

Article

Equilibrium Figures for a Rotating Compressible Capillary Two-Layer Liquid

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Abstract: The paper proves the existence of a family of axisymmetric equilibrium figures as solutions of a stationary problem with unknown boundaries for the Navier–Stokes equations corresponding to the slow rotation of a viscous compressible two-layer liquid mass about some axis. It is assumed that the liquids are barotropic and capillary, and have different viscosities, the internal fluid being bounded by a closed surface. This interface does not intersect with the external boundary of the cloud. The proof is based on implicit function theorem and carried out in the Hölder spaces.

Keywords: equilibrium figures; viscous compressible two-layer fluid; capillary forces; interface problem for the Navier–Stokes system; the Hölder spaces

MSC: 35Q30; 76T17; 76N06; 35R35; 76N10



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1. Introduction

Existence of an equilibrium surface for an isolated compressible liquid mass rotating about a fixed axis was first proved in [1]. Our aim is to prove the existence of equilibrium figures for a rotating compressible two-layer fluid.

The problem of the rotation of an isolated incompressible liquid mass about a fixed axis as a rigid body was considered by many famous mathematicians, including Newton, Maclaurin, Jacobi, Kovalevskaya, Lyapunov, Poincare, and others [2–4], who mainly studied the movement without capillarity. The capillary fluids were first investigated by Globa-Mikhailenko [5], Boussinesq, and Charrueau in the beginning of 20th century. The latter gave a detailed analysis of the problem, calculated the shape of equilibrium figures, including the toroidal case, and considered some aspects of the stability [6,7]. These results were included in a big review on this subject, presented in the book of Appell [8]. The stability problem for various ellipsoidal equilibrium figures is analyzed in monograph [9].

The existence of equilibrium figures for a two-layer incompressible self-gravitating capillary liquid (oblate embedded spheroids) was obtained by V.A. Solonnikov in [10].

Now we state, in a complete setting, the problem on unsteady motion of two compressible barotropic fluids of finite volume separated by a closed unknown interface.

At the initial instant $t = 0$, let a fluid with dynamic viscosities μ^+ , μ_1^+ be in a bounded domain $\Omega_0^+ \subset \mathbb{R}^3$, and in the domain Ω_0^- , surrounding it, let there be a fluid with dynamic viscosities μ^- , μ_1^- ;

$$\mu^\pm > 0, \quad 2\mu^\pm + 3\mu_1^\pm \geq 0.$$

The domain $\Omega_0 \equiv \overline{\Omega_0^+} \cup \Omega_0^-$ is bounded by the free surface Γ_0^- and includes the closed interface $\Gamma_0^+ \equiv \partial\Omega_0^+$; Γ_0^\pm are given. This two-component cloud rotates about the vertical axis x_3 with an angular velocity ω .

For $t > 0$, it is necessary to find the surfaces Γ_t^-, Γ_t^+ , as well as the velocity vector field $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and the density $\rho(x, t) > 0$ of the fluids satisfying the diffraction problem for the Navier–Stokes system

$$\begin{aligned} &\rho(\mathcal{D}_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) - \nabla \cdot \mathbb{T} = 0, \quad \mathcal{D}_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{in } \Omega_t^- \cup \Omega_t^+, \quad t > 0, \\ &\mathbf{v}|_{t=0} = \mathbf{v}_0(x), \quad \rho|_{t=0} = \rho_0(x) \quad \text{in } \Omega_0^- \cup \Omega_0^+, \\ &\mathbb{T}(\mathbf{v}, p) \mathbf{n}|_{\Gamma_t^-} = \sigma^- H^- \mathbf{n} \quad \text{on } \Gamma_t^-, \\ &[\mathbf{v}]|_{\Gamma_t^+} \equiv \lim_{\substack{x \rightarrow x_0 \in \Gamma_t^+, \\ x \in \Omega_t^+}} \mathbf{v}(x, t) - \lim_{\substack{x \rightarrow x_0 \in \Gamma_t^+, \\ x \in \Omega_t^-}} \mathbf{v}(x, t) = 0, \quad [\mathbb{T}(\mathbf{v}, p) \mathbf{n}]|_{\Gamma_t^+} = \sigma^+ H^+ \mathbf{n} \quad \text{on } \Gamma_t^+, \\ &V_n = \mathbf{v} \cdot \mathbf{n} \quad \text{on } \Gamma_t \equiv \Gamma_t^+ \cup \Gamma_t^-, \end{aligned} \tag{1}$$

where

$$\mathbb{T} = (-p(\rho) + \mu_1^\pm \nabla \cdot \mathbf{v}) \mathbb{I} + \mu^\pm \mathbb{S}(\mathbf{v})$$

is the stress tensor, $(\mathbb{S}(\mathbf{v}))_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$ is the double strain rate tensor, and \mathbb{I} is the identity matrix; μ^\pm, μ_1^\pm are step functions of dynamic viscosities, equal to μ^+, μ_1^+ in Ω_t^+ and μ^-, μ_1^- in Ω_t^- , respectively; $p(\rho)$ is the fluid pressure given by a known smooth density function; \mathbf{v}_0 and $\rho_0 > 0$ are initial distributions of velocity and density of the liquids, and \mathbf{n} is the outward normal vector to boundary union Γ_t ; $H^\pm(x, t)$ are twice the mean curvatures of the surfaces Γ_t^\pm (moreover, $H^+ < 0$ at points of convexity Γ_t^+ towards Ω_t^-); $\sigma^-, \sigma^+ > 0$ are surface tension coefficients on Γ_t^- and Γ_t^+ , respectively; and V_n is the rate of evolution of Γ_t in the direction \mathbf{n} . We assume that the Cartesian coordinate system $\{x\}$ is introduced in the space \mathbb{R}^3 . The central dot denotes the Cartesian scalar product.

We mean summation over repeated indices from 1 to 3 if they are denoted by Latin letters, and from 1 to 2 if they are Greek. We mark vectors and vector spaces in bold. The notation $\nabla \cdot \mathbb{T}$ denotes the vector with the components $(\nabla \cdot \mathbb{T})_j = \frac{\partial T_{ij}}{\partial x_i}, j = 1, 2, 3$.

The kinematic boundary condition $V_n = \mathbf{v} \cdot \mathbf{n}$ excludes mass transfer across fluid boundaries. It follows from our assumption that the fluid particles do not leave the boundaries Γ_t^\pm during the time.

The evolution problem for two viscous compressible immiscible liquids with an unknown interface belongs to the class of free boundary problems being intensively studied. The theory of these problems for the Navier–Stokes equations has only been in development for about three to four decades, although their setting goes back to the classical works of the 19th century.

The main difficulty of such nonlinear problems is due to the fact that the surfaces of the fluids are unknown. Another obstacle is surface tension. So, most of the authors study the problem without the capillarity. The case where surface tension on a free boundary is taken into account is essentially more difficult to investigate because the capillary forces generate noncoercive boundary conditions. The latter do not allow us to apply the methods developed for the classical problems with fixed boundaries. Interface conditions in system (1) follow from the continuity of a velocity vector field and momentum conservation when passing across the interface between the media. Similar conditions appear on the free boundary. One can find the explanations, for example, in the textbook of Pukhnachov [11] or in [12].

First, local (in time) solvability was proved for a problem similar to (1) in the whole space \mathbb{R}^3 , with a closed interface between the fluids. The result was obtained both in the Sobolev–Slobodetskiĭ and Hölder classes of functions [13]. One can prove similar results for a two-component domain bounded by a free surface if one takes into account the estimates for a model problem in a half-space [14]. In the paper of the Japanese researchers [15], solvability theory was developed for the problem in the anisotropic Sobolev spaces $L_p - L_q$, with different orders of summability with respect to the spatial and time variables. In the same spaces, the authors of [16] studied a compressible two-fluid problem without surface

tension and finding interface between the fluids. They showed local and global solvability of the problem under additional smallness assumptions on initial data in the global case.

The next step of the study will be to prove global unique solvability of the unsteady problem (1) for small data. If we have equilibrium two-layer figures, we can hope to obtain the convergence of the global solution to a stationary one. Thus, the motion of a two-component compressible liquid mass $\cup \Omega_t^\pm$ slow rotating about a fixed axis will tend during the time to the rigid rotation of the corresponding equilibrium figure $\cup \mathcal{F}^\pm$. Another question is the stability of equilibrium figures obtained. A similar study for two viscous incompressible immiscible liquids was published in our recent papers, one of which is [17].

As we have mentioned, we suppose the liquids to be barotropic, which implies that the pressure p is a known increasing function of the density: $p'(\rho) > 0$. Let, in addition, $\rho = \rho(|x'|)$, $x' = (x_1, x_2, 0)$.

Next, we assume that equilibrium figures \mathcal{F}^+ , $\mathcal{F} \equiv \overline{\mathcal{F}^+} \cup \mathcal{F}^-$ are nearly globular domains with the radiuses R_0^\pm ($R_0^+ < R_0^-$), respectively, and the motion of fluids is close to the state of rest, i.e., the velocity is small, and the density differs little from a step function $\rho^\pm > 0$. This picture is schematically presented in Figure 1. We denote the balls $\{x \in \mathbb{R}^3 : |x| \leq R_0^\pm\}$ by $B_{R_0^\pm}$.

We are going to prove the existence of \mathcal{G}^+ and \mathcal{G}^- , the boundaries of the figures \mathcal{F}^+ and \mathcal{F} , respectively. We follow the plan of paper [1].

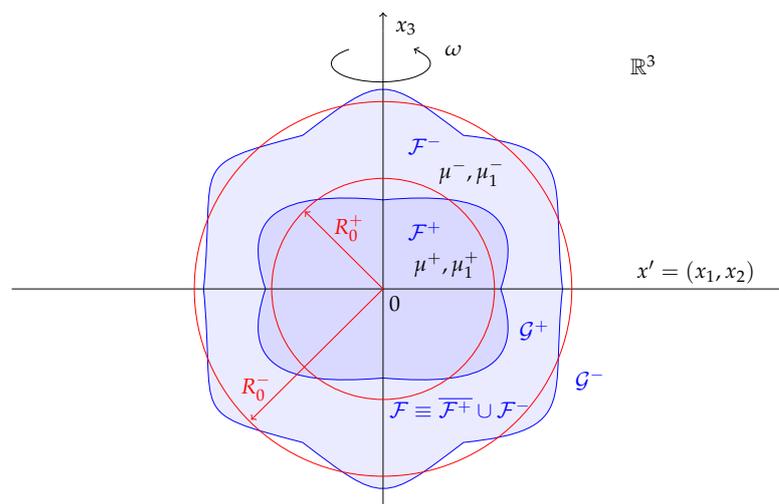


Figure 1. Equilibrium figures for a two-layer compressible fluid.

At rest, the bubble consists of the nested spherical two layers $B_{R_0^+}$ and $B_{R_0^-} \setminus B_{R_0^+}$ with the uniform distributions of densities ρ^\pm and with the piecewise constant pressure:

$$\begin{aligned}
 p(\rho^-) &= \frac{2\sigma^-}{R_0^-} \quad \text{in } B_{R_0^-} \setminus B_{R_0^+}, \\
 p(\rho^+) &= \frac{2\sigma^+}{R_0^+} + \frac{2\sigma^-}{R_0^-} \quad \text{in } B_{R_0^+}.
 \end{aligned}
 \tag{2}$$

The masses of the layers are

$$\rho^+ |B_{R_0^+}| = m^+, \quad \rho^- (|B_{R_0^-}| - |B_{R_0^+}|) = m^-, \tag{3}$$

where $|B_{R_0^\pm}| = \frac{4}{3}\pi R_0^\pm{}^3$.

Steady motion of a two-layer gaseous body \mathcal{F} uniformly rotating about the axis x_3 with a constant angular velocity ω is governed by the homogeneous stationary Navier–Stokes equations

$$\rho(\mathbf{V} \cdot \nabla)\mathbf{V} - \nabla \cdot \mathbb{T} = 0, \quad \nabla \cdot (\rho\mathbf{V}) = 0 \quad \text{in } \cup \mathcal{F}^\pm \tag{4}$$

(here, the density ρ and velocity \mathbf{V} depend only on x) and the boundary conditions

$$\begin{aligned} \mathbb{T}(\mathbf{V}, \mathcal{P})\mathbf{n}|_{\mathcal{G}^-} &= \sigma^- \mathcal{H}^- \mathbf{n} \quad \text{on } \mathcal{G}^-, \\ [\mathbf{V}]|_{\mathcal{G}^+} &= 0, \quad [\mathbb{T}(\mathbf{V}, \mathcal{P})\mathbf{n}]|_{\mathcal{G}^+} = \sigma^+ \mathcal{H}^+ \mathbf{n} \quad \text{on } \mathcal{G}^+, \\ \mathbf{V} \cdot \mathbf{n} &= 0 \quad \text{on } \mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-, \end{aligned} \tag{5}$$

where $\mathcal{H}^-, \mathcal{H}^+$ are twice the mean curvatures of $\mathcal{G}^-, \mathcal{G}^+$, respectively. The last relation follows from the boundary condition $\mathbf{v} \cdot \mathbf{n} = V_n$. The pressure \mathcal{P} depends on ρ and should be given by an increasing function.

It is easily seen that the velocity vector field

$$\mathbf{V}(x) = \omega e_3 \times x \equiv \omega(-x_2, x_1, 0) \tag{6}$$

satisfies (4) together with the pressure function gradient

$$\nabla \mathcal{P}(\rho) = \rho \omega^2 x' \equiv \rho \frac{\omega^2}{2} \nabla |x'|^2, \tag{7}$$

where e_i is the i th basis vector, $|x'|^2 = x_1^2 + x_2^2$.

First, we consider the simple case when equality (7) coincides with the following one

$$\nabla \mathcal{P}(\rho) = \mathcal{P}'(\rho) \nabla \left(\frac{\omega^2}{2} |x'|^2 \right),$$

whence $\mathcal{P}'(\rho) = \rho(x)$ and $\mathcal{P}^\pm(\rho) \equiv \frac{\rho^2(x)}{2} + p^\pm$ in \mathcal{F}^\pm with constants p^\pm , because pressure functions can differ from each other by a constant in different domains. These constants can be found from relations (2):

$$\begin{aligned} p^- &= \frac{2\sigma^-}{R_0^-} - \frac{\rho^{-2}}{2}, \\ p^+ &= \frac{2\sigma^+}{R_0^+} + \frac{2\sigma^-}{R_0^-} - \frac{\rho^{+2}}{2}. \end{aligned} \tag{8}$$

Let S_1 be the unit sphere in \mathbb{R}^3 with the center in zero, $\xi = \frac{x}{|x|} \in S_1$. We suppose \mathcal{G}^\pm to be given by functions $R^\pm(\xi)$ on S_1 . In addition, let $R^\pm(\xi)$ be rotationally symmetric, i.e., they depend only on $|\xi'| = \sqrt{\xi_1^2 + \xi_2^2}$ and ξ_3 , and be even in ξ_3 .

By substituting \mathbf{V} given by (6) and $\mathcal{P} = \mathcal{P}^\pm$ into boundary conditions (5), we obtain the equations for the surface \mathcal{G}^- of domain \mathcal{F} and for the interface \mathcal{G}^+ between the fluids:

$$\begin{aligned} \sigma^- \mathcal{H}^-(x) + \mathcal{P}(\rho) &= 0, \quad x \in \mathcal{G}^-, \\ \sigma^+ \mathcal{H}^+(x) + [\mathcal{P}(\rho)]|_{\mathcal{G}^+} &= 0, \quad x \in \mathcal{G}^+. \end{aligned} \tag{9}$$

The rotationally symmetry implies that $R^\pm(\xi)$ do not depend on $\arctan(\frac{\xi_2}{\xi_1})$. It is clear that $\mathbf{n} = (\frac{\xi_1}{c_0}, \frac{\xi_2}{c_0}, n_3)$, since the first two components of \mathbf{n} are proportional to those of the radius-vector of the circles, being horizontal sections of \mathcal{G}^\pm . Therefore, $\mathbf{V} \cdot \mathbf{n} = 0$ on \mathcal{G}^\pm .

Obviously, the density is given by the formulas

$$\begin{aligned} \rho(x) &= \frac{\omega^2}{2}|x'|^2 + c^+ \quad \text{in } \mathcal{F}^+, \\ \rho(x) &= \frac{\omega^2}{2}|x'|^2 + c^- \quad \text{in } \mathcal{F}^- \end{aligned} \tag{10}$$

with arbitrary positive constants c^+ and c^- . Equations (9) take the form

$$\begin{aligned} \sigma^- \mathcal{H}^-(x) + \mathcal{P}^-\left(\frac{\omega^2}{2}|x'|^2 + c^-\right) &= 0, \quad x \in \mathcal{G}^-, \\ \sigma^+ \mathcal{H}^+(x) + \mathcal{P}^+\left(\frac{\omega^2}{2}|x'|^2 + c^+\right) - \mathcal{P}^-\left(\frac{\omega^2}{2}|x'|^2 + c^-\right) &= 0, \quad x \in \mathcal{G}^+. \end{aligned} \tag{11}$$

One can determine the constants c^\pm by prescribing the masses of fluids to be the same as the masses of the nested spherical liquid layers (3):

$$\int_{\mathcal{F}^\pm} \rho(x) \, dx \equiv \int_{\mathcal{F}^\pm} \left(\frac{\omega^2}{2}|x'|^2 + c^\pm\right) \, dx = m^\pm. \tag{12}$$

We consider the angular momentum to be one more parameter of the problem. It is given:

$$\beta \equiv \int_{\mathcal{F}^+ \cup \mathcal{F}^-} \rho(x) \mathbf{v} \cdot \boldsymbol{\eta}_3 \, dx = \omega \int_{\mathcal{F}^\pm} \left(\frac{\omega^2}{2}|x'|^2 + c^\pm\right) |x'|^2 \, dx, \tag{13}$$

where $\boldsymbol{\eta}_i = \mathbf{e}_i \times \mathbf{x}$ is the i th rigid rotation vector, $i = 1, 2, 3$. Then, the angular velocity ω is a function of β .

We denote by C^s , $s > 0$, $s \notin \mathbb{N}$, the Hölder space of functions f on the sphere S_1 with the norm

$$|f|_{C^s(S_1)} \equiv \max_{k=\{1, \dots, N\}} \left(\sum_{|j| < s} \sup_{\xi \in \zeta_k} |\mathcal{D}^j f(\xi)| + \sum_{|j|=s} \sup_{\xi, \bar{\xi} \in \zeta_k} |\xi - \bar{\xi}|^{-(s-[s])} |\mathcal{D}^j f(\xi) - \mathcal{D}^j f(\bar{\xi})| \right),$$

where $\mathcal{D}^j f$ is the $|j|$ th derivative of f , calculated in local coordinates on the subdomain $\zeta_k \subset S_1$, $\cup_{k=1}^N \zeta_k = S_1$. Under $\tilde{C}^s(S_1)$, we mean the subspace of $C^s(S_1)$, consisting of rotationally symmetric functions that are even with respect to ξ_3 .

Theorem 1. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and let the data of problem (4), (5) be such that condition (26) holds. Then, for an arbitrary β satisfying the estimate*

$$|\beta| < \varepsilon \tag{14}$$

with small enough ε , there exists a unique solution $(R^\pm, \omega, c^\pm) \in \tilde{C}^{2+\alpha}(S_1) \times \tilde{C}^{2+\alpha}(S_1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to system (11)–(13). It obeys the inequality

$$\sum_{\pm} \{ |R^\pm - R_0^\pm|_{C^{2+\alpha}(S_1)} + |c^\pm - \rho^\pm| \} + |\omega| < c|\beta|. \tag{15}$$

2. Proof of Theorem 1

Proof. In order to linearize system (11), we apply the formula for the first variation of a functional $\delta_0 R[r] = \frac{d}{ds} R[sr] \Big|_{s=0}$. According to [10,18], the first variation of twice the mean curvature of \mathcal{G}^\pm with respect to the double curvature of the sphere $S_{R_0^\pm}$ is

$$\delta_0(\mathcal{H}^\pm[r(\xi)] + \frac{2}{R_0^\pm}) = \frac{1}{R_0^{\pm 2}} (\Delta_{S_1} r^\pm + 2r^\pm),$$

where Δ_{S_1} is the Laplace–Beltrami operator on S_1 and $r^\pm \equiv R^\pm - R_0^\pm$; $\mathcal{G}^\pm = \{x = R_0^\pm \xi + r^\pm(\xi)N, \xi \in S_1\}$, and N is the outward normal to S_1 . This yields

$$\begin{aligned} \frac{\sigma^-}{R_0^{-2}}(\Delta_{S_1} r^- + 2r^-) &= k^- + f^-, \quad \xi \in S_1, \\ \frac{\sigma^+}{R_0^{+2}}(\Delta_{S_1} r^+ + 2r^+) &= k^+ + f^+, \quad \xi \in S_1, \end{aligned} \tag{16}$$

where $k^- = \frac{2\sigma^-}{R_0^-} - \mathcal{P}^-(c^-) \equiv \mathcal{P}^-(\rho^-) - \mathcal{P}^-(c^-)$, $k^+ = \frac{2\sigma^+}{R_0^+} - \mathcal{P}^+(c^+) + \mathcal{P}^-(c^-) \equiv \mathcal{P}^+(\rho^+) - \mathcal{P}^+(c^+) + \mathcal{P}^-(c^-) - \mathcal{P}^-(\rho^-)$ in view of (2);

$$\begin{aligned} f^- &= \sigma^- \left(\delta_0(\mathcal{H}^-(x) + \frac{2}{R_0^-}) - (\mathcal{H}^-(x) + \frac{2}{R_0^-}) \right) - \mathcal{P}^-\left(\frac{\omega^2}{2}|x'|^2 + c^-\right) \\ &\quad + \mathcal{P}^-(c^-), \\ f^+ &= \sigma^+ \left(\delta_0(\mathcal{H}^+(x) + \frac{2}{R_0^+}) - (\mathcal{H}^+(x) + \frac{2}{R_0^+}) \right) - \mathcal{P}^+\left(\frac{\omega^2}{2}|x'|^2 + c^+\right) \\ &\quad + \mathcal{P}^-\left(\frac{\omega^2}{2}|x'|^2 + c^-\right) + \mathcal{P}^+(c^+) - \mathcal{P}^-(c^-). \end{aligned} \tag{17}$$

We integrate Equation (16) over S_1 by parts, then we obtain

$$\frac{2\sigma^\pm}{R_0^{\pm 2}} \int_{S_1} r^\pm \, d\xi = \int_{S_1} (k^\pm + f^\pm) \, d\xi. \tag{18}$$

The integrals $\int_{S_1} r^\pm \, d\xi$ can be expressed in terms of the differences of the volumes $|\mathcal{F}| - |B_{R_0^-}|$ and $|\mathcal{F}^+| - |B_{R_0^+}|$. For example,

$$|\mathcal{F}| - |B_{R_0^-}| = \frac{1}{3} \int_{S_1} (R^{-3} - R_0^{-3}) \, d\xi = R_0^{-2} \int_{S_1} r^- \, d\xi + R_0^- \int_{S_1} r^{-2} \, d\xi + \frac{1}{3} \int_{S_1} r^{-3} \, d\xi,$$

and hence

$$\begin{aligned} \int_{S_1} r^+ \, d\xi &= \frac{1}{R_0^{+2}} (|\mathcal{F}^+| - |B_{R_0^+}|) + Q[r^+], \\ \int_{S_1} r^- \, d\xi &= \frac{1}{R_0^{-2}} (|\mathcal{F}| - |B_{R_0^-}|) + Q[r^-], \end{aligned} \tag{19}$$

where $Q[r^\pm] \equiv -\frac{1}{R_0^\pm} \int_{S_1} r^{\pm 2} \, d\xi - \frac{1}{3R_0^{\pm 2}} \int_{S_1} r^{\pm 3} \, d\xi$.

We rewrite (12) as follows:

$$\begin{aligned} m^+ &= \int_{S_1} d\xi \int_0^{R^+(\xi)} \frac{\omega^2}{2} |\xi'|^2 s^4 \, ds + c^+ |\mathcal{F}^+| = \frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi) |\xi'|^2 \, d\xi + c^+ |\mathcal{F}^+|, \\ m^- &= \int_{S_1} d\xi \int_{R^+(\xi)}^{R^-(\xi)} \frac{\omega^2}{2} |\xi'|^2 s^4 \, ds + c^- |\mathcal{F}^-| = \frac{\omega^2}{10} \int_{S_1} (R^{-5} - R^{+5}) |\xi'|^2 \, d\xi + c^- |\mathcal{F}^-|, \end{aligned}$$

where $|\xi'|^2 = \xi_1^2 + \xi_2^2$. On the other hand, $m^+ = \rho^+ |B_{R_0^+}|$, $m^- = \rho^- (|B_{R_0^-}| - |B_{R_0^+}|)$. That is why

$$\begin{aligned} 0 &= \frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi) |\xi'|^2 \, d\xi + \rho^+ (|\mathcal{F}^+| - |B_{R_0^+}|) + |\mathcal{F}^+| (c^+ - \rho^+), \\ 0 &= \frac{\omega^2}{10} \int_{S_1} (R^{-5} - R^{+5}) |\xi'|^2 \, d\xi + \rho^- (|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|) + |\mathcal{F}^-| (c^- - \rho^-). \end{aligned} \tag{20}$$

We express the differences $|\mathcal{F}^\pm| - |B_{R_0^\pm}|$ from (20) and substitute them into (19). Then equalities (18) imply that

$$\frac{2\sigma^+}{R_0^{+2}} \left\{ \frac{-1}{\rho^+ R_0^{+2}} \left(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi) |\xi'|^2 d\xi + |\mathcal{F}^+|(c^+ - \rho^+) \right) + Q[r^+] \right\} = \int_{S_1} (k^+ + f^+) d\xi \quad (21)$$

and

$$\frac{2\sigma^-}{R_0^{-2}} \left(\frac{1}{R_0^{-2}} (|\mathcal{F}^+| + |\mathcal{F}^-| - |B_{R_0^-}| \pm |B_{R_0^+}|) + Q[r^-] \right) = \int_{S_1} (k^- + f^-) d\xi.$$

And finally, we have

$$\begin{aligned} \frac{2\sigma^-}{R_0^{-2}} \left\{ \frac{-1}{\rho^- R_0^{-2}} \left\{ \frac{\omega^2}{10} \int_{S_1} (R^{-5} - R^{+5}) |\xi'|^2 d\xi + |\mathcal{F}^-|(c^- - \rho^-) \right. \right. \\ \left. \left. + \frac{\rho^-}{\rho^+} \left(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi) |\xi'|^2 d\xi + |\mathcal{F}^+|(c^+ - \rho^+) \right) \right\} \right. \\ \left. + Q[r^-] \right\} = \int_{S_1} (k^- + f^-) d\xi. \end{aligned} \quad (22)$$

In order to prove the solvability of system (13), (16), (21), and (22), we use implicit function theorem. We represent this system as a nonlinear vector equation:

$$\Phi(\varphi) = 0, \quad (23)$$

where $\Phi(\varphi) = (\Phi_1^\pm, \Phi_2, \Phi_3^\pm)$, $\varphi \equiv (r^\pm, \omega, \lambda^\pm)$, $\lambda^\pm \equiv c^\pm - \rho^\pm$,

$$\begin{aligned} \Phi_1^\pm(\varphi) &= \frac{\sigma^\pm}{R_0^{\pm 2}} (\Delta_{S_1} r^\pm + 2r^\pm) - k^\pm - f^\pm, \\ \Phi_2(\varphi) &= \omega \int_{S_1} |\xi'|^2 d\xi \int_0^{R_0^+ + r^+} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + c^+ \right) s^4 ds \\ &\quad + \omega \int_{S_1} |\xi'|^2 d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + c^- \right) s^4 ds - \beta, \\ \Phi_3^+(\varphi) &= \frac{2\sigma^+}{R_0^{+2}} \left\{ \frac{1}{\rho^+ R_0^{+2}} \left(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi) |\xi'|^2 d\xi + |\mathcal{F}^+|\lambda^+ \right) - Q[r^+] \right\} \\ &\quad + \int_{S_1} (k^+ + f^+) d\xi, \\ \Phi_3^-(\varphi) &= \frac{2\sigma^-}{R_0^{-2}} \left\{ \frac{1}{\rho^- R_0^{-2}} \left\{ \int_{S_1} d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \frac{\omega^2}{2} |\xi'|^2 s^4 ds + |\mathcal{F}^-|\lambda^- \right. \right. \\ &\quad \left. \left. + \frac{\rho^-}{\rho^+} \left(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\xi) |\xi'|^2 d\xi + |\mathcal{F}^+|\lambda^+ \right) \right\} \right. \\ &\quad \left. - Q[r^-] \right\} + \int_{S_1} (k^- + f^-) d\xi. \end{aligned}$$

We linearize (23) at zero and show that the derivative of Φ' at this point is not equal to zero. Thus,

$$\Phi'(0)\varphi + \psi(\varphi) = 0, \quad (24)$$

where $\Phi'(0) = (\Phi_1^{\pm'}, \Phi_2', \Phi_3^{\pm'})(0)$ is the Fréchet derivative of Φ , that is,

$$\begin{aligned} \Phi_1^{+'}(0)\varphi &= \frac{\sigma^+}{R_0^{+2}} (\Delta_{S_1} r^+ + 2r^+) + \rho^+ \lambda^+ - \rho^- \lambda^-, \\ \Phi_1^{-'}(0)\varphi &= \frac{\sigma^-}{R_0^{-2}} (\Delta_{S_1} r^- + 2r^-) + \rho^- \lambda^-, \\ \Phi_2'(0)\varphi &= \omega \rho^+ \int_{S_1} |\zeta'|^2 d\zeta \int_0^{R_0^+} s^4 ds + \omega \rho^- \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^+}^{R_0^-} s^4 ds \\ &= \frac{8\pi}{15} \omega \left\{ \rho^+ R_0^{+5} + \rho^- (R_0^{-5} - R_0^{+5}) \right\}, \\ \Phi_3^{+'}(0)\varphi &= \frac{2\sigma^+}{R_0^{+2}} \frac{|B_{R_0^+}| \lambda^+}{\rho^+ R_0^{+2}} - 4\pi \rho^+ \lambda^+ + 4\pi \rho^- \lambda^- \\ &= 4\pi \left\{ \left(\frac{\mathcal{P}^+(\rho^+) - \mathcal{P}^-(\rho^-)}{3\rho^+} - \rho^+ \right) \lambda^+ + \rho^- \lambda^- \right\}, \\ \Phi_3^{-'}(0)\varphi &= \frac{2\sigma^-}{R_0^{-2}} \frac{1}{\rho^- R_0^{-2}} \left\{ (|B_{R_0^-}| - |B_{R_0^+}|) \lambda^- + \frac{\rho^-}{\rho^+} |B_{R_0^+}| \lambda^+ \right\} - 4\pi \rho^- \lambda^- \\ &= 4\pi \left\{ \frac{\mathcal{P}^-(\rho^-) R_0^{+3}}{3\rho^+ R_0^{-3}} \lambda^+ + \left(\frac{R_0^{-3} - R_0^{+3}}{3\rho^- R_0^{-3}} \mathcal{P}^-(\rho^-) - \rho^- \right) \lambda^- \right\}, \end{aligned}$$

and $\psi(\varphi) = \Phi(\varphi) - \Phi'(0)\varphi \equiv (\psi_1^{\pm}, \psi_2, \psi_3^{\pm})$ with

$$\begin{aligned} \psi_1^+(\varphi) &= \mathcal{P}^+(c^+) - \mathcal{P}^+(\rho^+) - \mathcal{P}^-(c^-) + \mathcal{P}^-(\rho^-) - \rho^+ \lambda^+ + \rho^- \lambda^- - f^+, \\ \psi_1^-(\varphi) &= \mathcal{P}^-(c^-) - \mathcal{P}^-(\rho^-) - \rho^- \lambda^- - f^-, \\ \psi_2(\varphi) &= \omega \int_{S_1} |\zeta'|^2 d\zeta \int_0^{R_0^+} \left(\frac{\omega^2}{2} |\zeta'|^2 s^2 + \lambda^+ \right) s^4 ds \\ &\quad + \omega \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^+}^{R_0^+ + r^+} \left(\frac{\omega^2}{2} |\zeta'|^2 s^2 + c^+ \right) s^4 ds \\ &\quad + \omega \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^+}^{R_0^-} \left(\frac{\omega^2}{2} |\zeta'|^2 s^2 + \lambda^- \right) s^4 ds \\ &\quad + \omega \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^-}^{R_0^- + r^-} \left(\frac{\omega^2}{2} |\zeta'|^2 s^2 + c^- \right) s^4 ds - \beta, \\ \psi_3^+(\varphi) &= \frac{2\sigma^+}{R_0^{+2}} \left\{ \frac{1}{\rho^+ R_0^{+2}} \left(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\zeta) |\zeta'|^2 d\zeta + (|\mathcal{F}^+| - |B_{R_0^+}|) \lambda^+ \right) - Q[r^+] \right\} \\ &\quad + \int_{S_1} (k^+ - \rho^+ \lambda^+ + \rho^- \lambda^- + f^+) d\zeta, \\ \psi_3^-(\varphi) &= \frac{2\sigma^-}{R_0^{-2}} \left\{ \frac{1}{\rho^- R_0^{-2}} \left\{ \frac{\omega^2}{10} \int_{S_1} (R^{-5} - R^{+5}) |\zeta'|^2 d\zeta + \frac{\rho^-}{\rho^+} \left(\frac{\omega^2}{10} \int_{S_1} R^{+5}(\zeta) |\zeta'|^2 d\zeta \right. \right. \right. \\ &\quad \left. \left. \left. + (|\mathcal{F}^+| - |B_{R_0^+}|) \lambda^+ \right) + (|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|) \lambda^- \right\} - Q[r^-] \right\} \\ &\quad + \int_{S_1} (k^- - \rho^- \lambda^- + f^-) d\zeta. \end{aligned}$$

Let $\mathcal{X} = \tilde{C}^{2+\alpha} \times \tilde{C}^{2+\alpha} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\mathcal{Y} = \tilde{C}^\alpha \times \tilde{C}^\alpha \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Using fixed point theorem, we will show that Equation (23) is solvable in \mathcal{X} .

In [19] (see the corollary to Theorem 2.1), it was proved that for any $f \in \tilde{C}^\alpha(S_1)$ the equation

$$\Delta_{S_1} r + 2r = f \tag{25}$$

has the unique solution $r \in \tilde{C}^{2+\alpha}(S_1)$ and

$$|r|_{C^{2+\alpha}(S_1)} \leq c_1 |f|_{C^\alpha(S_1)}.$$

Moreover, due to the assumptions of the theorem, the determinant of the matrix in the left-hand sides of two equations $\Phi_3^{\pm'}(0)\varphi = -\psi_3^\pm(\varphi)$ is not equal to zero:

$$\left(\frac{2\sigma^+}{3\rho^+R_0^+} - \rho^+\right) \left(\frac{2\sigma^-(R_0^{-3} - R_0^{+3})}{3\rho^-R_0^{-4}} - \rho^-\right) - \frac{2\sigma^-\rho^-R_0^{+3}}{3\rho^+R_0^{-4}} \neq 0. \tag{26}$$

(Here we have taken into account relations (8)). Hence, this matrix is invertible. Thus, the whole operator $\Phi'(0) : \mathcal{X} \rightarrow \mathcal{Y}$ is also invertible and

$$\|\varphi\|_{\mathcal{X}} \leq c_2 \|\Phi'(0)\varphi\|_{\mathcal{Y}}, \tag{27}$$

where $\|\varphi\|_{\mathcal{X}} = |r^+|_{C^{2+\alpha}(S_1)} + |r^-|_{C^{2+\alpha}(S_1)} + |\omega| + |\lambda^+| + |\lambda^-|$ and $\|\Phi'(0)\varphi\|_{\mathcal{Y}} = |\Phi_1^{\pm'}(0)\varphi|_{C^\alpha(S_1)} + |\Phi_1^{-'}(0)\varphi|_{C^\alpha(S_1)} + |\Phi_2'(0)\varphi|_{\mathbb{R}} + |\Phi_3^{+'}(0)\varphi|_{\mathbb{R}} + |\Phi_3^{-'}(0)\varphi|_{\mathbb{R}}$.

Therefore, Equation (24) can be written in the form:

$$\varphi + \mathcal{A}\psi(\varphi) = 0, \quad \mathcal{A} = \Phi'(0)^{-1}. \tag{28}$$

Let us now estimate the non-linear operator ψ . The term f^- given by (17) can be written as

$$f^- = -\sigma^- \int_0^1 (1-s) \frac{d^2}{ds^2} \mathcal{H}^-[sr] ds - \mathcal{P}'(\bar{c}^-) \frac{\omega^2}{2} R^{+2}(\bar{\xi}) |\bar{\xi}'|^2, \tag{29}$$

where $\mathcal{H}^-[sr]$ is twice the mean curvature of the surface \mathcal{G}_s^- ; $\mathcal{G}_s^\pm = \{x = R_0^\pm \xi + sr^\pm(\xi)N, \xi \in S_1\}$, $N \equiv \frac{\xi}{|\xi|}$; and \bar{c}^- is some point of the interval $[c^-, \frac{\omega^2}{2} R^{+2}(\bar{\xi}) |\bar{\xi}'|^2 + c^-]$. One can write a similar formula for f^+ . The terms f^\pm have the second order of smallness with respect to φ . After simple calculations for two values of the arguments $\varphi = (r^\pm, \omega, \lambda^\pm)$ and $\varphi' = (r^{\pm'}, \omega', \lambda^{\pm'}) \in \mathcal{X}$ such that

$$\|\varphi\|_{\mathcal{X}}, \|\varphi'\|_{\mathcal{X}} \leq \eta, \quad \eta \leq \min\{1, R_0^+/2, (R_0^- - R_0^+)/2\}, \tag{30}$$

it can be shown that the estimates

$$\begin{aligned} \|\psi(\varphi)\|_{\mathcal{Y}} &\leq c \|\varphi\|_{\mathcal{X}}^{1+\alpha} + |\beta| \leq c_3 \eta \|\varphi\|_{\mathcal{X}}^\alpha + |\beta|, \\ \|\psi(\varphi) - \psi(\varphi')\|_{\mathcal{Y}} &\leq c_4 \eta^\alpha \|\varphi - \varphi'\|_{\mathcal{X}} \end{aligned} \tag{31}$$

hold. This implies that $\mathcal{A}\psi(\varphi)$ maps the ball $\{\varphi \in \mathcal{X} : \|\varphi\|_{\mathcal{X}} \leq \eta\}$ into itself and it is a contraction operator if

$$c_2(c_3\eta^{1+\alpha} + |\beta|) \leq \eta, \quad c_2c_4\eta^\alpha < 1.$$

These inequalities are satisfied for η such that

$$2c_2|\beta| \leq \eta \leq \min\{(2c_2c_3)^{-1/\alpha}, (c_2c_4)^{-1/\alpha}\}. \tag{32}$$

So if

$$|\beta| \leq \frac{1}{2c_2} \min\{(2c_2c_3)^{-1/\alpha}, (c_2c_4)^{-1/\alpha}\}, \tag{33}$$

then, by fixed point theorem, Equation (28) has a unique solution in the ball $\{\|\varphi\|_{\mathcal{X}} \leq \eta\}$ with η satisfying (32). Thus, Equation (23) is also uniquely solvable for β , obeying inequality (33), which gives the estimate of ε in condition (14).

We now prove estimate (15). Since

$$\|\varphi\|_X \leq \|A\| \|\psi(\varphi)\|_Y \leq c_2(c_3\eta^\alpha \|\varphi\|_X + |\beta|), \tag{34}$$

then, for $c_2c_3\eta^\alpha \leq \frac{1}{2}$,

$$\|\varphi\|_X \leq 2c_2|\beta|,$$

which coincides with inequality (15). \square

Thus, we have shown that, for certain data of the problem, there are equilibrium figures for a two-layer compressible fluid, the pressure function being $\mathcal{P}^\pm = \frac{\rho^2(x)}{2} + p^\pm$ and the density ρ being determined by Formula (10).

3. General Case

Now, we study the general case of pressure function. We assume only that $\mathcal{P}'(\rho)$ is positive and may have a jump across \mathcal{G}^+ .

The velocity vector field remains the same as above:

$$\mathcal{V}(x) = \omega(-x_2, x_1, 0).$$

It satisfies (4), together with the pressure function gradient $\nabla\mathcal{P}(\rho) = \rho\frac{\omega^2}{2}\nabla|x'|^2$.

We introduce $\mathcal{Q}(\rho)$, such that

$$\nabla\mathcal{Q}(\rho) \equiv \frac{\mathcal{P}'(\rho)\nabla\rho}{\rho} = \nabla\left(\frac{\omega^2}{2}|x'|^2\right). \tag{35}$$

The function $\mathcal{Q}(\rho) = \int_{\rho_1}^{\rho} \frac{\mathcal{P}'(s)}{s} ds$, $\rho_1 \geq 0$. Since $\mathcal{Q}'(\rho) = \frac{\mathcal{P}'(\rho)}{\rho} > 0$, there is an inverse function \mathcal{Q}^{-1} . And (35) implies

$$\rho = \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|x'|^2 + C^\pm\right) \text{ in } \mathcal{F}^\pm \tag{36}$$

with arbitrary constants C^\pm . Substituting ρ into Equation (9), we have

$$\begin{aligned} \sigma^- \mathcal{H}^-(x) + \mathcal{P}^-\left(\mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|x'|^2 + C^-\right)\right) &= 0, \quad x \in \mathcal{G}^-, \\ \sigma^+ \mathcal{H}^+(x) + \mathcal{P}^+\left(\mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|x'|^2 + C^+\right)\right) - \mathcal{P}^-\left(\mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|x'|^2 + C^-\right)\right) &= 0, \quad x \in \mathcal{G}^+. \end{aligned} \tag{37}$$

We prescribe the masses of fluids and apply (36):

$$m^\pm = \int_{\mathcal{F}^\pm} \rho(x) dx \equiv \int_{\mathcal{F}^\pm} \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|x'|^2 + C^\pm\right) dx. \tag{38}$$

Then, one has the equations for determining the constants C^\pm .

Similarly, a given angular momentum β defines the angular velocity ω :

$$\beta \equiv \int_{\mathcal{F}^+ \cup \mathcal{F}^-} \rho(x) \mathcal{V} \cdot \eta_3 dx = \omega \int_{\mathcal{F}^\pm} \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|x'|^2 + C^\pm\right) |x'|^2 dx. \tag{39}$$

Let us state the main theorem.

Theorem 2. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, and let $\mathcal{P}^\pm(\rho) \in C^\alpha(\mathbb{R}_+)$ be positive increasing functions, such that equalities (2) are satisfied for it. Here, $\mathbb{R}_+ \equiv \{x \in \mathbb{R} | x > 0\}$. We assume also that the data of problem (4), (5) are subjected to condition (50). Then, for an arbitrary β satisfying the estimate*

$$|\beta| < \varepsilon \tag{40}$$

with ε small enough, there exists a unique solution $(R^\pm, \omega, C^\pm) \in \tilde{C}^{2+\alpha}(S_1) \times \tilde{C}^{2+\alpha}(S_1) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ to system (37)–(39), and the inequality

$$\sum_{\pm} \{ |R^\pm - R_0^\pm|_{C^{2+\alpha}(S_1)} + |\mathcal{Q}^{-1}(C^\pm) - \rho^\pm| \} + |\omega| < c|\beta| \tag{41}$$

holds.

Proof. After linearisation, system (37) takes the form of (16) with

$$\begin{aligned} k^- &= \frac{2\sigma^-}{R_0^-} - \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) \equiv \mathcal{P}^-(\rho^-) - \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)), \\ k^+ &= \frac{2\sigma^+}{R_0^+} - \mathcal{P}^+(\mathcal{Q}^{-1}(C^+)) + \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) \\ &\equiv \mathcal{P}^+(\rho^+) - \mathcal{P}^+(\mathcal{Q}^{-1}(C^+)) + \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) - \mathcal{P}^-(\rho^-) \end{aligned} \tag{42}$$

due to (2) and

$$\begin{aligned} f^- &= \sigma^- \left(\delta_0(\mathcal{H}^-(x) + \frac{2}{R_0^-}) - (\mathcal{H}^-(x) + \frac{2}{R_0^-}) \right) - \mathcal{P}^-(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|x'|^2 + C^-)) \\ &\quad + \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)), \\ f^+ &= \sigma^+ \left(\delta_0(\mathcal{H}^+(x) + \frac{2}{R_0^+}) - (\mathcal{H}^+(x) + \frac{2}{R_0^+}) \right) - \mathcal{P}^+(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|x'|^2 + C^+)) \\ &\quad + \mathcal{P}^-(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|x'|^2 + C^-)) + \mathcal{P}^+(\mathcal{Q}^{-1}(C^+)) - \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)). \end{aligned} \tag{43}$$

Following Section 2, we obtain (18) by integrating Equation (16) by parts but now with new k^\pm (42) and f^\pm (43). We substitute (19) in equalities (18):

$$\begin{aligned} \frac{2\sigma^+}{R_0^{+2}} \left(\frac{1}{R_0^{+2}} (|\mathcal{F}^+| - |B_{R_0^+}|) + \tilde{Q}[r^+] \right) &= \int_{S_1} (k^+ + f^+) \, d\xi, \\ \frac{2\sigma^-}{R_0^{-2}} \left(\frac{1}{R_0^{-2}} (|\mathcal{F}^+| + |\mathcal{F}^-| - |B_{R_0^-}| \pm |B_{R_0^+}|) + \tilde{Q}[r^-] \right) &= \int_{S_1} (k^- + f^-) \, d\xi, \end{aligned} \tag{44}$$

where $\tilde{Q}[r^\pm] \equiv -\frac{1}{R_0^\pm} \int_{S_1} r^{\pm 2} \, d\xi - \frac{1}{3R_0^{\pm 2}} \int_{S_1} r^{\pm 3} \, d\xi$.

We can write (38) as follows

$$\begin{aligned} m^+ &= \int_{S_1} d\xi \int_0^{R^+(\xi)} \left(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+) - \mathcal{Q}^{-1}(C^+) \right) s^2 \, ds + \mathcal{Q}^{-1}(C^+) |\mathcal{F}^+|, \\ m^- &= \int_{S_1} d\xi \int_{R^+(\xi)}^{R^-(\xi)} \left(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^-) - \mathcal{Q}^{-1}(C^-) \right) s^2 \, ds + \mathcal{Q}^{-1}(C^-) |\mathcal{F}^-|. \end{aligned}$$

In view of (3), we have

$$\begin{aligned} 0 &= \int_{S_1} d\xi \int_0^{R^+(\xi)} \left(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^+) - \mathcal{Q}^{-1}(C^+) \right) s^2 \, ds \\ &\quad + \rho^+ (|\mathcal{F}^+| - |B_{R_0^+}|) + |\mathcal{F}^+| (\mathcal{Q}^{-1}(C^+) - \rho^+), \\ 0 &= \int_{S_1} d\xi \int_{R^+(\xi)}^{R^-(\xi)} \left(\mathcal{Q}^{-1}(\frac{\omega^2}{2}|\xi'|^2 s^2 + C^-) - \mathcal{Q}^{-1}(C^-) \right) s^2 \, ds \\ &\quad + \rho^- (|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|) + |\mathcal{F}^-| (\mathcal{Q}^{-1}(C^-) - \rho^-). \end{aligned} \tag{45}$$

By expressing the differences $|\mathcal{F}^+| - |B_{R_0^+}|$ and $|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|$ from equalities (45), and substituting them in (44), we arrive at

$$\begin{aligned} \frac{2\sigma^+}{R_0^{+2}} \left\{ \frac{-1}{\rho^+ R_0^{+2}} \left(\int_{S_1} d\xi \int_0^{R^+(\xi)} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^+ \right) - \mathcal{Q}^{-1}(C^+) \right) s^2 ds \right. \right. \\ \left. \left. + |\mathcal{F}^+| (\mathcal{Q}^{-1}(C^+) - \rho^+) \right) + \tilde{Q}[r^+] \right\} = \int_{S_1} (k^+ + f^+) d\xi \end{aligned} \tag{46}$$

and

$$\begin{aligned} \frac{2\sigma^-}{R_0^{-2}} \left\{ \frac{-1}{\rho^- R_0^{-2}} \left\{ \int_{S_1} d\xi \int_{R^+(\xi)}^{R^-(\xi)} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^- \right) - \mathcal{Q}^{-1}(C^-) \right) s^2 ds \right. \right. \\ \left. \left. + |\mathcal{F}^-| (\mathcal{Q}^{-1}(C^-) - \rho^-) \right. \right. \\ \left. \left. + \frac{\rho^-}{\rho^+} \left(\int_{S_1} d\xi \int_0^{R^+(\xi)} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^+ \right) - \mathcal{Q}^{-1}(C^+) \right) s^2 ds \right. \right. \right. \\ \left. \left. \left. + |\mathcal{F}^+| (\mathcal{Q}^{-1}(C^+) - \rho^+) \right) \right\} + \tilde{Q}[r^-] \right\} = \int_{S_1} (k^- + f^-) d\xi. \end{aligned} \tag{47}$$

Next, we represent system, (16), (39), (46), and (47) in the form of vector Equation (23) with $\Phi(\varphi) = (\Phi_1^\pm, \Phi_2, \Phi_3^\pm)$, $\varphi \equiv (r^\pm, \omega, \lambda^\pm)$, $\lambda^\pm \equiv \mathcal{Q}^{-1}(C^\pm) - \rho^\pm$,

$$\begin{aligned} \Phi_1^\pm(\varphi) &= \frac{\sigma^\pm}{R_0^{\pm 2}} (\Delta_{S_1} r^\pm + 2r^\pm) - k^\pm - f^\pm, \\ \Phi_2(\varphi) &= \omega \int_{S_1} |\xi'|^2 d\xi \int_0^{R_0^+ + r^+} \mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^+ \right) s^4 ds \\ &\quad + \omega \int_{S_1} |\xi'|^2 d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^- \right) s^4 ds - \beta, \\ \Phi_3^+(\varphi) &= \frac{2\sigma^+}{R_0^{+2}} \left\{ \frac{1}{\rho^+ R_0^{+2}} \left(\int_{S_1} d\xi \int_0^{R_0^+ + r^+} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^+ \right) - \mathcal{Q}^{-1}(C^+) \right) s^2 ds \right. \right. \\ &\quad \left. \left. + |\mathcal{F}^+| \lambda^+ \right) - \tilde{Q}[r^+] \right\} + \int_{S_1} (k^+ + f^+) d\xi, \\ \Phi_3^-(\varphi) &= \frac{2\sigma^-}{R_0^{-2}} \left\{ \frac{1}{\rho^- R_0^{-2}} \left\{ \int_{S_1} d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^- \right) - \mathcal{Q}^{-1}(C^-) \right) s^2 ds + |\mathcal{F}^-| \lambda^- \right. \right. \right. \\ &\quad \left. \left. + \frac{\rho^-}{\rho^+} \left(\int_{S_1} d\xi \int_0^{R_0^+ + r^+} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^+ \right) - \mathcal{Q}^{-1}(C^+) \right) s^2 ds \right. \right. \right. \\ &\quad \left. \left. \left. + |\mathcal{F}^+| \lambda^+ \right) \right\} - \tilde{Q}[r^-] \right\} + \int_{S_1} (k^- + f^-) d\xi. \end{aligned} \tag{48}$$

Then, we apply again implicit function theorem to (23). To this end, we calculate the Fréchet derivative of Φ at zero and linearize (23) at this point. As a result, one has (24) with $\Phi'(0) = (\Phi_1^{\pm'}, \Phi_2', \Phi_3^{\pm'})(0)$,

$$\begin{aligned} \Phi_1^{+'}(0)\varphi &= \frac{\sigma^+}{R_0^{+2}}(\Delta_{S_1}r^+ + 2r^+) + \mathcal{P}^{+'}(\rho^+)\lambda^+ - \mathcal{P}^{-'}(\rho^-)\lambda^-, \\ \Phi_1^{-'}(0)\varphi &= \frac{\sigma^-}{R_0^{-2}}(\Delta_{S_1}r^- + 2r^-) + \mathcal{P}^{-'}(\rho^-)\lambda^-, \\ \Phi_2'(0)\varphi &= \omega\rho^+ \int_{S_1} |\zeta'|^2 d\zeta \int_0^{R_0^+} s^4 ds + \omega\rho^- \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^+}^{R_0^-} s^4 ds \\ &= \frac{8\pi}{15}\omega\left\{\rho^+R_0^{+5} + \rho^-(R_0^{-5} - R_0^{+5})\right\}, \\ \Phi_3^{+'}(0)\varphi &= \frac{2\sigma^+}{R_0^{+2}}\frac{|B_{R_0^+}|\lambda^+}{\rho^+R_0^{+2}} - 4\pi\mathcal{P}^{+'}(\rho^+)\lambda^+ + 4\pi\mathcal{P}^{-'}(\rho^-)\lambda^- \\ &= 4\pi\left\{\left(\frac{2\sigma^+}{3\rho^+R_0^+} - \mathcal{P}^{+'}(\rho^+)\right)\lambda^+ + \mathcal{P}^{-'}(\rho^-)\lambda^-\right\}, \\ \Phi_3^{-'}(0)\varphi &= \frac{2\sigma^-}{R_0^{-2}}\frac{1}{\rho^-R_0^{-2}}\left\{(|B_{R_0^-}| - |B_{R_0^+}|)\lambda^- + \frac{\rho^-}{\rho^+}|B_{R_0^+}|\lambda^+\right\} - 4\pi\mathcal{P}^{-'}(\rho^-)\lambda^- \\ &= 4\pi\left\{\frac{2\sigma^-R_0^{+3}}{3\rho^+R_0^{-4}}\lambda^+ + \left(\frac{2\sigma^-(R_0^{-3} - R_0^{+3})}{3\rho^-R_0^{-4}} - \mathcal{P}^{-'}(\rho^-)\right)\lambda^-\right\}, \end{aligned} \tag{49}$$

and $\psi(\varphi) = \Phi(\varphi) - \Phi'(0)\varphi \equiv (\psi_1^{\pm}(\varphi), \psi_2(\varphi), \psi_3^{\pm}(\varphi))$, where

$$\begin{aligned} \psi_1^+(\varphi) &= \mathcal{P}^+(\mathcal{Q}^{-1}(C^+)) - \mathcal{P}^+(\rho^+) - \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) + \mathcal{P}^-(\rho^-) - \mathcal{P}^{+'}(\rho^+)\lambda^+ \\ &\quad + \mathcal{P}^{-'}(\rho^-)\lambda^- - f^+, \\ \psi_1^-(\varphi) &= \mathcal{P}^-(\mathcal{Q}^{-1}(C^-)) - \mathcal{P}^-(\rho^-) - \mathcal{P}^{-'}(\rho^-)\lambda^- - f^-, \\ \psi_2(\varphi) &= \omega \int_{S_1} |\zeta'|^2 d\zeta \int_0^{R_0^+} \left\{ \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|\zeta'|^2s^2 + \mathcal{Q}(\rho^+ + \lambda^+)\right) - \rho^+ \right\} s^4 ds \\ &\quad + \omega \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^+}^{R_0^+ + r^+} \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|\zeta'|^2s^2 + \mathcal{Q}(\rho^+ + \lambda^+)\right) s^4 ds \\ &\quad + \omega \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^+}^{R_0^-} \left\{ \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|\zeta'|^2s^2 + \mathcal{Q}(\rho^- + \lambda^-)\right) - \rho^- \right\} s^4 ds \\ &\quad + \omega \int_{S_1} |\zeta'|^2 d\zeta \int_{R_0^-}^{R_0^- + r^-} \mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|\zeta'|^2s^2 + \mathcal{Q}(\rho^- + \lambda^-)\right) s^4 ds - \beta, \\ \psi_3^+(\varphi) &= \frac{2\sigma^+}{R_0^{+2}}\left\{\frac{1}{\rho^+R_0^{+2}}\left(\int_{S_1} d\zeta \int_0^{R_0^+ + r^+} \left(\mathcal{Q}^{-1}\left(\frac{\omega^2}{2}|\zeta'|^2s^2 + C^+\right) - \mathcal{Q}^{-1}(C^+)\right) s^2 ds \right.\right. \\ &\quad \left.\left. + (|\mathcal{F}^+| - |B_{R_0^+}|)\lambda^+\right) - \tilde{Q}[r^+]\right\} \\ &\quad + \int_{S_1} (k^+ - \mathcal{P}^{+'}(\rho^+)\lambda^+ + \mathcal{P}^{-'}(\rho^-)\lambda^- + f^+) d\zeta, \end{aligned}$$

$$\begin{aligned} \psi_3^-(\varphi) = & \frac{2\sigma^-}{R_0^{-2}} \left\{ \frac{1}{\rho^- R_0^{-2}} \left\{ \int_{S_1} d\xi \int_{R_0^+ + r^+}^{R_0^- + r^-} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^- \right) - \mathcal{Q}^{-1}(C^-) \right) s^2 ds \right. \right. \\ & + \frac{\rho^-}{\rho^+} \left(\int_{S_1} d\xi \int_0^{R_0^+ + r^+} \left(\mathcal{Q}^{-1} \left(\frac{\omega^2}{2} |\xi'|^2 s^2 + C^+ \right) - \mathcal{Q}^{-1}(C^+) \right) s^2 ds \right. \\ & \left. \left. + (|\mathcal{F}^+| - |B_{R_0^+}|) \lambda^+ \right) + (|\mathcal{F}^-| - |B_{R_0^-}| + |B_{R_0^+}|) \lambda^- \right\} - \tilde{Q}[r^-] \Big\} \\ & + \int_{S_1} (k^- - \mathcal{P}'(\rho^-) \lambda^- + f^-) d\xi. \end{aligned}$$

We recall the notation: $\mathcal{X} = \tilde{C}^{2+\alpha} \times \tilde{C}^{2+\alpha} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $\mathcal{Y} = \tilde{C}^\alpha \times \tilde{C}^\alpha \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. By means of fixed point theorem, we are going to prove the solvability of system (23) with (48) in \mathcal{X} .

The existence of a unique solution $r \in \tilde{C}^{2+\alpha}(S_1)$ to (25) and the estimate for it are discussed in Section 2. The operator $\Phi_2'(0)$ is invertible. In addition, we have assumed that the determinant of two equations $\Phi_3^{\pm'}(0)\varphi = -\psi_3^\pm(\varphi)$ with relations (49) is not equal to zero:

$$\left(\frac{2\sigma^+}{3\rho^+ R_0^+} - \mathcal{P}'(\rho^+) \right) \left(\frac{2\sigma^- (R_0^{-3} - R_0^{+3})}{3\rho^- R_0^{-4}} - \mathcal{P}'(\rho^-) \right) - \frac{2\sigma^- \mathcal{P}'(\rho^-) R_0^{+3}}{3\rho^+ R_0^{-4}} \neq 0. \tag{50}$$

Therefore, the vector value operator $\Phi'(0) : \mathcal{X} \rightarrow \mathcal{Y}$ is invertible too, and the solution obeys inequality (27). Hence, one can write Equation (24) in the form (28).

We estimate the operator ψ in the same way as in Section 2. Using formulas similar to (29) for the new functions f^\pm , we deduce inequalities similar to (31) for two values of the arguments $\varphi = (r^\pm, \omega, \lambda^\pm)$ and $\varphi' = (r^{\pm'}, \omega', \lambda^{\pm'}) \in \mathcal{X}$, which satisfy estimates (30).

By repeating the arguments of Section 2, we conclude that $\mathcal{A}\psi(\varphi)$ is a contraction operator if inequality (33) holds. Hence, fixed point theorem guarantees the existence of a unique solution to Equation (28) in the ball $\{\|\varphi\|_{\mathcal{X}} \leq \eta\}$, with η satisfying (32). Therefore, (23) with $\Phi(\varphi)$ given by (48) is also uniquely solvable for β such that condition (33) holds. This inequality implies an estimate for ε in (40).

Estimate (41) follows from (34) if $c_2 c_3 \eta^\alpha \leq \frac{1}{2}$. \square

4. Conclusions

Thus, we have studied the stationary Navier–Stokes equations with interface conditions that follow from the continuity of the velocity vector field and momentum conservation when passing across the interface between the media. Similar conditions are posed on the free boundary. We have shown that, if the pressure is given by a smooth increasing function of fluid densities, and problem data are small and satisfy a certain condition, then for a two-layer compressible fluid under low angular momentum there exist axisymmetric equilibrium figures close to embedded balls (Theorem 2).

The next stage of investigation of the problem will consist in proving the existence of a global solution to the nonstationary problem for small initial data and their tendency to the stationary solution (\mathcal{V}, ρ) , as well as in studying the stability of equilibrium figures obtained. We hope that new our papers will deal with these investigations.

In addition, we note that our analysis has been carried out for the case of neglecting the gravity of liquids. This situation is realized in space, and our two-layer gas cloud can be considered, for example, as a gaseous planet or another cosmic rotating body.

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