m,\lambda}(z)\right) = \prod_{i=1}^{l}\left(D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)\right)^{\gamma_i},\ z \in U. \tag{46}$$

This equality suggests that

$$\ln D_q\left(G_q^{n,m,\lambda}(z)\right) = \gamma_1 \ln\left(D_q\left(\mathcal{R}_q^{n,m,\lambda} f_1(z)\right)\right) + \ldots + \gamma_l \ln\left(D_q\left(\mathcal{R}_q^{n,m,\lambda} f_l(z)\right)\right).$$

By q-differentiating both sides of the aforementioned equality, we obtain

$$\frac{D_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)} = \sum_{i=1}^{l} \gamma_i \left(\frac{D_q^2\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}\right). \tag{47}$$

Therefore,

$$\begin{aligned} 1 + \frac{zD_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)} &= \sum_{i=1}^{l} \gamma_i \left(1 + \frac{zD_q^2\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)} - \alpha_i\right) + \\ &+ \sum_{i=1}^{l} \gamma_i(\alpha_i - 1) + 1. \end{aligned} \tag{48}$$

This relation is equivalent to

$$\begin{aligned} Re\left\{1 + \frac{zD_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)}\right\} &= 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1) + \\ &+ \sum_{i=1}^{l} \gamma_i Re\left(1 + \frac{zD_q^2\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)} - \alpha_i\right). \end{aligned} \tag{49}$$

As $f_i \in \mathcal{UC}_q^n(\alpha_i, k_i)$, for all $i \in \{1, 2, \ldots, l\}$, from (49), we deduce

$$\begin{aligned} Re\left\{1 + \frac{zD_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)}\right\} &\geq 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1) + \\ &+ \sum_{i=1}^{l} \gamma_i k_i \left|\frac{zD_q^2\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}\right|. \end{aligned}$$

Since

$$\sum_{i=1}^{l} \gamma_i k_i \left|\frac{zD_q^2\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}{D_q\left(\mathcal{R}_q^{n,m,\lambda} f_i(z)\right)}\right| > 0,$$

for all $i \in \{1, 2, \ldots, l\}$, we obtain

$$Re\left\{1 + \frac{zD_q^2\left(G_q^{n,m,\lambda}(z)\right)}{D_q\left(G_q^{n,m,\lambda}(z)\right)}\right\} \geq 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1).$$

Therefore, the integral operator $G_q^{n,m,\lambda}(z)$ is a convex of order λ, with $\lambda = 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1)$. The proof is now finished. □

Corollary 4. *Let $n, m \in \mathbb{N}$, $\lambda \geq 0$, $-1 \leq \alpha_i < 1$, $k_i > 0$, $\gamma_i > 0$, $i \in \{1, 2, \ldots, l\}$, $f_i \in \mathcal{US}(\alpha_i, k_i)$. If $0 \leq 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1) < 1$, then the integral operator $\int_0^z \prod_{i=1}^{l} \left(\frac{\mathcal{R}^{n,m,\lambda} f_i(t)}{t} \right)^{\gamma_i} dt$ is a convex of order λ, with $\lambda = 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1)$, where $\mathcal{R}^{n,m,\lambda}$ is the generalised Darus and Al-Shaqsi derivative operator [27].*

Proof. By allowing $q \to 1$ in Theorem 6, we obtain the corresponding corollary. □

Corollary 5. *Let $n, m \in \mathbb{N}$, $\lambda \geq 0$, $-1 \leq \alpha_i < 1$, $k_i > 0$, $\gamma_i > 0$, $i \in \{1, 2, \ldots, l\}$, $f_i \in \mathcal{UC}(\alpha_i, k_i)$. If $0 \leq 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1) < 1$; then, the integral operator $\int_0^z \prod_{i=1}^{l} \left(\left(\mathcal{R}^{n,m,\lambda} f_i(t)\right)' \right)^{\gamma_i} dt$ is a convex of order λ, with $\lambda = 1 + \sum_{i=1}^{l} \gamma_i(\alpha_i - 1)$, where $\mathcal{R}^{n,m,\lambda}$ is the generalised Darus and Al-Shaqsi derivative operator [27].*

Proof. Letting $q \to 1$ in Theorem 7, the corollary follows. □

3. Conclusions

By employing the recently introduced q-operator, denoted as $\mathcal{R}_q^{n,m,\lambda}$, we employ the methods of the theory of differential subordination to conduct this study. This exploration results in the identification of novel and compelling differential subordination relationships, along with the determination of the best dominant. By making use of the new defined q-operator, and inspired by the operators introduced in [31,32], two new q-integral operators $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$ are introduced in this work. Using these operators, specific classes of functions are presented and analyzed, and convexity properties of the operators $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$ are examined. We anticipate that this research provides a groundwork for future exploration into various classes of analytic functions. This can be achieved by employing the previously introduced q-difference operator $\mathcal{R}_q^{n,m,\lambda}$ and the q-integral operators $F_q^{n,m,\lambda}$ and $G_q^{n,m,\lambda}$, exploring their diverse geometric properties, including associated coefficient estimates, sufficiency criteria, radii of starlikeness, convexity, close to convexity, extreme points, and distortion bounds. The expected outcome is the application of these considerations to explore additional classes of analytic functions.

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