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Proximal Analytic Center Cutting Plane Algorithms for Variational Inequalities and Nash Economic Equilibrium

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Abstract: In this study, we proposed proximal analytic center cutting plane algorithms for solving variational inequalities whose domains are normal regions. Our algorithms stop with a solution of the variational inequality after a finite number of iterations, or we may find a sequence that converges to the solution of the variational inequality. We introduced the definition of the Nash economic equilibrium solution over a normal region and proved a sufficient condition for our Nash economic solution. An example of Nash equilibrium over a normal region is also provided. Our proximal analytic center cutting plane algorithms are constructive proofs of our Nash equilibrium problems.

Keywords: normal region; variational inequality; proximal analytic center cutting plane algorithms; Nash economic equilibrium

MSC: 65K15; 90C33; 91A10; 49J40

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1. Preliminaries

Cutting plane methods for optimization have a long history that goes back at least to a fundamental paper by Kelley [1]. The theoretical approach of the analytic center cutting plane methods started from Gon and Vial [2]. du Merle [3] developed an implementation of the method of the prototype, which was successfully applied to solve several nontrivial convex optimization problems [4,5]. Some later developments of the analytic center cutting plane methods have been proposed for solving various variational inequalities, e.g., [6–9].

We present two proximal analytic center cutting plane algorithms for solving variational inequalities whose domains are normal regions.

Our proximal analytic center cutting plane algorithms are also constructive solutions to our Nash economic equilibrium problems.

This study contains a detailed description of computational schemes of algorithms and provides the theoretical proofs of their convergence to the desired solution.

Suppose X is a non-empty subset of the n-dimensional Euclidean space R^n, and F: $X \to R^n$ is a function. We call that a point $x^* \in X$ is a solution of the variational inequality $VI\,[F, X]$ if

$$F(x^*)^T(x - x^*) \geq 0, \forall x \in X. \quad (1)$$

The point $x^* \in X$ is a solution of the dual variational inequality $VID\,(F, X)$ if

$$F(x)^T(x - x^*) \geq 0, \forall x \in X.$$

Given $VI[F, X]$ ($VID[F, X]$), the gap function is defined as

$$g_X(x) = \max_{y \in X} F(x)^T(x - y), x \in X \; (f_X(x) = \max_{y \in X} F(y)^T(x - y), x \in X).$$

A point $x^* \in X$ is said to be a ε-solution of the variational inequality (1) if $g_X(x^*) \geq \varepsilon$, for given $\varepsilon > 0$.

A function F: $X \to R^n$ is pseudomonotone on X if $\forall\, x \in X$

$$F(x)^T(y-x) \geq 0 \Rightarrow F(y)^T(y-x) \geq 0, \forall y \in X.$$

It is known that (see Auslender [10]), if F is continuous, then a solution $x^* \in X$ of *VID* (F, X) is a solution of *VI* (F, X); and if F is continuous pseudomonotone, then $x^* \in X$ is a solution of *VI* (F, X) if and only if it is a solution of *VID* (F, X).

It is known that the following Lemma 1 holds.

Lemma 1. *A point $x^* \in X$ is a solution of VI[F, X] (VID[F, X]) if and only if $g_X(x^*) = 0$ ($f_X(x^*) = 0$).*

The convex hull of a set $B \subseteq R^n$ is the set

$$con(B) = \{\sum_{i=1}^{l} \alpha_i x_i; x_i \in B, \sum_{i=1}^{l} \alpha_i = 1, \alpha_i \geq 0\}.$$

A polytope is a set $P \subseteq R^n$ which is the convex hull of a finite set.

A polyhedron is a set

$$\{x \in R^n; A^T x \leq b\} \subseteq R^n,$$

where $b \in R^n$, and A is an $m \times n$ matrix.

A polytope is a polyhedron. The following intuitively clear but nontrivial to prove result is essentially due to Farkas [11], Minkowski [12], and Weyl [13]:

Lemma 2. *P is a polytope if and only if it is a bounded polyhedron.*

In the sequel, we assume that a polytope always has a non-empty interior.

Definition 1. *A subset $X \subseteq R^n$ is said to be a normal region if it is a closed bounded set and if there exists a sequence of polytopes $\{C_j\}$, which satisfies*

$$C_j \subseteq C_{j+1}\ (j = 1, \cdots),$$

such that

$$(\cup_{j=1}^{\infty} C_j)^c = X.$$

The proof of the following Theorem 1 is trivial.

Theorem 1. *A closed, bounded, convex region $X \subseteq R^n$ is a normal region.*

We denote X^c the topological closure of X.

Theorem 2. *A subset $X \subseteq R^n$ is a normal region if there exists a uniformly bounded sequence of polytopes $\{C_j\}$, $C_j \subseteq X$ ($j = 1, \cdots$), such that*

$$(\cup_{j=1}^{\infty} C_j)^c = X.$$

Proof. Actually, let

$$C_j{}' = con(\cup_{i=1}^{j} C_j).$$

Then, the sequence $\{C_j{}'\}$ satisfies the conditions in Definition 1. □

Definition 2. *A subset $X \subseteq R^n$ is a unbounded normal region if there exists a sequence of bounded normal regions $X_j \subseteq X (j = 1, 2, \ldots)$, such that each X_j contains all boundary points of X, and*

$$\cup_{j=1}^{\infty} X_j = X.$$

Actually, a subset $X \subseteq R^n$ is a bounded or unbounded normal region if and only if it is a bounded or unbounded convex region.

2. Proximal Analytic Center Cutting Plane Algorithms

This section modifies the method in Shen and Pang [6] and presents proximal analytic center cutting plane algorithms for solving variational inequality $VI[F, X]$, whose domains are normal regions.

From now on, we make the following assumptions: for each x, $y \in X$, given any $\bar{\varepsilon} = (\varepsilon, \varepsilon, \cdots, \varepsilon)^T \in R^n$, $\bar{\delta} = (\delta, \delta, \cdots, \delta)^T \in R^n$, where $\varepsilon, \delta \in (0,1)$, we can always find $\overline{F}_x \in R^n$ and $\overline{F}_y \in R^n$ such that

(i) $F(x) \leq \overline{F}_x \leq F(x) + \bar{\varepsilon}$, $F(y) \leq \overline{F}_y \leq F(y) + \bar{\delta}$
(ii) $\overline{F}_y \to \overline{F}_x$ if $y \to x$, no matter the relationship between $\bar{\varepsilon}$ and $\bar{\delta}$,
(iii) $\|\overline{F}_y - \overline{F}_x\| \leq L \|y - x\|$, where L is a constant.

Assume the auxiliary $\Gamma(x, y) : R^n \times R^n \to R^m$ be a mapping that is continuous in x and y, strong monotone with respect to y with a constant $M > 0$, i.e.,

$$(\Gamma(y, x) - \Gamma(z, x))^T (y - z) \geq M\|y - z\|^2, \forall y, z \in X,$$

Consider the auxiliary variational inequality associated with Γ, whose solution $\omega(x)$ satisfies

$$(\Gamma(\omega(x), x) - \Gamma(x, x) + \overline{F}_x)^T (y - \omega(x)) \geq 0, \forall y, z \in X.$$

In view of the strong monotonicity of $\Gamma(x, y)$ with respect to y, this auxiliary variational inequality has a unique solution (Goffin, Luo, and Ye [7]).

For any polytope $\{x \in R^n; A^T x \leq b\}$, $\{x \in R^n; A^T x + s = b, s = (s_1, s_2, \cdots, s_n), s_i \geq 0\}$ is associated with the potential function $\varphi = \sum_{i=1}^{n} \ln s_i$. An approximate analytic center, introduced by Goffin, Luo, and Ye [7], is the maximizer of the potential function φ and the unique solution of the system

$$\begin{aligned} & A^T z = 0 \\ & A^T x + s = b \\ & \|Z^T s - e\| \leq \eta < 1, \end{aligned}$$

where z is a dual vector, and Z the diagonal matrix built upon z.

Let $X \subseteq R^n$ be a normal region, there exists a sequence of polytopes $\{C_j\}$ satisfying $C_j \subseteq C_{j+1}$ ($j = 1, 2, \cdots$) such that

$$(\cup_{j=1}^{\infty} C_j)^c = X.$$

Then, there is a sequence of variational inequalities $VI[F, C_j]$ ($j = 1, 2, \cdots$) from the original variational inequality $VI[F, X]$, where the polytope C_j is given by the linear inequalities $A_j x = b_j$, $x, b_j \in R^n$, and A_j is an $m \times n$ matrix, ($j = 1, 2, \cdots$).

The following Algorithm 1 extends Algorithm 3.1 in Shen and Pang [6] from a feasible region of polytope to a normal region.

Theorem 3. *Algorithm 1 either stops with a solution of the variational inequality VI(F, X) after a finite number of iterations, or there exists a sequence in X that converges to the solution of VI(F, X).*

Algorithm 1

Assume X is a bounded normal region. Let $\varepsilon \in (0,1)$ and $\alpha \in (0, M)$ be two constant. Set $k = 0$, $j = 1$, $A^k = A_j$, $b^k = b_j$, $\varepsilon^k = \varepsilon$, and
$C_j^k = \{x \in R^n : A_j^k x \le b_j^k\}$.
Step 1 (Computation of the approximate analytic center). Find an approximate analytic center x_j^k of $C_j^k = \{x \in R^n : A_j^k x \le b_j^k\}$.
Step 2 (Stopping criterion). If $g_X(x_j^k) = 0$, then stop.
Else go to Step 3.
Step 3 (Computation of a $1/j$ solution). Find x_j^k of $C_j^k = \{x \in R^n : A_j^k x \le b_j^k\}$
If $g_{C_j^k}(x_j^k) \le 1/j$ then increase j by 1 and go to Step 1.
Else go to Step 4.
Step 4 (Solving the approximate auxiliary variational inequality problem). Find $\omega(x_j^k)$, such that
$(\Gamma(\omega(x_j^k), x_j^k) - \Gamma(x_j^k, x_j^k) + \overline{F}_{x^k})^T (y - \omega(x_j^k)) \ge 0, \forall y \in X$,
where $\overline{F}_{x_j^k}$ satisfies $F(x_j^k) \le \overline{F}_{x_j^k} \le F(x_j^k) + \overline{\varepsilon}_{x_j^k}$, $\overline{\varepsilon}_{x_j^k} = (\varepsilon^k, \varepsilon^k, \cdots, \varepsilon^k)^T \in R^n$.
Step 5 (construction of the approximate cutting plane). Let
$y_j^k = x_j^k + \rho^{l_{k,j}}(\omega(x_j^k) - x_j^k)$ and $\overline{G}_{x_j^k} = \overline{F}_{y_j^k}$, where $l_{k,j}$ is the smallest integer that satisfies
$\overline{F}^T_{x_j^k + \rho^{l_{k,j}(\omega(x_j^k) - x_j^k)}} (x_j^k - \omega(x_j^k)) \ge \alpha \left\| \omega(x_j^k) - x_j^k \right\|^2$,
where $\overline{F}^T_{x_j^k + \rho^{l_{k,j}(\omega(x_j^k) - x_j^k)}}$ satisfies
$F(x_j^k) + \rho^{l_k}(\omega(x_j^k) - x_j^k)) \le \overline{F}_{x^k + \rho^{l_{k,j}}(\omega(x_j^k) - x_j^k)} \le F(x_j^k + \rho^{l_{k,j}}(\omega(x_j^k) - x_j^k)) + \overline{\varepsilon}_k$
where $\overline{\varepsilon}_k = \varepsilon_{x^k + \rho^{l_k}(\omega(x^k) - x^k)} = (\varepsilon^k, \varepsilon^k, \cdots, \varepsilon^k)^T$
Let $H_j^k = \{x : G^T_{x_j^k}(x - x_j^k) = 0\}$,
$A_j^{k+1} = \begin{pmatrix} A_j^k \\ \overline{G}_{x_j^k} \end{pmatrix}, b_j^{k+1} = \begin{pmatrix} b_j^k \\ \overline{G}_{x_j^k} x_j^k \end{pmatrix}$.
Increase k by 1 and go to Step 1.
End of Algorithm 1.

Proof. For any given j, $\exists x_j \in C_j$, such that $g_{C_j}(x_j) < 1/j$, ($j = 1, 2, \ldots$), then

$$g_{C_j}(x_j) \to 0, j \to \infty.$$

Because X is a bounded set, there exists a subsequence of $\{x_j\}$ and a point $x^* \in X$ such that the subsequence converges to $x^* \in X$. We may assume that $\lim_{j \to \infty} x_j = x^*$.

Since X is closed, $\exists y^0 \in X$ such that $g_X(x_j) = \max_{y \in X} F(x_j)(x_j - y) = F(x_j)(x_j - y^0)$. And noting that $(\cup_{j=1}^{\infty} C_j)^c = X$, $\exists y^i \in \cup_{j=1}^{\infty} C_j$ such that $\lim_{i \to \infty} y^i = y^0$, and for given j

$$\to F(x_j)(x_j - y^0), i \to \infty.$$

So, $\forall \varepsilon > 0, \exists N$, for $i \ge N$

$$0 \le F(x_j)(x_j - y^0) - F(x_j)(x_j - y^i) < \varepsilon.$$

There is a subsequence of $\{C_j\}$, and without a loss of generosity, we may assume $\{C_j\}$ itself, which satisfies that $y^j \in C_j, j = 1, 2, \cdots$. Therefore, for $j \ge N$

$$\begin{aligned} &0 \le g_X(x_j) - g_{Cj}(x_j) \\ &= F(x_j)(x_j - y^0) - \max_{y \in C_j} F(x_j)(x_j - y) \\ &\le g_X(x_j) - F(x_j)(x_j - y^j) < \varepsilon. \end{aligned}$$

On the other hand, the continuity of g_X implies that

$$\| g_X(x^*) - g_X(x_j) \| \to 0, j \to \infty.$$

Consequently,

$$\| g_X(x^*) \| \le \| g_X(x^*) - g_X(x_j) \| + \| g_X(x_j) - g_{C_j}(x_j) \| + \| g_{C_j}(x_j) \| \to 0, j \to \infty.$$

Therefore,

$$g_X(x^*) = \max_{y \in X} F(x^*)(x^* - y) = 0.$$

Which concludes that x^* is a solution of $VI(F, X)$. The proof is complete. □

By use of our Algorithm 1, we are going to present the following Algorithm 2 to solve variational inequality $VI(F, X)$ over an unbounded normal region X.

We can find a sequence of bounded normal regions X_j (j = 1, 2, . . .), such that each X_j contains all boundary points of X and

$$\cup_{j=1}^{\infty} X_j = X.$$

We note that each variational inequality VI [F, X_j] has a unique solution $x_j^* \in X_j$ (j = 1, 2, . . .). And, when j is large enough, say $j > N$, the unique solution x^* of the VI[F, X] satisfies that $x^* \in X_j, j > N$.

Algorithm 2

Assume X is an unbunded normal region. Set j = 1.
Step 1 (Computation of a sequence $x_j^l \subseteq X_j$ such that $\| x_j^1 - x_j^l \| \le 1/j, (l = 2, 3, \cdots)$)
By use of Algorithm 1, computing a sequence $\{x_j^l\}_{l=1}^{\infty} \subseteq X_j$ such that
$\lim\limits_{l \to \infty} x_j^l = x_j^*$ and $\| x_j^1 - x_j^l \| \le 1/j, (l = 2, 3, \cdots .)$, (and let $l \to \infty$, one
has $\| x_j^1 - x_j^* \| \le 1/j$, although x_j^* is still unknown yet).
Step 2 (Stopping criterion). If $g_X(x_j^l)$ = 0 for any of $l = 1, 2, \cdots$, then stop.
Else go to Step 1.
End of Algorithm 2.

Theorem 4. *Algorithm 2 either stops with a solution of the variational inequality VI(F, X) after a finite number of iterations, or there exists a sequence in X that converges to the solution of VI(F, X).*

Proof. Algorithm 2 stops with a solution of $VI(F, X)$, or we can find sequences $\{x_j^l\}_{l=1}^{\infty} \subseteq X_j \subseteq X$ such that $\lim\limits_{l \to \infty} x_j^l = x_j^*$ (j = 1, 2, . . .), and $x_j^* = x^*$ if $j > N$, i.e., $\lim\limits_{k \to \infty} x_j^k = x^*$ (although x^* is still unknown) if $j > N$. Therefore, from Step 1, one obtains

$$\| x_j^1 - x^* \| \le 1/j, \forall j > N. \tag{2}$$

Then, take the sequence $\{x_j^1\}_{j=1}^{\infty} \subseteq X$ (consisting of the first term of each sequence $\{x_j^l\}_{l=1}^{\infty} \subseteq X_j$, j = 1, 2, . . .), and one has

$$\| x_j^1 - x_k^1 \| \le \| x_j^1 - x^* \| + \| x^* - x_k^1 \| \le 1/j + 1/k \to 0, j, k \to \infty.$$

Hence, $\{x_j^1\}_{j=1}^{\infty} \subseteq X$ is a Cauchy sequence and so is convergent. From (2), one obtains $\lim\limits_{j \to \infty} x_j^1 = x^*$, which completes the proof. □

3. Nash Economic Equilibrium Application

Economic equilibrium is a condition or state in which economic forces are balanced. These economic variables remain unchanged from their equilibrium values in the absence of external influences. Economic equilibrium may also be defined as the point at which supply equals demand for a product, with the equilibrium price existing where the hypothetical supply and demand curves intersect. Economists can usually explain the past and sometimes predict the future, but not without help. One of the most important tools at their disposal is the Nash equilibrium, named after John Nash [14], who won a Nobel Prize in 1994 for his discovery. There were plenty of discussions followed after Nash, e.g., some recent approaches can be seen in Fischer [15], Faraci [16], and Boilan [17].

In this section, we explain that our proximal analytic center cutting plane algorithms can be used to solve some practical problems whose domains are normal regions, e.g., Nash economic equilibrium problems over normal regions.

Consider an oligopolistic economy in which a homogeneous product is supplied by n firms. Let $p(\sigma)$ denote the inverse demand function, which is the price at which consumers will purchase a quantity σ. If each firm supplies q_i units of the product, then the total supply is

$$\sigma_q = \sum_{i=1}^{n} q_i. \tag{3}$$

We denote by $h_i(q_i)$ the i-th firm's total cost of supplying q_i units of the product; the profit of the i-th firm is given by

$$\varphi_i(q) = q_i p(\sigma_q) - h_i(q_i).$$

A vector $q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ is a said to be a Nash equilibrium solution if it is an optimal solution to the problem

$$\max_{q\in \mathrm{K}} \varphi_i(q) = \max_{q\in \mathrm{K}} [q_i p(\sigma_q + \sigma_i^*) - h_i(q_i)].$$

where K is the box-constrained set (see Konnov and Volotskaya [18]),

$$\mathrm{K} = \{x \in R^n; 0 \le a_j \le x_j \le b_j \le +\infty\},$$

a_j are constants, b_j are either constants or $+\infty$ ($j = 1, \ldots, n$). And $\sigma_i^* = \sum_{j=1, j\neq i}^{n} q_j^*$ ($i = 1, 2, \ldots, n$).

The Clarke–Rokafellar generalized derivatives of h at p_0 in the direction d is defined by

$$h^{\uparrow}(p_0, d) = \sup_{\varepsilon>0} \limsup_{\substack{p \to (h, p_0) \\ t \to 0+}} \inf_{e\in B_\varepsilon(d)} \frac{h(p+te) - h(p)}{t},$$

where $B_\varepsilon(d) = \{e \in X : \|e - d\| < \varepsilon\}$, $t \to 0+$ means that $t > 0$ and $t \to 0$, $p \to (h, p_0)$ indicates that both $p \to p_0$ and $h(p) \to h(p_0)$.

The Clarke–Rokafellar subdifferential of h at p_0 is given by

$$\partial h(p_0) = \{p \in X : (p, d) \le h^{\uparrow}(p_0, d), \forall d \in X\}.$$

Let

$$G_i(q) = -p(\sigma_q) - q_i p'(\sigma_q), (i = 1, 2, \ldots, m).$$

From Golshtein and Tretyakov [19], as well as Murphy, Sherali, and Soyster [20], a vector is a Nash equilibrium solution in the oligopolistic economy if and only if it is a solution to the problem

$$G(q^*)^T(q - q^*) + \sum_{i=1}^{n} \partial h_i(q_i^*)(q_i - q_i^*) \geq 0, \forall q \in \mathrm{K}.$$

where K = $\{x \in R^n; 0 \leq a_j \leq x_j \leq b_j \leq +\infty\}$.

Write $F(q) = G(q) + \partial h(q)$, then

$$F(q^*)^T(q - q^*) \geq 0, \forall q \in \mathrm{K}.$$

Similarly, we can introduce the Nash equilibrium solution in the oligopolistic economy if the domain is a normal region X:

A Nash equilibrium solution in the oligopolistic economy can be defined as a solution of the variational inequality $VI[F, X]$

$$F(q^*)^T(q - q^*) \geq 0, \forall q \in X, \tag{4}$$

if X is a normal region.

Example 1. *The following mathematical model is used to calculate the remaining loan balance of a fixed mortgage loan. The mortgage payment amount should be paid periodically for m periods on a mortgage amount L at a periodic interest rate of r_1. After r_2 periods for full amortization m periods ($r_2 \leq m$), the remaining balance B of the loan is given by*

$$B = L\frac{(1 + r_1)^m - (1 + r_1)^{r_2}}{(1 + r_1)^m - 1}.$$

Assume one would like to consider a selling price r_3 that satisfies

$$\begin{aligned} r_3 \geq D + L - B &= D + L - L\frac{(1 + r_1)^m - (1 + r_1)^{r_2}}{(1 + r_1)^m - 1} \\ &\geq L - L\frac{(1 + r_1)^m - (1 + r_1)^{r_2}}{(1 + r_1)^m - 1}, \end{aligned} \tag{5}$$

where D is the down payment at the beginning of the mortgage.

If there are $n = (l_1, l_2, \ldots, l_{\mathrm{n}})$ mortgage providers in the same mortgage rate, $p = r_1 = p(\sigma)$. Mortgage provider l_i provides a mortgage amount of q_{i} ($i = 1, 2, \ldots, n$). Then, q_i are all functions of r_1, r_2, and r_3. Let $Y = Y(r_1, r_2, r_3)$ be the convex region given by (5), where $0 < q_1 \leq \mathrm{M} < \infty$ for a given M. Then,

$$\max_{q \in \mathrm{K}} \varphi_i(q) = \max_{q \in \mathrm{K}} [q_i p(q_i + \sigma_i^*) - h_i(q_i)] = \max_{y \in \mathrm{Y}} [q_i p(q_i + \sigma_i^*) - h_i(q_i)]$$

which is an example of Nash equilibrium over a convex region.

Therefore, by Example 1, a domain of a Nash equilibrium problem may be a normal region, and so our proximal analytic center cutting plane methods can be applied to solve it.

A function F: $X \to R^n$ is said to be strictly pseudomonotone on X if $\forall q^* \in X$

$$F(q^*)^T(q - q^*) \geq 0 \Rightarrow F(q)^T(q - q^*) > 0, \forall q \in X, q \neq q^*.$$

Theorem 5. *Let $X \subseteq R^n$ be a normal region and F: $X \to R^n$ a continuous and strictly pseudomonotone function, then the Nash equilibrium (4) has one and only one solution.*

Proof. $\forall q', q'' \in \cup_{j=1}^{\infty} C_j$, $\exists i, j$ such that $q'' \in C_i$ and $q' \in C_j$. Without loss of generality, we suppose that $i \geq j$, then $q', q'' \in C_i$. Due to the convexity of C_i, we have

$$\alpha q' + (1 - \alpha) q'' \in C_i \subseteq \cup_{j=1}^{\infty} C_j, \forall \alpha \in [0, 1].$$

Which means that $\cup_{j=1}^{\infty} C_j$ is convex. It is easy to see that the closure of any convex set is convex. Therefore, $X = (\cup_{j=1}^{\infty} C_j)^c$ is a convex and compact set in R^n. From Hartman and Stampacchia [21], the Nash equilibrium (4) has solutions.

On the other hand, assume that q^* is a solution of the Nash equilibrium (4); then,

$$F(q^*)^T (q - q^*) \geq 0, \forall q \in X.$$

Due to the strict pseudomonotonicity of F, we have

$$F(q)^T (q - q^*) > 0, \forall q \in X, q \neq q^*.$$

i.e.,

$$F(q)^T (q^* - q) < 0, \forall q \in X, q \neq q^*$$

Which indicates that $\forall q \in X^*$ with $q \neq q^*$ is not a solution of (2). Therefore, the Nash equilibrium (2) has at most one solution. We complete the proof. □

4. Final Remarks

Remark 1. *The study is connected with the application of the proposed proximal analytic center cutting plane techniques to the analysis of the Nash equilibrium problems in models of oligopolistic economy stated as problems of variational inequalities. Our proximal analytic center cutting plane algorithms are constructive proof of the existence of our Nash equilibrium solutions in Section 3.*

Remark 2. *We presented proximal analytic center cutting plane algorithms for solving variational inequalities, which extended the algorithms over polytopes in [6] to normal regions.*

Remark 3. *Compared with [22], in this article, we dropped off the conditions of "Lipschitz continuous", "pseudomonotone plus", and/or "strongly pseudomonotone" in corresponding results.*

Remark 4. *Similar to [23,24], our algorithms can be used in Machine Learning and Artificial Intelligence.*

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