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Article

Extension of the Kantorovich Theorem to Equations in Vector Metric Spaces: Applications to Functional Differential Equations

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Abstract: The equation $G(x,x) = \tilde{y}$, where $G : X \times X \to Y$, and X, Y are vector metric spaces (meaning that the values of a distance between the points in these spaces belong to some cones E_+, M_+ of a Banach space E and a linear space M, respectively), is considered. This operator equation is compared with a "model" equation, namely, $g(t,t) = 0$, where a continuous map $g : E_+ \times E_+ \to M_+$ is orderly covering in the first argument and antitone in the second one. The idea to study equations comparing them with "simpler" ones goes back to the Kantorovich fixed-point theorem for an operator acting in a Banach space. In this paper, the conditions under which the solvability of the "model" equation guarantees the existence of solutions to the operator equation are obtained. The statement proved extends the recent results about fixed points and coincidence points to more general equations in more general vector metric spaces. The results obtained for the operator equation are then applied to the study of the solvability, as well as to finding solution estimates, of the Cauchy problem for a functional differential equation.

Keywords: operator equation in vector metric space; existence and estimates of solutions; functional differential equation

MSC: 47J05; 54H25; 34K05; 34K32

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1. Introduction

In the theory of differential equations, an important role belongs to the methods and results based on the comparison of the equations under consideration with "model" ones. They allow for judging the properties of solutions of the studied equations by the properties (quite easily determined) of the solutions to the "model" equations. For example, to obtain estimates and approximate solutions, the Chaplygin comparison theorem and its generalizations establishing a connection between solutions of the equations $\dot{x} = f_1(t,x)$ and $\dot{x} = f_2(t,x)$, where $f_1(t,x) < f_2(t,x)$ for all t, x, are widely used. Another example is the study of quasi-linear differential equations of the form $\mathcal{L}x = f(t,x)$ with a linear differential operator L. The corresponding results explicitly or implicitly use a comparison of this equation with the "model" linear differential equation $\mathcal{L}x = y(t)$. A great contribution to the development of the comparison method for ordinary differential equations and to its extension to various classes of functional differential equations (FDEs) was made by N.V. Azbelev and the members of his mathematical school (see the articles [1–3], monographs [4] (Ch. 9,10) and [5] (Ch. VII)). In the contemporary research of L.M. Berezansky, E.Ya. Braverman, E.I. Bravyi, A.I. Bulgakov, A. Domoshnitsky, V.P. Maksimov, V.V. Malygina, A.V. Ponossov, P.M. Simonov, and A.I. Shindyapin on various issues of the theory of FDEs, the idea of comparing equations is effectively used and, in particular, approaches based on distinguishing families of equations and defining in them a "model" equation by which it is possible to judge the properties of all the equations in the family.

In this article, a method for determining a "model" FDE is proposed. The idea of the method goes back to Kantorovich's fixed-point theorem. Kantorovich's theorem (see [6] or [7] (Ch. XVIII, §1.2, Theorem 1)) uses a comparison of a continuously differentiable map S of a Banach space X with a "model" nondecreasing differentiable function φ having a fixed point t^* in some interval $[0, R)$, $R > 0$. It is assumed that this function majorizes the map S, i.e., the following relations take place:

$$\|S(x_0) - x_0\| \leq \varphi(0) \text{ and } \forall t \in [0, R) \ \forall x \in X \ \|x - x_0\| \leq t \implies \|S'(x)\| \leq \varphi'(t).$$

Under these conditions, S has a fixed point x^* satisfying the estimate $\|x^* - x_0\| \leq t^*$.

In recent studies [8–11] by A.V. Arutyunov, O.E. Zubelevich, S.E. Zhukovsky, and the authors of this work, analogs of Kantorovich's theorem for fixed points and coincidence points are obtained, as well as those for solutions of general operator equations not only in normed but also in metric spaces. In [8], Kantorovich's theorem is extended to the coincidence points of maps acting in Banach spaces. In [9], Kantorovich's theorem is generalized for coincidence points of multi-valued (and hence single-valued) maps acting in metric spaces and in [10] for those acting in vector metric spaces. In [11], based on a comparison with a majorizing equation in $\mathbb{R}_+$, conditions for the existence and estimates of the solutions of general equations in metric space are obtained. In the present paper, we study an operator equation in a space with a vector metric taking values in some cone E_+ of a Banach space E. The equation under discussion is compared with a majorizing equation, which is defined in the cone E_+. We also demonstrate the application of the statement obtained to the study of a nonlinear implicit (not resolved with respect to the derivative) first-order FDE.

2. Space with a Vector Metric

Let M be a linear space (over the field $\mathbb{R}$ of real numbers), in which there is defined an acute cone M_+, i.e., a set containing 0, and such that for any element $\nu \in M_+$, $\nu \neq 0$, the relation $c\,\nu \in M_+$ holds for any $c \geq 0$, and $c\,\nu \notin M_+$ for any $c < 0$. We also let M_+ be convex. In the space M, we define a "natural" order assuming that $\nu \leq \mu$ (or $\mu \geq \nu$, which is the same) for $\nu, \mu \in M$ if $\mu - \nu \in M_+$. We write $\nu < \mu$ (or $\mu > \nu$, which is the same) when $\nu \leq \mu$ and $\nu \neq \mu$.

Let $\underline{\mu}, \overline{\mu} \in M$, $\underline{\mu} \leq \overline{\mu}$. In a standard way, we denote the intervals

$$[\underline{\mu}, \overline{\mu}]_M := \{\nu \in M : \underline{\mu} \leq \nu \leq \overline{\mu}\}, \ (\underline{\mu}, \overline{\mu})_M := \{\nu \in M : \underline{\mu} < \nu < \overline{\mu}\},$$
$$(\underline{\mu}, \overline{\mu}]_M := \{\nu \in M : \underline{\mu} < \nu \leq \overline{\mu}\}, \ [\underline{\mu}, \overline{\mu})_M := \{\nu \in M : \underline{\mu} \leq \nu < \overline{\mu}\}$$

(in the case of $\mu := \underline{\mu} = \overline{\mu}$, we set $[\mu, \mu]_M = \{\mu\}$, $(\mu, \mu)_M = [\mu, \mu)_M = (\mu, \mu]_M = \varnothing$).

Let us be given a nonempty set Y and a map $\mathcal{P}_Y^M : Y \times Y \to M_+$ such that for any $y, z, w \in Y$, the relations

$$\mathcal{P}_Y^M(y, z) = 0 \ \Leftrightarrow \ y = z; \ \ \mathcal{P}_Y^M(y, z) = \mathcal{P}_Y^M(z, y); \ \ \mathcal{P}_Y^M(y, w) \leq \mathcal{P}_Y^M(y, z) + \mathcal{P}_Y^M(z, w)$$

hold. Then, the map $\mathcal{P}_Y^M$ is called a *vector metric* or, for shortness, a v-*metric*, and a pair $(Y, \mathcal{P}_Y^M)$ is a *vector metric* (v-*metric*) *space*. For similar definitions of vector metric spaces, as well as for theorems on fixed points of maps acting in such spaces, see, e.g., [12–14].

The concept of the v-metric is a natural generalization of a metric, namely, if M is the space of real numbers $\mathbb{R}$ with the cone $\mathbb{R}_+ = [0, \infty)$, then a v-metric becomes a "classical" metric.

By analogy with metric spaces, in the v-metric space $(Y, \mathcal{P}_Y^M)$, a *ball of radius* $\nu \in M_+$ *centered at* $y_0 \in Y$ is defined as

$$B_Y(y_0, \nu) := \{y \in Y : \mathcal{P}_Y^M(y, y_0) \leq \nu\}.$$

Let us give an example of a v-metric space important for the further discussion.

Example 1. *Given $T > 0$, we denote by $W^n = W([0,T],\mathbb{R}^n)$ the linear space of (Lebesgue) measurable on $[0,T]$ functions (classes of functions). In the case $n = 1$, we omit the upper index (equal to 1) in this notation. In the space W, we define the cone $W_+ = W([0,T],\mathbb{R}_+)$ of the scalar measurable nonnegative functions and a vector metric $\mathcal{P}^W_{W^n} : W^n \times W^n \to W_+$ on W^n by the relation*

$$\forall y, w \in W^n \ \ \big(\mathcal{P}^W_{W^n}(y,w)\big)(t) = \big|y(t) - w(t)\big|_{\mathbb{R}^n}, \ \ t \in [0,T].$$

Obviously, a ball in the space W^n is the set

$$B_{W^n}(y_0,\nu) = \{y \in W^n : \ \big|y(t) - y_0(t)\big|_{\mathbb{R}^n} \leq \nu(t)\}, \ \ y_0 \in W^n, \ \ \nu \in W_+.$$

Along with the v-metric space $(Y,\mathcal{P}^M_Y)$, we also use a v-metric space $(X,\mathcal{P}^E_X)$, where a v-metric $\mathcal{P}^E_X$ takes values in an acute closed convex cone E_+ of a Banach space E. Note that from the closedness of E_+, it follows that for any nonincreasing sequence $\{\varsigma_i\} \subset E$ (meaning $\varsigma_{i+1} \leq \varsigma_i$ for every i), in the case of its convergence $\varsigma_i \to \varsigma$, the inequality $\varsigma \leq \varsigma_i$ holds for every i. In addition, we assume the cone E_+ to be regular. The cone is called regular (see [15] (p. 257)) if any nonincreasing sequence $\{\varsigma_i\} \subset E$ converges if and only if it is bounded from below. Moreover, as it is shown in [16] (Proposition 6), due to the regularity of the cone E_+, for such a bounded nonincreasing sequence $\{\varsigma_i\} \subset E$, there exists an infimum and $\lim_{i\to\infty}\varsigma_i = \inf\{\varsigma_i\}$. From the regularity of the cone E_+, it also follows that for any chain $S \subset E$ bounded from below, there exists a nonincreasing sequence $\{\varsigma_i\} \subset S$ that is coinitial to the chain S, i.e., for any element ς of the chain S, there exists an element ς_i of this sequence such that $\varsigma_i \leq \varsigma$. So, the chain S has an infimum, and $\inf S = \inf\{\varsigma_i\} = \lim_{i\to\infty}\varsigma_i$ (see [16] (Proposition 7)).

We also note that the regularity of the cone E_+ guarantees that any element of an arbitrary nonempty closed and bounded one from below set $U \subset E$ is subordinate to some minimal element of this set, i.e., for any $\varsigma \in U$, there exists a minimal (probably nonunique) element $\tau \in U$ such that $\tau \leq \varsigma$ (for the proof of this property, see [10] (pp. 399, 400)).

It is obvious that in the considered Banach space E ordered by the closed regular cone E_+, the sets bounded from above, chains, and nondecreasing sequences have similar properties to those listed above.

The regularity of a cone leads to its normality (see [15] (p. 257)), meaning

$$\exists \mathcal{C} \geq 0 \ \forall \varsigma,\tau \in E_+ \ \ \varsigma \leq \tau \ \Rightarrow \ \|\varsigma\|_E \leq \mathcal{C}\|\tau\|_E. \tag{1}$$

Let us give one more example of a v-metric space that will be used further on.

Example 2. *Given $T > 0$, denote by $L^n = L([0,T],\mathbb{R}^n)$ the set of (Lebesgue) summable on $[0,T]$ functions (classes of functions). In the case $n = 1$, we omit the upper index in this notation and write L. In L, we consider the norm $\|\varsigma\|_L = \int_0^T |\varsigma(s)|, ds$, $\varsigma \in L$. Then, L is a Banach space, and the cone $L_+ = L([0,T],\mathbb{R}_+)$ of the nonnegative functions in this space is closed and regular (see [15] (p. 257)). In L^n, we define a vector metric $\mathcal{P}^L_{L^n} : L^n \times L^n \to L_+$ by the relation*

$$\forall u, v \in L^n \ \ \big(\mathcal{P}^L_{L^n}(u,v)\big)(t) = \big|u(t) - v(t)\big|_{\mathbb{R}^n}, \ \ t \in [0,T].$$

In connection with the considered example, we recall that, unlike the cone L_+, the cone C_+ of the nonnegative functions in the space C of the continuous functions (with the "standard" norm) is not regular (see [15] (p. 257)). But, at the same time, the cone C_+, as well as the cone L_+, are normal; moreover, relation (1) is valid with the constant $\mathcal{C} = 1$.

Let us denote by $\overline{\mathcal{C}}$ the smallest of the constants $\mathcal{C}$ satisfying (1). For $\overline{\mathcal{C}} = 1$, we have

$$\forall \varsigma,\tau \in E_+ \ \ \varsigma \leq \tau \ \Rightarrow \ \|\varsigma\|_E \leq \|\tau\|_E,$$

and the norm in E is called monotone. It is easy to see that if the norm in E is monotone, then a v-metric $\mathcal{P}^E_X$ defines a "classical" metric via $\|\mathcal{P}^E_X\|_E$. However, as it is noted in [13] (Remark 1), using a v-metric instead of the corresponding "classical" one allows for,

for example, obtaining less burdensome conditions of the existence of solutions and more accurate estimates for them. Moreover, in the case of $\overline{\mathcal{C}} > 1$, the map defined as $\|\mathcal{P}_X^E\|_E$ cannot be considered a metric (it does not satisfy the triangle inequality).

The simplest example of a nonmonotone norm gives the Euclidean norm $|\cdot|_{\mathbb{R}^2}$ on the plane $\mathbb{R}^2$ with respect to the order $\preceq$ generated by the cone $A(\mathbb{R}^2_+)$, where A is a 2×2 nonsingular matrix. Indeed, in this case, for elements $\varsigma, \tau \in A(\mathbb{R}^2_+)$, the relation $\varsigma \preceq \tau$ is equivalent to the "usual" inequalities $0 \le A^{-1}\varsigma \le A^{-1}\tau$, so for the Euclidean norm of the elements ς, τ, we have

$$\left|\varsigma\right|_{\mathbb{R}^2} = \left|AA^{-1}\varsigma\right|_{\mathbb{R}^2} \le \|A\|\left|A^{-1}\varsigma\right|_{\mathbb{R}^2}| \le \|A\|\left|A^{-1}\tau\right|_{\mathbb{R}^2} \le \|A\|\|A^{-1}\|\left|\tau\right|_{\mathbb{R}^2}.$$

It is obvious that for any nonsingular matrix A, the inequality $\|A\|\|A^{-1}\| \ge 1$ holds, and it is quite easy to present a matrix A such that $\overline{\mathcal{C}} > 1$.

Example 3. *Set* $A = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$, $A^{-1} = 4^{-1}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$. *Suppose that the order on* $\mathbb{R}^2$ *is generated by the cone*

$$A(\mathbb{R}^2_+) = \{\varsigma = (\varsigma_1, \varsigma_2) : \varsigma_1 + 2\varsigma_2 \ge 0, \ -\varsigma_1 + 2\varsigma_2 \ge 0\}.$$

With respect to this order, for the elements $\varsigma = (-2,1)$, $\tau = (0,2)$, *we have* $\varsigma \preceq \tau$, *but* $\left|\varsigma\right|_{\mathbb{R}^2} = \sqrt{5} > 2 = \left|\tau\right|_{\mathbb{R}^2}$. *So,* $\overline{\mathcal{C}} \ge \sqrt{5}/2 > 1$, *and for the given cone* $A(\mathbb{R}^2_+)$ *in* $\mathbb{R}^2$, *the Euclidean norm is not monotone. Now, consider the product* $X = X_1 \times X_2$, *where* X_i *are metric spaces with metrics* $\rho_i : X_i \to \mathbb{R}_+$, $i = 1, 2$. *Define on* X *a* v-*metric* $\mathcal{P}_X^{\mathbb{R}^2} : X \times X \to A(\mathbb{R}^2_+)$ *by using the formula*

$$\forall u = (u_1, u_2),\ v = (v_1, v_2) \in X \ \ \mathcal{P}_X^{\mathbb{R}^2}(u, v) = A\big(\rho_1(u_1, v_1), \rho_2(u_2, v_2)\big).$$

Then, the corresponding map $\left|\mathcal{P}_X^{\mathbb{R}^2}\right|_{\mathbb{R}^2}$ *is not a metric.*

Concluding the discussion on the metrizability of v-metric spaces, note that the v-metric considered in Example 1 takes values in the cone of the linear space of measurable functions not equipped with a norm, so it is basically impossible to define a "classical" metric here using the given v-metric.

By analogy with "classical" metric spaces, in the v-metric space $(X, \mathcal{P}_X^E)$, the following concepts related to convergence are defined.

A sequence $\{x_i\} \subset X$ *converges* (as $i \to \infty$) to x in the space X, if $\mathcal{P}_X^E(x_i, x) \to 0$ in the Banach space E, which basically means the convergence $\|\mathcal{P}_X^E(x_i, x)\|_E \to 0$. The limit x, if it exists, is obviously unique. We also note that the map $\mathcal{P}_X^E : X \times X \to E_+$ is continuous, i.e., if $x_i \to x$ and $u_i \to u$, then $\mathcal{P}_X^E(x_i, u_i) \to \mathcal{P}_X^E(x, u)$. Indeed,

$$\pm\big(\mathcal{P}_X^E(x_i, u_i) - \mathcal{P}_X^E(x, u)\big) \le \mathcal{P}_X^E(x_i, x) + \mathcal{P}_X^E(u_i, u),$$

and from this inequality, according to (1), it follows that

$$\big\|\mathcal{P}_X^E(x_i, u_i) - \mathcal{P}_X^E(x, u)\big\|_E \le \mathcal{C}\big\|\mathcal{P}_X^E(x_i, x) + \mathcal{P}_X^E(u_i, u)\big\|_E.$$

A sequence $\{x_i\} \subset X$ is called *fundamental* if for any $\varepsilon > 0$, there exists a number I such that for every $i, j > I$, the inequality $\|\mathcal{P}(x_i, x_j)\|_E < \varepsilon$ holds. A v-metric space X is *complete* if any fundamental sequence in X converges.

A set $U \subset X$ is *closed* if for any convergent sequence $\{x_i\} \subset U$, $x_i \to x$, the relation $x \in U$ holds.

Example 4. *Let us show that the set* $U := B_X(x_0, e) := \{x \in X : \mathcal{P}_X^E(x, x_0) \le e\}$, *a ball of radius* $e \in E_+$ *centered at* $x_0 \in X$, *is closed in the space* $\big(X, \mathcal{P}_X^E\big)$. *Take an arbitrary sequence* $\{x_i\} \subset U$

convergent to some $x \in X$. We have $\mathcal{P}_X^E(x, x_0) \le e$. From the continuity of $\mathcal{P}_X^E$, it follows that the sequence $\{\mathcal{P}_X^E(x_i, x_0)\}$ converges to $\mathcal{P}_X^E(x, x_0)$, and $\|\mathcal{P}_X^E(x_i, x)\|_E \to 0$. But for any $i \in \mathbb{N}$, the inequality $\mathcal{P}_X^E(x_i, x_0) \le e$ holds; hence, $e - \mathcal{P}_X^E(x_i, x_0) \in E_+$. From this and the closedness of the cone E_+, we obtain $e - \mathcal{P}_X^E(x, x_0) \in E_+$. Thus, $\mathcal{P}_X^E(x, x_0) \le e$, and the ball $B_X(x_0, e)$ is closed.

3. Existence of Solutions to Equations in Vector Metric Spaces

Let $(X, \mathcal{P}_X^E)$, $(Y, \mathcal{P}_Y^M)$ be the v-metric spaces defined above (where E is a Banach space with closed regular convex and acute cone E_+, and M is a linear space with convex and acute cone M_+). We assume that the space $(X, \mathcal{P}_X^E)$ is complete. Let there be given maps $G : X \times X \to Y$, $g : E_+ \times E_+ \to M$, and an element $\tilde{y} \in Y$. We consider the equation

$$G(x, x) = \tilde{y} \tag{2}$$

with respect to $x \in X$ and the corresponding "model" equation

$$g(\varsigma, \varsigma) = 0 \tag{3}$$

with the unknown $\varsigma \in E_+$. We are interested in the conditions under which the existence of a solution to "model" Equation (3) guarantees the existence of a solution to Equation (2).

We start by formulating the solvability conditions for "model" Equation (3).

Definition 1. *Given a nonempty closed set $\mathfrak{I} \subset E_+$, we say that the map g is closed with respect to the sets $\mathfrak{I}$ and $\{0\} \subset M$, if for any two sequences $\{e_i\}, \{e_i'\} \subset \mathfrak{I}$ convergent to the same limit $e = \lim_{i\to\infty} e_i = \lim_{i\to\infty} e_i'$, and such that $g(e_i', e_i) = 0$ $\forall i \in \mathbb{N}$, the limit e satisfies the equality $g(e, e) = 0$.*

Note that, in Definition 1, $e \in \mathfrak{I}$, because the set $\mathfrak{I}$ is closed in E. We also point out the following properties of the map g satisfying this definition.

Property (i): *For any closed set $\mathfrak{I}^* \subset \mathfrak{I}$, the closedness of the map g with respect to the sets $\mathfrak{I} \subset E_+$ and $\{0\} \subset M$ implies its closedness with respect to the sets $\mathfrak{I}^*$ and $\{0\}$.* The validity of this statement is quite obvious.

Property (ii): *If the map g is closed with respect to the sets $\mathfrak{I}$ and $\{0\}$, then*

$$\forall \{e_i\} \subset \mathfrak{I} \ \lim_{i\to\infty} e_i = e \text{ and } \forall i \in \mathbb{N} \ g(e_{i+1}, e_i) = 0 \implies g(e, e) = 0. \tag{4}$$

In fact, let g satisfy Definition 1. Given a sequence $\{e_i\} \subset \mathfrak{I}$, we denote a sequence $\{e_i'\} \subset \mathfrak{I}$ by the relation $e_i' = e_{i+1}$. Then, by virtue of Definition 1, relation (4) is satisfied.

Now, denote by $\mathrm{Sol}_{\mathfrak{I}}(g)$ the set of solutions to Equation (3) belonging to a closed set $\mathfrak{I} \subset E_+$. The set $\mathrm{Sol}_{\mathfrak{I}}(g)$ is closed in E. Indeed, for any convergent sequence $\{\varsigma_i\} \subset \mathrm{Sol}_{\mathfrak{I}}(g)$, $\lim_{i\to\infty} \varsigma_i = \varsigma$, assuming $e_i = e_i' = \varsigma_i$ in the definition of the closedness of g with respect to the sets $\mathfrak{I} \subset E$ and $\{0\} \subset M$, we obtain

$$g(e_i, e_i') = g(\varsigma_i, \varsigma_i) = 0 \implies g(\varsigma, \varsigma) = 0 \implies \varsigma \in \mathrm{Sol}_{\mathfrak{I}}(g).$$

Let $\mathrm{Sol}_{\mathfrak{I}}(g) \neq \emptyset$. Then, from the regularity of the cone E_+, it follows that each element ς of the closed nonempty set $\mathrm{Sol}_{\mathfrak{I}}(g)$ is subordinate to some minimal element $\varsigma^* \in \mathrm{Sol}_{\mathfrak{I}}(g)$, i.e., $\varsigma^* \le \varsigma$.

Recall that a map $f : E_+ \to M$ is called *antitone* on a set $\mathfrak{I} \subset E_+$, if for any $e, e' \in \mathfrak{I}$, the relation $e \le e'$ implies $f(e) \ge f(e')$. We also recall the definition (given in [16] (Definition 1)) of an orderly covering map in relation to the sets that appear in the statements below.

Definition 2. *We say that a map $f : E_+ \to M$ orderly covers the one-point set $\{0\} \subset M$ on a set $\mathfrak{I} \subset E_+$ if*

$$\forall e \in \mathfrak{I} \ \ f(e) \le 0 \implies \exists e' \in \mathfrak{I} \ \ e' \ge e \text{ and } f(e') = 0.$$

Lemma 1. *Let the set* $\mathrm{Sol}_{E_+}(g)$ *of solutions to Equation* (3) *be nonempty and* $\varsigma^* \in E_+$ *be a minimal element in this set. Suppose that the following conditions hold:* $g(0,0) \leq 0$; *the map* g *is closed with respect to the sets* $\mathfrak{I}^* := [0,\varsigma^*]_E$ *and* $\{0\} \subset M$; *and for any* $\nu \in \mathfrak{I}^*$, *the map* $g(\cdot,\nu) : E_+ \to M$ *orderly covers the set* $\{0\} \subset M$ *on the interval* $\mathfrak{I}^*$ *and the map* $g(\nu,\cdot) : E_+ \to M$ *is antitone on* $\mathfrak{I}^*$. *Then, there exists a nondecreasing sequence* $\{\varsigma_i\} \subset \mathfrak{I}^*$ *with the initial element* $\varsigma_0 = 0$ *such that*

$$g(\varsigma_i, \varsigma_{i-1}) = 0, \;\; i \in \mathbb{N}, \tag{5}$$

and $\varsigma_i \to \varsigma^*$.

Proof. In view of the assumptions made, $\varsigma_0 = 0$ satisfies the inequality $g(\varsigma_0,\varsigma_0) \leq 0$. Because the map $g(\cdot,\varsigma_0)$ orderly covers $\{0\} \subset M$ on the interval $\mathfrak{I}^*$, there exists a $\varsigma_1 \in [\varsigma_0,\varsigma^*]_E$ such that $g(\varsigma_1,\varsigma_0) = 0$. Next, for the found element ς_1, the map $g(\varsigma_1,\cdot)$ is antitone, so the inequality $g(\varsigma_1,\varsigma_1) \leq g(\varsigma_1,\varsigma_0) = 0$ takes place. And again, from the orderly covering property of the map $g(\cdot,\varsigma_1)$, it follows that there exists a $\varsigma_2 \in [\varsigma_1,\varsigma^*]_E$ such that $g(\varsigma_2,\varsigma_2) = 0$. Continuing such reasoning, we obtain a nondecreasing sequence satisfying recurrent relation (5).

Due to the regularity of the cone E_+, the constructed nondecreasing sequence converges. Let $\varsigma = \lim_{i\to\infty} \varsigma_i$, and then $\varsigma \leq \varsigma^*$. Moreover, because the map g is closed with respect to the sets $[0,\varsigma^*]_E \subset \mathfrak{I}$ and $\{0\} \subset M$, from relation (5), according to the property (ii), it follows that ς is a solution to Equation (3). But ς^* is a minimal point in the set $\mathrm{Sol}_{\mathfrak{I}}(g)$ of solutions to (3); hence, from the inequality $\varsigma \leq \varsigma^*$, we obtain $\varsigma = \varsigma^*$. $\square$

We obtain now the conditions for the existence of a solution to Equation (2) in the form of a comparison theorem with "model" Equation (3).

Definition 3. *Given a nonempty closed set* $\mathfrak{B} \subset X$, *we say that the map* G *is closed with respect to the sets* $\mathfrak{B}$ *and* $\{\tilde{y}\} \subset Y$ *if for any two sequences* $\{x_i\}, \{x_i'\} \subset \mathfrak{B}$ *convergent to the same limit* $x = \lim_{i\to\infty} x_i = \lim_{i\to\infty} x_i'$, *and such that* $G(x_i, x_i') = \tilde{y} \;\forall i \in \mathbb{N}$, *the limit* x *satisfies the equality* $G(x,x) = \tilde{y}$.

The next definition extends [11] (Definition 2) to maps acting in v-metric spaces.

Definition 4. *Let* $x_0 \in X$, $\bar{e} \in E_+$. *We say that the map* g *majorizes the map* G *on the ball* $\mathfrak{B} := B_X(x_0,\bar{e})$, *if* G, g, *as the maps of the first argument, satisfy the relation*

$$\begin{aligned}&\forall e \in \mathfrak{I} := [0,\bar{e}]_E \;\; \forall x \in B_X(x_0,e) \;\; \forall \Delta \in [0,\bar{e}-e]_E \\ &\quad \mathcal{P}_Y^M\big(\tilde{y}, G(x,x)\big) \leq g(e+\Delta,e) - g(e,e) \implies \exists\, u \in \mathfrak{B} \;\; G(u,x) = \tilde{y}, \;\; \mathcal{P}_X^E(u,x) \leq \Delta,\end{aligned} \tag{6}$$

and, as the maps of the second argument, the relation

$$\begin{aligned}&\forall e \in \mathfrak{I} \;\; \forall \varsigma \in [0,e]_E \;\; \forall x \in B_X(x_0,e) \;\; \forall u \in B_X(x_0,\varsigma) \\ &\qquad\qquad \mathcal{P}_X^E(u,x) \leq e - \varsigma \implies \mathcal{P}_Y^M\big(G(x,u), G(x,x)\big) \leq g(e,\varsigma) - g(e,e).\end{aligned} \tag{7}$$

It is easy to see that the property of the maps g and G described in Definition 4 remains valid when the interval $\mathfrak{I}$ is "reduced". More precisely, the following statement takes place.

Property (iii): *If the map* g *majorizes the map* G *on the ball* $\mathfrak{B} := B_X(x_0,\bar{e})$, *then for every* $e' \in [0,\bar{e}]_E$, *the map* g *majorizes the map* G *on the ball* $\mathfrak{B}^* := B_X(x_0,e')$ as well.

Theorem 1. *Let the set* $\mathrm{Sol}_{E_+}(g)$ *of the solutions to Equation* (3) *be nonempty,* $\varsigma^* \in E_+$ *be a minimal element in this set, and all the assumptions of Lemma* 1 *be satisfied. Suppose that the map* g *majorizes the map* G *on the ball* $\mathfrak{B}^* := B_X(x_0,\varsigma^*)$, *the map* G *is closed with respect to the sets* $\mathfrak{B}^*$ *and* $\{\tilde{y}\}$, *and the inequality*

$$\mathcal{P}_Y^M\big(\tilde{y}, G(x_0,x_0)\big) \leq -g(0,0) \tag{8}$$

is valid. Then, there exists a solution $x^* \in \mathfrak{B}^*$ *to Equation* (2).

Proof. According to Lemma 1, there exists a nondecreasing sequence $\{\varsigma_i\} \subset \mathfrak{I}^*$, $\varsigma_0 = 0$, satisfying relation (5) and convergent to ς^*.

Let us show that there exists a sequence $\{x_i\} \subset \mathfrak{B}^*$ satisfying for every $i \in \mathbb{N}$ the relations

$$\begin{aligned} G(x_i, x_{i-1}) = \tilde{y}, \ \ \mathcal{P}_Y^M(\tilde{y}, G(x_i, x_i)) \le -g(\varsigma_i, \varsigma_i), \\ \mathcal{P}_X^E(x_0, x_i) \le \varsigma_i, \ \ \mathcal{P}_X^E(x_{i-1}, x_i) \le \varsigma_i - \varsigma_{i-1}. \end{aligned} \tag{9}$$

From assumption (8) and equality (5), we obtain

$$\mathcal{P}_Y^M(\tilde{y}, G(x_0, x_0)) \le -g(\varsigma_0, \varsigma_0) = g(\varsigma_1, \varsigma_0) - g(\varsigma_0, \varsigma_0).$$

Hence, by virtue of condition (6), in which we set $x := x_0$, $e := \varsigma_0$, $\Delta := \varsigma_1 - \varsigma_0$, there exists an $x_1 \in \mathcal{B}$ such that

$$G(x_1, x_0) = \tilde{y}, \ \ \mathcal{P}_X^E(x_1, x_0) \le \varsigma_1 - \varsigma_0 = \varsigma_1.$$

The map g majorizes the map G, so from condition (7), in which we set $e := \varsigma_1$, $\varsigma := \varsigma_0$, $x := x_1$, $u := x_0$, and from relation (5), it follows that

$$\mathcal{P}_Y^M(y, G(x_1, x_1)) = \mathcal{P}_Y^M(G(x_1, x_0), G(x_1, x_1)) \le g(\varsigma_1, \varsigma_0) - g(\varsigma_1, \varsigma_1) = g(\varsigma_2, \varsigma_1) - g(\varsigma_1, \varsigma_1).$$

Because $\varsigma_1 \ge \mathcal{P}_X^E(x_0, x_1)$, we have $x_1 \in B_X(x_0, \varsigma_1)$. From condition (6), with $x := x_1$, $e := \varsigma_1$, $\Delta := \varsigma_2 - \varsigma_1$, it follows that there exists an $x_2 \in \mathcal{B}$ such that

$$G(x_2, x_1) = \tilde{y}, \ \ \mathcal{P}_X^E(x_2, x_1) \le \varsigma_2 - \varsigma_1, \ \ \mathcal{P}_X^E(x_2, x_0) \le \varsigma_2 - \varsigma_1 + \varsigma_1 - \varsigma_0 = \varsigma_2.$$

Thus, we define the first two members of a sequence $\{x_i\} \subset \mathfrak{B}^*$ satisfying relation (9). Using the similar reasoning, we construct by induction all the other members of this sequence.

From the last inequality in (9), we obtain

$$\forall i, n \in \mathbb{N} \ \ \mathcal{P}_X^E(x_i, x_{i+n}) \le \sum_{l=0}^{n-1} \mathcal{P}_X^E(x_{i+l}, x_{i+l+1}) \le \sum_{l=0}^{n-1} (\varsigma_{i+l+1} - \varsigma_{i+l}) = \varsigma_{i+n} - \varsigma_i. \tag{10}$$

The sequence $\{\varsigma_i\} \subset \mathfrak{I}^*$ is fundamental; then, according to (10), the sequence $\{x_i\} \subset \mathfrak{B}^*$ is also fundamental and hence converges in the complete space $(X, \mathcal{P}_X^E)$ to some element $x^* \in X$. Because the ball $\mathfrak{B}^*$ is a closed set (see Example 4), we obtain $x^* \in \mathfrak{B}^*$. And due to the closedness of the map G with respect to the sets $\mathfrak{B}^*$ and $\{\tilde{y}\}$, from the first relation in (9), we obtain that $G(x^*, x^*) = \tilde{y}$. □

4. Applications to Functional Differential Equations

Let mes denote the Lebesgue measure on $[0, T]$, $T > 0$, and let $(W^m, \mathcal{P}_{W^m}^W)$, $(L^n, \mathcal{P}_{L^n}^L)$ be the v-metric spaces of the measurable and, respectively, integrable (in the sense of Lebesgue) functions defined in Examples 1 and 2. The v-metrics of these spaces take values in the cone W_+ of the linear space W of measurable scalar functions and in the cone L_+ of the Banach space L of summable scalar functions, respectively.

For a wide class of FDEs, the Cauchy problem, as well as the boundary value problems, can be reduced to an integral equation of the form

$$\Phi(t, (Ku)(t), (S_h u)(t), u(t)) = 0, \ \ t \in [0, T], \tag{11}$$

$$\text{where } \ (Ku)(t) := \int_0^T \mathcal{K}(t, s) u(s) ds, \ \ (S_h u)(t) := \begin{cases} u(h(t)), \text{ if } h(t) \in [0, T], \\ 0, \text{ if } h(t) \notin [0, T], \end{cases}$$

with respect to the derivative $u = \dot{x} \in L^n$ of the required absolutely continuous function x. The function $\Phi : [0,T] \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ here satisfies the Caratheodory conditions, i.e., it is measurable in the first argument and continuous in the totality of the remaining arguments; the function $\mathcal{K} : [0,T] \times [0,T] \to \mathbb{R}^{k\times n}$ is measurable and with respect to the second argument, as the function $\mathcal{K}(t,\cdot)$, is essentially bounded for a.e. $t \in [0,T]$; and the function $h : [0,T] \to \mathbb{R}$ is measurable, and the condition

$$\forall \mathfrak{V} \subset [0,T] \;\; \mathrm{mes}\, \mathfrak{V} = 0 \implies \mathrm{mes}\, h^{-1}(\mathfrak{V}) = 0 \tag{12}$$

is valid.

In the literature, the FDEs reducible to Equation (11) are studied in detail in the case of solvability with respect to the derivative, that is, if $n = m$ and the function $\Phi : [0,T] \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is of the form $\Phi(t,u,v) = v - F(t,u)$ (see [4] (§§1.1, 7.1–10.2), [5] (Ch. VII), the bibliographic lists in these books, as well as later articles by L.M. Berezansky, E.Ya. Braverman, E.I. Bravyi, A.I. Bulgakov, A. Domoshnitsky, V.P. Maksimov, V.V. Malygina, A.V. Ponossov, P.M. Simonov, and A.I. Shindyapin). However, in the general case, "implicit" FDEs remain practically unexamined because many classical methods of analysis, fixed-point theorems in particular, cannot be applied here. It is possible that statements about the existence of solutions and their estimates and dependence on parameters can be obtained with the use of contemporary results on covering maps (see [17–20]), which have recently been successfully applied to ordinary differential equations unsolved with respect to the derivative (see, e.g., articles [21–25] and other works by the same authors). But until now, such studies of "implicit" FDEs were fragmentary; we note only the works [26–28]. Here, it is proposed to apply the above Theorem 1 on equations in vector metric spaces to the study of Equation (11). Note that the use of vector metrics allows for obtaining more accurate conditions for the existence and evaluation of solutions than that of a "classical" scalar metric. And Theorem 1, with a proper choice of a majorizing equation, allows for investigating, for example, equations with maps that are not covering.

As a majorizing equation, we will use the equation

$$\varphi\big(t, (\widehat{K}\varsigma)(t), (S_h\varsigma)(t), \varsigma(t)\big) = 0, \;\; t \in [0,T], \tag{13}$$

with respect to the function $\varsigma \in L_+$, where the function $\varphi : [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the Caratheodory conditions, the integral operator $\widehat{K}$ is defined by the formula

$$(\widehat{K}\varsigma)(t) := \int_0^T \kappa(t,s)\varsigma(s)ds,$$

and the function $\kappa : [0,T] \times [0,T] \to \mathbb{R}$ is measurable and as a function of the second argument $\kappa(t,\cdot)$ is essentially bounded for a.e. $t \in [0,T]$, and also satisfies the inequality

$$\kappa(t,s) \le \big|\mathcal{K}(t,s)\big|_{\mathbb{R}^n\times\mathbb{R}^n}, \;\; (t,s) \in [0,T] \times [0,T]$$

(in particular, it can be assumed that $\kappa(t,s) = \big|\mathcal{K}(t,s)\big|_{\mathbb{R}^n\times\mathbb{R}^n}$).

The following statement gives the conditions under which Equation (13) is solvable, and in the set of solutions, there exists not even a minimal but the smallest element. Note that similar results in the theory of differential equations are called the statements of Chaplygin's theorem on the differential inequality type. For implicit (i.e., unsolved with respect to the highest derivative) ordinary differential equations, similar results were obtained in [29,30].

Lemma 2. *Let the function $\varphi(t,\cdot,\cdot,\varsigma) : \mathbb{R}_+ \to \mathbb{R}$ be nonincreasing for a.e. $t \in [0,T]$ and any $\varsigma \in \mathbb{R}_+$ (meaning $\sigma_1 > \sigma_2$, $\nu_1 > \nu_2 \implies \varphi(t,\sigma_1,\nu_1,\varsigma) \le \varphi(t,\sigma_2,\nu_2,\varsigma)$). Suppose that the inequality $\varphi(t,0,0,0) \le 0$ is valid almost everywhere on $[0,T]$, and there exists a function $\overline{\varsigma} \in L_+$ such that*

$$\varphi\big(t, (\widehat{K}\overline{\varsigma})(t), (S_h\overline{\varsigma})(t), \overline{\varsigma}(t)\big) \ge 0, \;\; t \in [0,T]. \tag{14}$$

Then, Equation (13) has a solution $\varsigma \in L_+$ such that $\varsigma \leq \overline{\varsigma}$, and in the solution set, there exists the smallest element.

Proof. First, we show that there exists a nondecreasing sequence $\{\varsigma_i\} \subset [0, \overline{\varsigma}]_L$ (i.e., $\forall i \in \mathbb{N}$ $\varsigma_i \geq \varsigma_{i-1}$) satisfying the relation

$$\varsigma_0(t) = 0, \ \ \forall i \in \mathbb{N} \ \ \varphi\big(t, (\widehat{K}\varsigma_{i-1})(t), (S_h\varsigma_{i-1})(t), \varsigma_i(t)\big) = 0, \ \ t \in [0, T]. \tag{15}$$

Set $\varsigma_0(t) \equiv 0$. According to the assumptions made, we have

$$\varphi\big(t, (\widehat{K}\varsigma_0)(t), (S_h\varsigma_0)(t), \varsigma_0(t)\big) \leq 0, \ \ t \in [0, T],$$

and moreover, because the function $\varphi(t, \cdot, \cdot, \overline{\varsigma}(t))$ is nonincreasing, the inequalities

$$\varphi\big(t, (\widehat{K}\varsigma_0)(t), (S_h\varsigma_0)(t), \overline{\varsigma}(t)\big) \geq \varphi\big(t, (\widehat{K}\overline{\varsigma})(t), (S_h\overline{\varsigma})(t), \overline{\varsigma}(t)\big) \geq 0$$

take place. From the continuity of the function $\varphi\big(t, (\widehat{K}\varsigma_0)(t), (S_h\varsigma_0)(t), \cdot\big)$ and from the obtained inequalities, it follows that for a.e. $t \in [0, T]$, the inclusion

$$0 \in \varphi\big(t, (\widehat{K}\varsigma_0)(t), (S_h\varsigma_0)(t), \ [\varsigma(t), \overline{\varsigma}(t)]\big)$$

holds. And due to this inclusion, according to the Filippov measurable selection lemma (see, e.g., [31] (Theorem 1.5.15)), there exists a measurable function $\varsigma_1 \in [\varsigma_0, \overline{\varsigma}]_L$ such that

$$\varphi\big(t, (\widehat{K}\varsigma_0)(t), (S_h\varsigma_0)(t), \varsigma_1(t)\big) = 0, \ \ t \in [0, T]. \tag{16}$$

So, for $i = 1$, relation (15) is proved.

Because the function $\varphi(t, \cdot, \cdot, \varsigma_1(t))$ is nonincreasing, in (16), we obtain

$$\varphi\big(t, (\widehat{K}\varsigma_1)(t), (S_h\varsigma_1)(t), \varsigma_1(t)\big) \leq 0, \ \ t \in [0, T].$$

Hence, ς_1 satisfies the same inequality as ς_0 does. Repeating the above reasoning, which defined the function ς_1 by the function ς_0, we find, by the function ς_1, a function ς_2 so that inequality (15) is valid for $i = 2$, etc. As a result, we obtain by induction a nondecreasing sequence $\{\varsigma_i\} \subset [\varsigma_{i-1}, \overline{\varsigma}]_L \subset [\varsigma_0, \overline{\varsigma}]_L$ such that relation (15) holds for all $i \in \mathbb{N}$.

Due to the regularity of the cone L_+, the nondecreasing and bounded sequence $\{\varsigma_i\}$ converges in the Banach space L to some function $\varsigma \in [\varsigma_0, \overline{\varsigma}]_L$. In view of the monotonicity of this sequence, $\varsigma_i(t)$ converges to $\varsigma(t)$ for a.e. $t \in [0, T]$. By the Lebesgue theorem on the passage to the limit under the integration sign, the convergence $\int_0^T \kappa(t,s)\varsigma_i(s)ds \to \int_0^T \kappa(t,s)\varsigma(s)ds$ takes place for a.e. $t \in [0, T]$. And because φ satisfies the Caratheodory conditions, we finally obtain

$$0 = \varphi\big(t, (\widehat{K}\varsigma_{i-1})(t), (S_h\varsigma_{i-1})(t), \varsigma_i(t)\big) \to \varphi\big(t, (\widehat{K}\varsigma)(t), (S_h\varsigma)(t), \varsigma(t)\big), \ \ t \in [0, T].$$

So, it is established that there exists a solution $\eta \in [0, \overline{\varsigma}]_L$ to Equation (13).

Next, let us show that the set of solutions to Equation (13) belonging to $[0, \overline{\varsigma}]_L$ is closed. Consider a sequence $\{\eta_i\}$ of solutions to this equation that converges in L to some function η. There exists a subsequence $\{\eta_{i_j}\} \subset [0, \overline{\varsigma}]_L$ convergent to η almost everywhere on $[0, T]$. Then, for a.e. $t \in [0, T]$, according to the Lebesgue theorem, the convergence $\int_0^T \kappa(t,s)\eta_i(s)ds \to \int_0^T \kappa(t,s)\eta(s)ds$ takes place. Because φ satisfies the Caratheodory conditions, we obtain

$$0 = \varphi\big(t, \int_0^T \kappa(t,s)\eta_i(s)ds, \eta_i(t)\big) \to \varphi\big(t, \int_0^T \kappa(t,s)\eta(s)ds, \eta(t)\big), \ \ t \in [0, T].$$

So, $\eta \in [0, \overline{\varsigma}]_L$ is a solution to Equation (13), and the closedness of the solution set is proved.

Due to the regularity of the cone L_+, in the set of solutions to Equation (13) belonging to the interval $[0, \overline{\varsigma}]_L$, there exists a minimal element; denote this element by ς^* and prove that it is the smallest element in the set of all solutions to Equation (13). Suppose the assertion is not true and there exists a solution $\varsigma \in L_+$ such that $\varsigma^* \not\leq \varsigma$. Define the sets $\mathfrak{T} := \{t \in [0,T] : \varsigma^*(t) \geq \varsigma(t)\}$, $\mathfrak{T}^* := \{t \in [0,T] : \varsigma^*(t) \leq \varsigma(t)\}$, and the function

$$\overline{\eta} \in [0, \overline{\varsigma}]_L, \;\; \overline{\eta}(t) = \inf\{\varsigma^*(t), \varsigma(t)\} = \begin{cases} \varsigma(t), & t \in \mathfrak{T}, \\ \varsigma^*(t), & t \in \mathfrak{T}^*. \end{cases}$$

For this function, the inequalities $\overline{\eta} < \varsigma^*$, $\overline{\eta} \leq \varsigma$ are valid. For $t \in \mathfrak{T}$, from the fact that the function φ is nonincreasing with respect to the second argument, we obtain

$$\begin{aligned} \varphi\big(t, (\widehat{K}\overline{\eta})(t), (S_h\overline{\eta})(t), \overline{\eta}(t)\big) &= \varphi\big(t, (\widehat{K}\overline{\eta})(t), (S_h\overline{\eta})(t), \varsigma(t)\big) \\ &\geq \varphi\big(t, (\widehat{K}\varsigma)(t), (S_h\varsigma)(t), \varsigma(t)\big) = 0. \end{aligned}$$

By analogy, for $t \in \mathfrak{T}^*$, we obtain

$$\begin{aligned} \varphi\big(t, (\widehat{K}\overline{\eta})(t), (S_h\overline{\eta})(t), \overline{\eta}(t)\big) &= \varphi\big(t, (\widehat{K}\overline{\eta})(t), (S_h\overline{\eta})(t), \varsigma^*(t)\big) \\ &\geq \varphi\big(t, (\widehat{K}\varsigma^*)(t), (S_h\varsigma^*)(t), \varsigma^*(t)\big) = 0. \end{aligned}$$

So, for the function $\overline{\eta}$, the inequality

$$\varphi\big(t, (\widehat{K}\overline{\eta})(t), (S_h\overline{\eta})(t), \overline{\eta}(t)\big) \geq 0$$

holds on the entire interval $[0, T]$. But according to the proof above, there should exist a solution $\nu \in [0, \overline{\eta}]_L$ to Equation (13) for which the inequalities $\nu \leq \overline{\eta} < \varsigma^*$ should hold. And this contradicts the fact that ς^* is a minimal element in the solution set. □

Let us demonstrate how the statement proved can be applied to the study of FDEs.

Example 5. *Consider the Cauchy problem with the initial condition $x(0) = 0$ for the following FDE:*

$$\sqrt[3]{\dot{x}(t)} - \dot{x}(\sqrt[3]{t}) - t^{-1}x^3(t/16) = y(t), \;\; t \in [0,1]. \tag{17}$$

Note that this equation contains a nonsummable on $[0,1]$ coefficient, namely the function t^{-1}, and, moreover, the divergent argument in the unknown function x is delayed, and the one in its derivative $\dot{x}$ is advanced. Show that for any measurable function $y : [0,1] \to \mathbb{R}$ such that

$$0 \leq y(t) \leq \frac{4\sqrt[3]{2} - 3}{8\sqrt[4]{t}}, \;\; t \in [0,1], \tag{18}$$

the problem has an absolutely continuous on $[0,1]$ solution x with derivative $\dot{x} \in L_+ := L([0,1], \mathbb{R}_+)$ satisfying the inequality

$$0 \leq \dot{x}(t) \leq \frac{1}{4\sqrt[4]{t^3}}, \;\; t \in [0,1],$$

and in the set of derivatives, there exists the smallest element.

An integral equation equivalent to this Cauchy problem with respect to the unknown $\varsigma = \dot{x} \in L_+$ has the form

$$\sqrt[3]{\varsigma(t)} - \varsigma(\sqrt[3]{t}) - \frac{1}{t}\Big(\int_0^{t/16} \varsigma(s)ds\Big)^3 - y(t) = 0, \;\; t \in [0,1]. \tag{19}$$

In the equation under consideration, the function κ is defined by the formula

$$\kappa(t,s) := \begin{cases} 1, & 0 \leq s \leq t/16 \leq 1, \\ 0, & 0 \leq t/16 < s \leq 1, \end{cases} \tag{20}$$

and hence, it is measurable, nonnegative, and essentially bounded. The function

$$\varphi(t,\sigma,\nu,\varsigma) = \sqrt[3]{\varsigma} - \nu - t^{-1}\sigma^3 - y(t) \tag{21}$$

generated by Equation (19) is measurable in t, continuous in the other arguments, decreases with respect to ν and σ, and satisfies the relation

$$\varphi(t,0,0,0) = -y(t) \le 0.$$

We complete the verification of the assumptions of Lemma 2 by proving the validity of inequality (14). Set

$$\bar{\varsigma}(t) := \frac{1}{4\sqrt[4]{t^3}}, \quad t \in [0,1].$$

For this function, in view of assumption (18) for $t \in [0,1]$, we have

$$\begin{aligned}\sqrt[3]{\bar{\varsigma}(t)} - \bar{\varsigma}(\sqrt[3]{t}) - \frac{1}{t}\Big(\int_0^{t/16} \bar{\varsigma}(s)ds\Big)^3 - y(t) &= \frac{1}{\sqrt[3]{4}\sqrt[4]{t}} - \frac{1}{4\sqrt[4]{t}} - \frac{1}{8\sqrt[4]{t}} - y(t)\\ &= \frac{4\sqrt[3]{2}-3}{8}\sqrt[4]{t} - y(t) \ge 0.\end{aligned}$$

So, all the conditions of Lemma 2 are fulfilled. According to this lemma, integral Equation (19) has a solution $\varsigma \in [0,\bar{\varsigma}]_L$, and among the solutions, there is the smallest one. Therefore, the Cauchy problem under consideration has an absolutely continuous on $[0,1]$ solution x with derivative $\dot{x} \in [0,\bar{\varsigma}]_L$, and in the set of derivatives to the solutions, there is the smallest element.

Let us now get back to the study of the general form of integral Equation (13). This equation can be written as operator Equation (3) with the map $g : L_+ \times L_+ \to W$ defined by relation

$$\forall \varsigma,\nu \in L_+ \ \ \big(g(\varsigma,\nu)\big)(t) := \varphi\big(t,(\widehat{K}\nu)(t),(S_h\nu)(t),\varsigma(t)\big), \ \ t \in [0,T]. \tag{22}$$

As it is shown in the statements below, the conditions of the proven here Lemma 2 guarantee, along with the solvability of Equation (13), the fulfillment of the assumptions of Lemma 1.

Lemma 3. *Let the function $\varphi(t,\cdot,\cdot,\varsigma) : \mathbb{R}_+ \to \mathbb{R}$ be nonincreasing for a.e. $t \in [0,T]$ and any $\varsigma \in \mathbb{R}_+$, and let $\varsigma^* \in L_+$ be the smallest solution of Equation (13). Then, the map $g : L_+ \times L_+ \to W$ defined by Formula (22) is closed with respect to the sets $\mathfrak{I}^* := [0,\varsigma^*]_L$ and $\{0\} \subset W$, for any $\nu \in \mathfrak{I}^*$, this map, as the map of the first argument, $g(\cdot,\nu) : L_+ \to W$ orderly covers the set $\{0\} \subset W$ on the set $\mathfrak{I}^*$, and as the map of the second argument, $g(\nu,\cdot) : L_+ \to W$ is antitone on $\mathfrak{I}^*$.*

Proof. Show that the map g is closed with respect to the sets $\mathfrak{I}^* := [0,\varsigma^*]_L$ and $\{0\} \subset W$. Take any two sequences $\{\eta_i\}, \{\eta_i'\} \subset \mathfrak{I}^* \subset L_+$ convergent in L to some function η and such that $g(\eta_i',\eta_i) = 0$. There exists a subsequence $\{\eta_{i_j}\} \subset \{\eta_i\}$ convergent to η almost everywhere on $[0,T]$. And from the sequence $\{\eta_{i_j}'\}$ convergent to η in L, we can choose a subsequence $\{\eta_{i_{j_p}}'\}$ convergent to η almost everywhere on $[0,T]$. The corresponding sequence $\{\eta_{i_{j_p}}\} \subset \{\eta_{i_j}\}$ also converges to η almost everywhere on $[0,T]$. For simplicity, we denote the subsequences constructed in this way by $\{\eta_p\}$ and $\{\eta_p'\}$.

In view of condition (12), from the convergence $\eta_p(t) \to \eta(t)$, $t \in [0,T]$, it follows that $(S_h\eta_p)(t) \to (S_h\eta)(t)$, $t \in [0,T]$. Next, according to the Lebesgue theorem, the convergence

$$\int_0^T \kappa(t,s)\eta_p(s)ds \to \int_0^T \kappa(t,s)\eta(s)ds$$

takes place for a.e. $t \in [0,T]$, and, because φ satisfies the Caratheodory conditions, the convergence

$$0 = g(\eta_p', \eta_p)(t) = \varphi\big(t, (\widehat{K}\eta_p)(t), (S_h\eta_p)(t), \eta_p'(t)\big) \to \\ \to \varphi\big(t, (\widehat{K}\eta)(t), (S_h\eta)(t), \eta(t)\big) = g(\eta, \eta)(t)$$

as well. Thus, it is established that the map g is closed with respect to the sets $\mathfrak{I}^*$ and $\{0\}$.

Next, we prove that for any $\nu \in \mathfrak{I}^*$, the map $g(\cdot, \nu) : L_+ \to W$ orderly covers the set $\{0\} \subset W$ on the set $\mathfrak{I}^*$. Let the inequality $g(\varsigma, \nu) \le 0$ hold for some $\varsigma \in \mathfrak{I}^*$. Then, for a.e. $t \in [0,T]$, the relations

$$\varphi\big(t, (\widehat{K}\nu)(t), (S_h\nu)(t), \varsigma(t)\big) \le 0,$$
$$\varphi\big(t, (\widehat{K}\nu)(t), (S_h\nu)(t), \varsigma^*(t)\big) \ge \varphi\big(t, (\widehat{K}\varsigma^*)(t), (S_h\varsigma^*)(t), \varsigma^*(t)\big) = 0$$

take place, from which it follows that the inclusion

$$0 \in \varphi\big(t, (\widehat{K}\nu)(t), (S_h\nu)(t), \big[\varsigma(t), \varsigma^*(t)\big]\big)$$

is valid for a.e. $t \in [0,T]$. Due to this inclusion, according to the Filippov measurable selection lemma, there exists a measurable function $\varsigma' \in [\varsigma, \varsigma^*]_L$ such that

$$\varphi\big(t, (\widehat{K}\nu)(t), (S_h\nu)(t), \varsigma'(t)\big) = 0, \;\; t \in [0,T] \iff g(\varsigma', \nu) = 0.$$

So, the map $g(\cdot, \nu)$ orderly covers the set $\{0\} \subset W$ on $[0, \overline{\varsigma}]_L$.

In conclusion of the proof, we note that from the fact that the function φ is nonincreasing in the second and third arguments it follows directly that, for every $\nu \in \mathfrak{I}^*$, the map $g(\nu, \cdot) : L_+ \to W$ is antitone on $\mathfrak{I}^*$. $\square$

We consider now Equation (11). This equation can be written in the form of operator Equation (2), where $\tilde{y} \in W^m$, and the map $G : L^n \times L^n \times L^n \to W^m$ is defined by relation

$$\forall u, v \in L^n \;\; \big(G(u,v)\big)(t) := \Phi\big(t, (Kv)(t)(S_hv)(t), u(t)\big), \;\; t \in [0,T]. \tag{23}$$

Suppose the given functions $u_0 \in L_n, \overline{e} \in L_+$. Let us formulate the sufficient conditions on the functions φ, Φ, under which map (22) majorizes map (23) on the ball $B_{L^n}(u_0, \overline{e})$.

Define functions $z_0 \in W^k$, $w_0 \in W^n$, and $\overline{r}, \overline{\varrho} \in W_+$ as follows

$$z_0(t) := (Ku_0)(t), \;\; w_0(t) := (S_hu_0)(t), \;\; \overline{r}(t) := (\widehat{K}\overline{e})(t), \;\; \overline{\varrho}(t) := (S_h\overline{e})(t) \;\; t \in [0,T]. \tag{24}$$

Definition 5. *We say that the functions φ, Φ satisfy the condition* $\mathrm{M1}[u_0, \overline{e}]$, *if for a.e. $t \in [0,T]$, the relation*

$$\forall \varsigma \in [0, \overline{r}(t)] \;\; \forall z \in B_{\mathbb{R}^k}\big(z_0(t), \varsigma\big) \;\; \forall \sigma \in [0, \overline{\varrho}(t)] \;\; \forall w \in B_{\mathbb{R}^n}\big(w_0(t), \sigma\big)$$
$$\forall e \in [0, \overline{e}(t)] \;\; \forall u \in B_{\mathbb{R}^n}\big(u_0(t), e\big) \;\; \forall \Delta \in \big[0, \overline{e}(t) - e\big]$$
$$\big|\Phi(t, z, w, u)\big|_{\mathbb{R}^m} \le \varphi(t, \varsigma, \sigma, e + \Delta) - \varphi(t, \varsigma, \sigma, e)$$
$$\implies \exists v \in B_{\mathbb{R}^n}\big(u_0(t), \overline{e}(t)\big) \;\; \Phi(t, z, w, v) = 0, \;\; |u - v|_{\mathbb{R}^n} \le \Delta,$$

holds, and the condition $\mathrm{M2}[u_0,\bar{e}]$, *if for a.e.* $t \in [0,T]$, *the relation*

$$\begin{aligned}
&\forall e \in [0,\bar{e}(t)]\ \forall u \in B_{\mathbb{R}^n}(u_0(t),e) \\
&\forall \varsigma \in [0,\bar{r}(t)]\ \forall z \in B_{\mathbb{R}^k}(z_0(t),\varsigma)\ \forall \varsigma' \in [0,\varsigma]\ \forall z' \in B_{\mathbb{R}^k}(z_0(t),\varsigma') \\
&\forall \sigma \in [0,\bar{\varrho}(t)]\ \forall w \in B_{\mathbb{R}^n}(w_0(t),\sigma)\ \forall \sigma' \in [0,\sigma]\ \forall w' \in B_{\mathbb{R}^n}(w_0(t),\sigma')
\end{aligned}$$

$$|z-z'|_{\mathbb{R}^k} \le \varsigma - \varsigma',\ |w-w'|_{\mathbb{R}^k} \le \sigma - \sigma'$$
$$\Longrightarrow\ \big|\Phi(t,z,w,u) - \Phi(t,z',w',u)\big|_{\mathbb{R}^m} \le \varphi(t,r',\sigma',e) - \varphi(t,r,\sigma,e)$$

is valid.

Lemma 4. *Let the functions* $u_0 \in L_n, \bar{e} \in L_+$ *be given. If the functions* φ, Φ *satisfy the condition* $\mathrm{M1}[u_0,\bar{e}]$, *then for the maps* (22) *and* (23), *relation* (6) *holds. If the functions* φ, Φ *satisfy the condition* $\mathrm{M2}[u_0,\bar{e}]$, *then for the maps* (22) *and* (23), *relation* (7) *takes place. And if both conditions* $\mathrm{M1}[u_0,\bar{e}]$ *and* $\mathrm{M2}[u_0,\bar{e}]$ *are fulfilled, then map* (22) *majorizes map* (23) *on the ball* $B_{L^n}(u_0,\bar{e})$.

Proof. Let φ, Φ satisfy the condition $\mathrm{M1}[u_0,\bar{e}]$. We verify relation (6) for the maps g, G defined by Formulas (22) and (23), respectively.

Take arbitrary $e \in [0,\bar{e}]_{L_+}$, $u \in B_{L_n}(u_0,e)$, $\Delta \in [0,\bar{e}-e]_{L_+}$. Let the inequality

$$\mathcal{P}^{W}_{W^m}\big(\tilde{y}, G(u,u)\big) \le g(e+\Delta,e) - g(e,e)$$

hold; for Equation (11) in question, it takes the form

$$\big|\Phi(t,z(t),w(t),u(t))\big|_{\mathbb{R}^m} \le \varphi\big(t,\varsigma(t),\sigma(t),e(t)+\Delta(t)\big) - \varphi\big(t,\varsigma(t),\sigma(t),e(t)\big),$$

where $z(t) := (Ku)(t)$, $\varsigma(t) := (\widehat{K}e)(t)$, $w(t) := (S_h u)(t)$, $\sigma(t) := (S_h e)(t)$. Note that

$$\begin{aligned}
\big|z(t) - z_0(t)\big|_{\mathbb{R}^m} &= \Big|\int_0^T \mathcal{K}(t,s)\big(u(s)-u_0(s)\big)ds\Big|_{\mathbb{R}^m} \le \int_0^T \kappa(t,s)\big|u(s)-u_0(s)\big|_{\mathbb{R}^n}ds \\
&\le \int_0^T \kappa(t,s)\bar{e}(s)ds = \bar{r}(t), \qquad \varsigma(t) = \int_0^T \kappa(t,s)e(s)ds \le \int_0^T \kappa(t,s)\bar{e}(s)ds = \bar{r}(t), \\
\big|w(t) - w_0(t)\big|_{\mathbb{R}^n} &= \big|(S_h u)(t) - (S_h u_0)(t)\big|_{\mathbb{R}^n} \le (S_h \bar{e})(t) = \bar{\varrho}(t), \ \ \sigma(t) \le (S_h \bar{e})(t) = \bar{\varrho}(t).
\end{aligned}$$

Due to the condition $\mathrm{M1}[u_0,\bar{e}]$, for a.e. $t \in [0,T]$, there exists a $v \in B_{\mathbb{R}^n}\big(u_0(t),\bar{e}(t)\big)$ such that $\Phi(t,z(t),w(t),v) = 0$ and $|v - u(t)|_{\mathbb{R}^n} \le \Delta(t)$. This implies the inclusion

$$0 \in \Phi\big(t,(Ku)(t)ds,(S_h u)(t),B_{\mathbb{R}^n}\big(u(t),\Delta(t)\big)\big),\ \ t \in [0,T],$$

from which, according to the Filippov measurable selection lemma, it follows that for some measurable function $v \in B_{L^n}(u,\Delta)$, the relation

$$\big(G(v,u)\big)(t) = \Phi\big(t,(Ku)(t)ds,(S_h u)(t),v(t)\big) = 0,\ \ t \in [0,T],$$

holds. So, (6) holds for the maps g, G defined by Formulas (22) and (23).

The second assertion of the lemma is straightforward: from condition $\mathrm{M2}[u_0,\bar{e}]$ it follows directly that, for the maps g, G under consideration, (7) is valid. And meeting both conditions (6) and (7) means that the map g majorizes G. $\square$

Lemmas 3 and 4 allow for applying Theorem 1 to Equations (11) and (13). Thus, we obtain the following statement.

Theorem 2. *Given $u_0 \in L_n$ and $\bar{e} \in L_+$, let the functions φ, Φ satisfy the conditions* M1$[u_0, \bar{e}]$, M2$[u_0, \bar{e}]$, *let the function $\varphi(t, \cdot, \varsigma) : \mathbb{R}_+ \to \mathbb{R}$ be nonincreasing for a.e. $t \in [0, T]$ and any $\varsigma \in \mathbb{R}_+$, and let the inequality*

$$\left|\Phi\big(t, (Ku_0)(t), (S_h u_0)(t), u_0(t)\big)\right| \le -\varphi(t, 0, 0, 0) \tag{25}$$

hold. Then, if the set of solutions to Equation (13) is not empty, and $\varsigma^* \in E_+$ is its smallest element, then there exists a solution $u^* \in B_{L^n}(u_0, \varsigma^*)$ to Equation (11).

Let us give an example of applying Theorem 2 to the study of FDEs.

Example 6. *Consider the Cauchy problem with respect to the unknown function $x = (x_1, x_2)$ for a system of two equations of the form*

$$\begin{gathered}\sqrt[3]{\dot{x}_i(t)} - \sum_{j=1,2} p_{ij}(t)\dot{x}_j(\sqrt[3]{t}) - t^{-1} \sum_{j=1,2} q_{ij}(t)x_j(g_{ij}(t)) = \tilde{y}_i(t), \;\; t \in [0,1], \;\; i = 1,2, \\ x_i(s) = 0, \;\; s \notin [0,1], \;\; i = 1,2,\end{gathered} \tag{26}$$

with the homogeneous initial condition

$$x_1(0) = x_2(0) = 0. \tag{27}$$

Show that for any measurable functions $\tilde{y}_i, p_{ij}, q_{ij}, g_{ij} : [0,1] \to \mathbb{R}$, $i, j = 1,2$ satisfying for a.e. $t \in [0,1]$ the inequalities

$$|\tilde{y}_i(t)| \le \frac{4\sqrt[3]{2} - 3}{8\sqrt[4]{t}}, \;\; |p_{i1}(t) + p_{i2}(t)| \le 1, \;\; |q_{i1}(t) + q_{i2}(t)| \le 1, \;\; g_{ij}(t) \le \frac{t}{16},$$

the problem under consideration has an absolutely continuous on $[0,1]$ solution x with derivative $\dot{x} \in L^2$ whose components $\dot{x}_1, \dot{x}_2$ satisfy the inequality

$$|\dot{x}_i(t)| \le \dot{x}_i^*(t) \le \frac{1}{4\sqrt[4]{t^3}}, \;\; t \in [0,1], \;\; i = 1,2,$$

where $\dot{x}^* \in L_+$ is the smallest element in the set of derivatives of the solutions of the Cauchy problem in Example 5 for Equation (17) in the case of

$$y(t) = \frac{4\sqrt[3]{2} - 3}{8\sqrt[4]{t}}, \;\; t \in [0,1]. \tag{28}$$

The integral equation (with respect to the unknown $u = \dot{x}$, $u = (u_1, u_2)$) equivalent to the problems (26) and (27) has the form

$$\sqrt[3]{u(t)} - P(t)u(t^3) - \frac{1}{t}\Big(\int_0^1 \mathcal{K}(t,s)u(s)ds\Big)^3 - \tilde{y}(t) = 0, \;\; t \in [0,1], \tag{29}$$

where

$$\mathcal{K}(t,s) = \big(\mathcal{K}_{ij}(t,s)\big)_{2\times 2}, \mathcal{K}_{ij}(t,s) = \begin{cases} q_{ij}(t) & \text{if } 0 \le s \le g_{ij}^+(t), \\ 0 & \text{if } g_{ij}^+(t) < s \le 1, \end{cases} \;\; g_{ij}^+(t) = \max\{g_{ij}(t), 0\};$$

$$P(t) = \big(p_{ij}(t)\big)_{2\times 2}.$$

We compare this equation with "model" integral Equation (19) (see Example 5), the right-hand side of which is defined by relation (28).

In $\mathbb{R}^2$, *we define a norm of an element, a vector* $v = (v_1, v_2)$, *by the formula* $|v|_{\mathbb{R}^2} = \max\{|v_1|, |v_2|\}$. *Due to the assumptions made, we have*

$$|P(t)|_{\mathbb{R}^2\times\mathbb{R}^2} \le 1 \text{ and } |\mathcal{K}(t,s)|_{\mathbb{R}^2\times\mathbb{R}^2} \le \kappa(t,s),$$

where the function κ *is given by* (20). *The function* $\Phi : [0,1]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$ *generated by Equation* (29) *is defined as*

$$\Phi(t,z,w,u) = \sqrt[3]{u} - P(t)w - t^{-1}z^3 - \tilde{y}(t).$$

Let us show that this function and the function φ *given in Example* 5 *by* (21) *satisfy the conditions* $\mathrm{M1}[u_0,\bar{e}]$ *and* $\mathrm{M2}[u_0,\bar{e}]$ *for* $u_0 = 0 \in L^2$ *and any function* $\bar{e} \in L_+$.

For arbitrary $t \in [0,1]$, $e \in [0,\bar{e}(t)]$, *and* $u \in B_{\mathbb{R}^2}(0,e)$, *let the inequality*

$$|\Phi(t,z,w,u)|_{\mathbb{R}^2} \le \varphi(t,\varsigma,\sigma,e+\Delta) - \varphi(t,\varsigma,\sigma,e)$$

be valid. Then, according to the definition of the norm in $\mathbb{R}^2$, *the relation*

$$|\sqrt[3]{u_i} - A_i| \le \sqrt[3]{e+\Delta} - \sqrt[3]{e}$$

holds for $i = 1,2$, *where* A_i *is the i-th component* ($i = 1,2$) *of the vector* $A := P(t)w + t^{-1}z^3 + \tilde{y}(t)$. *Obviously, there exists a unique vector* $v = (v_1, v_2) \in \mathbb{R}^2$ *such that* $\sqrt[3]{v_i} = A_i$, *and therefore the latter inequality can be written as* $|\sqrt[3]{u_i} - \sqrt[3]{v_i}| \le \sqrt[3]{e+\Delta} - \sqrt[3]{e}$. *Because* $|u_i| \le e$, *due to the upward convexity of the function* $\mathbb{R}_+ \ni e \to \sqrt[3]{e} \in \mathbb{R}_+$, *we obtain* $|u_i - v_i| \le \Delta$. *So,* $|u - v|_{\mathbb{R}^2} \le \Delta$, *and condition* $\mathrm{M1}[u_0,\bar{e}]$ *is fulfilled for the functions* φ, Φ.

Now, let us verify the validity of condition $\mathrm{M2}[u_0,\bar{e}]$. *According to* (24), *define*

$$z_0(t) = 0,\;\; w_0(t) = 0,\;\; \bar{r}(t) = \int_0^{t/16} \bar{e}(s)ds,\;\; \bar{\varrho}(t) = \bar{e}(\sqrt[3]{t}),\;\; t \in [0,1].$$

Let for arbitrary $0 \le \varsigma' \le \varsigma \le \bar{r}(t)$, $z \in B_{\mathbb{R}^2}(0,\varsigma)$, $z' \in B_{\mathbb{R}^2}(0,\varsigma')$ *and arbitrary* $0 \le \sigma' \le \sigma \le \bar{\varrho}(t)$, $w \in B_{\mathbb{R}^2}(0,\sigma)$, $w' \in B_{\mathbb{R}^2}(0,\sigma')$, *the inequalities*

$$|z - z'|_{\mathbb{R}^2} \le \varsigma - \varsigma',\;\; |w - w'|_{\mathbb{R}^2} \le \sigma - \sigma'$$

hold. We have

$$\begin{aligned}|\Phi(t,z,w,u) - \Phi(t,z',w',u)|_{\mathbb{R}^2} &= |P(t)(w-w') + t^{-1}(z^3 - z'^3)|_{\mathbb{R}^2}\\ &\le |w-w'|_{\mathbb{R}^2} + t^{-1}|z^3 - z'^3|_{\mathbb{R}^2} \le \sigma - \sigma' + \max_{i=1,2}|z_i^3 - {z'}_i^3|\\ &\le \sigma - \sigma' + \max_{i=1,2}\left(|z_i - z_i'|\left(|z_i|^2 + |z_i||z_i'| + |z_i'|^2\right)\right)\\ &\le \sigma - \sigma' + (\varsigma - \varsigma')(\varsigma^2 + \varsigma\varsigma' + \varsigma'^2).\end{aligned}$$

Hence,

$$|\Phi(t,z,w,u) - \Phi(t,z',w',u)|_{\mathbb{R}^2} \le \sigma - \sigma' + \varsigma^3 - \varsigma'^3 = \varphi(t,r',\sigma',e) - \varphi(t,r,\sigma,e),$$

and therefore condition $\mathrm{M2}[u_0,\bar{e}]$ *is satisfied for the functions* φ, Φ.

Concluding the verification of the assumptions of Theorem 2, *note that inequality* (25) *is valid. Indeed,*

$$|\Phi(t,(Ku_0)(t),(S_h u_0)(t),u_0(t))| = |\tilde{y}(t)| \le \frac{4\sqrt[3]{2} - 3}{8\sqrt[4]{t}} = -\varphi(t,0,0,0).$$

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