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Global Existence, Blowup, and Asymptotic Behavior for a Kirchhoff-Type Parabolic Problem Involving the Fractional Laplacian with Logarithmic Term

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Abstract: In this paper, we studied a class of semilinear pseudo-parabolic equations of the Kirchhoff type involving the fractional Laplacian with logarithmic nonlinearity: $\begin{cases} u_t + M([u]_s^2)(-\Delta)^s u + (-\Delta)^s u_t = |u|^{p-2}u\ln|u|, & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$, where $[u]_s$ is the Gagliardo semi-norm of u, $(-\Delta)^s$ is the fractional Laplacian, $s \in (0,1)$, $2\lambda < p < 2_s^* = 2N/(N-2s)$, $\Omega \in \mathbb{R}^N$ is a bounded domain with $N > 2s$, and u_0 is the initial function. To start with, we combined the potential well theory and Galerkin method to prove the existence of global solutions. Finally, we introduced the concavity method and some special inequalities to discuss the blowup and asymptotic properties of the above problem and obtained the upper and lower bounds on the blowup at the sublevel and initial level.

Keywords: parabolic; Kirchhoff type; logarithmic; Galerkin method; potential wells

MSC: 35R11; 35K92; 47G20

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1. Introduction

We deal with the following fractional Kirchhoff-type semilinear pseudo-parabolic problem involving logarithmic nonlinearity:

$$\begin{cases} u_t + M([u]_s^2)\mathscr{L}_K u + \mathscr{L}_K u_t = f(u), & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $f(u) = |u|^{p-2}u\ln|u|$ and the Kirchhoff function $M(t) = t^{\lambda-1}$ with $t \in \mathbb{R}_0^+$ and $\lambda \in [1, \frac{2_s^*}{2})$ for $2_s^* = 2N/(N-2s)$. For convenience, we set the functions:

$$\mathscr{T}^{\varphi}(x,y) = |\varphi(x) - \varphi(y)|^2 K(x,y),$$

$$\mathscr{T}^{\varphi,\phi}(x,y) = (\varphi(x) - \varphi(y))(\phi(x) - \phi(y))K(x,y).$$

As a non-local integration operator, $\mathscr{L}_K$ satisfies:

$$\mathscr{L}_K\varphi(x) = 2\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus D_\omega(x)} \mathscr{T}^{\varphi}(x,y)dy,$$

$$[\varphi]_s = \left(\iint_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy\right)^{1/2},$$

for $\forall\varphi \in C_0^\infty(\mathbb{R}^N)$, where $D_\omega(x)$ refers to a sphere in $\mathbb{R}^N$ with $x \in \mathbb{R}^N$ as the center and $\omega > 0$ as the radius. The function $K : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^+$ satisfies: $K(x) \geq m|x|^{-(N+2s)}$ for $\forall x \in \mathbb{R}^N \backslash \{0\}$, where m is a positive number and $s \in (0,1)$, so that $K_0 K \in L^1(\mathbb{R}^N)$ when $K_0(x) = \min\{|x|^2, 1\}$. Usually, we set $K(x) = |x|^{-(N+2s)}$ to meet the above conditions. Ergo, it can be inferred that $\mathscr{L}_K u = (-\Delta)^s u$ for $\forall u \in C_0^\infty(\mathbb{R}^N)$. For more-relevant details about the fractional Laplacian and fractional Sobolev space, we can refer to the literature [1,2].

In recent years, research on the problem of parabolic equations with the fractional Laplacian and Kirchhoff term has been a hot topic. In [3], the prototype of the Kirchhoff termcan be traced back to 1883:

$$\chi \frac{\partial^2 u}{Bythedescriptionoft^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u(x)}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

which described the physical phenomenon of elastic string vibration. As a result, more and more scholars are attempting to introduce the Kirchhoff model into the study of parabolic equations, obtaining many interesting results and more-complex changes. In [4], the authors put forward the following Kirchhoff-type problems with a non-local integral operator:

$$-M(\|u\|_Z^2)\mathscr{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u; \tag{2}$$

here, 2_* is equal to 2_s^* in this article. (2) imposes a special constraint on f when proving the existence of non-negative solutions, while considered an auxiliary problem with

$$M_a(t) = \begin{cases} M(t), & \text{if } 0 \leq t \leq t_0, \\ 0, & \text{if } t \geq t_0. \end{cases}$$

Application and research on the Kirchhoff term can be found in [4–12], where we note that, in each of these papers, the authors gave the following restrictions to the Kirchhoff function:

$(M_0) M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous and non-decreasing function.
$(M_1) M(t) \geq a,$ where $a > 0$, for $\forall t \in \mathbb{R}_0^+$.

We let $M(t) = a + b\,t^{\lambda-1}$ $(t \geq 1)$ meet the conditions M_0 and M_1, where $a \geq 0$ and $b > 0$. Specifically, in this article, we set $a = 0$, $b = 1$, and $\lambda \in [1, \frac{2_s^*}{2})$.

In [13], since Sattinger introduced the theory of potential wells in the construction of the global existence of the solution for hyperbolic equations, a growing number of authors have introduced the theory of potential wells in the study of various properties of solutions of parabolic equations; see [5–8,14]. On the other hand, Levine established the concavity method in [15,16]. In [5], Pan and Zhang opened up a way of investigating the nature of Kirchhoff-type parabolic problems containing the fractional p-Laplacian when they investigated the existence of global solutions at sublevel ($\mathscr{H}(u_0) < d$) and critical energy level ($\mathscr{H}(u_0) = d$) for (3), combining, for the first time, the theory of the potential wells and the Galerkin method:

$$u_t + [u]_{s,p}^{(\lambda-1)p}(-\triangle)_p^s u = |u|^{q-2}u, \tag{3}$$

where $p < q < NP/(N - sp)$ with $1 < p < N/s$ and $1 \leq \lambda < N/(N - sp)$. In [9], Yang and Tian took a deeper look at (3) by letting p and q satisfy $2 < p\lambda < q < Np/(N - sp)$ with $1 \leq \lambda < N/(N - sp)$. They obtained the blowup properties and asymptotic behavior of the weak solutions at the sublevel and critical energy level by means of the potential well theory, the concavity method, and some inequality tricks. In [10], Zhang and Xiang investigated the burstiness of non-negative solutions at sublevel ($\mathscr{H}(u_0) < d$), critical ($\mathscr{H}(u_0) = d$), and supercritical ($\mathscr{H}(u_0) > 0$) in $p = 2$, in addition to obtaining the corresponding upper and lower bounds on the blowup at different energy levels. We can also see [11,12,17,18] for more details on the application of these two methods.

In [19], Ding and Zhou made $p = 2$ and replaced the polynomial term at the right of Equation (3) with the logarithmic nonlinear term:

$$u_t + M([u]_s^2)\mathscr{L}_K u = |u|^{q-2}u \ln |u|; \tag{4}$$

at this point, the Kirchhoff term $M(t) = a + bt^{\lambda-1}(a \geq 0, b > 0)$ was taken. In order to analyze the effect of the logarithmic terms on (4), the logarithmic fractional-order Sobolev spaces were introduced, and some inequality tricks were cleverly used to analyze the problem in depth and to obtain the global existence, invariance of the region, blowup, and asymptotic behavior. In [20], the authors also considered (4), with the difference that the Kirchhoff function is an unknown function, and they used differential inequality techniques to overcome these difficulties to obtain upper and lower bounds for the blowup.

For the problem:

$$u_t - \triangle u_t - \triangle u = u^p,$$

the authors studied the initial-boundary-value problem with subcritical level $\mathscr{H}(u_0) < d$ for $\mathscr{P}(u_0) < 0$ and $\mathscr{P}(u_0) > 0$, critical level $\mathscr{H}(u_0) = d$ with $\mathscr{P}(u_0) \geq 0$, and high initial energy $\mathscr{H}(u_0) > d$ and also introduced invariants for three sets $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{G}_0$. Moreover, to learn more about the nature of solutions and the definition of the sets, we can refer to [21]. In [22], Chen and Tian introduced a logarithmic term on the above model to obtain the following semilinear pseudo-parabolic equation:

$$u_t - \triangle u_t - \triangle u = u^p \ln |u|;$$

for the above model, the authors utilized a modified potential well theory and the definition of the logarithmic Sobolev space to obtain quite different results from parabolic equations containing polynomial nonlinear terms. The details with logarithmic Sobolev spaces can be found in [7,8,12,19,23–25].

Inspired by the above work, we added a fractional-order nonlinear dissipative term $(-\Delta)^s u_t$ to (4) and let $M(t) = t^{\lambda-1}$, different from the Kirchhoff function considered in [19]. In the subsequent proofs, we introduce the correlation function $\mathscr{P}_\iota(u)$, as well as the new set of potential wells Ψ_ι and a tighter control of the logarithmic terms. In this article, we considered the problem (1). In Section 2, we give the definition and related properties of the logarithmic fractional Sobolev space. In Section 3, we give the modified potential well theory and some necessary Lemmas. In Section 4, we construct an approximate solution to the problem (1) using the Galerkin method. In Section 5, we focus on proving the existence of global solutions when $\mathscr{H}(u_0) = d$ for $\mathscr{P}(u_0) > 0$ or $0 \leq \mathscr{H}(u_0) \leq d$ for $\mathscr{P}(u_0) = 0$. In Section 6, we prove the finite-time blowup at subcritical ($\mathscr{H}(u_0) < d$) and critical ($\mathscr{H}(u_0) = d$) energy levels and derive the corresponding upper and lower bounds. At the same time, we obtain the asymptotic behaviors of the global solutions. In Section 7, we give an example to illustrate our results. In Section 8, we provide a conclusion of the entire article.

2. Preliminaries

In the following, we first give some necessary definitions about fractional Sobolev spaces and related properties, and we can refer to [26,27] for more details.

Now, we introduce some definitions. We define $L^\gamma(\Omega)$ to be the usual Lebesgue space for $\gamma \geq 1$ with the norm:

$$\|u\|_\gamma = \left(\int_\Omega |u|^\gamma dx\right)^{1/\gamma};$$

in particular, when $\gamma = 2$, we define the inner-product in the following form:

$$(u, v) = \int_\Omega uv dx.$$

In the following, let $0 < s < 1$ and define the fractional critical exponent 2_s^* by

$$2_s^* = \begin{cases} \dfrac{2N}{N-2s}, & \text{if } 2s < N, \\ \infty, & \text{if } 2s \geq N. \end{cases}$$

Put $Q = \mathbb{R}^N \backslash \mathfrak{O}$, where $\mathfrak{O} = \mathfrak{C}(\Omega) \times \mathfrak{C}(\Omega) \subset \mathbb{R}^{2N}$ and $\mathfrak{C}(\Omega) = \mathbb{R}^N \backslash \Omega$. We considered the fractional Sobolev space Ψ satisfying the Lebesgue measurable functions u from $\mathbb{R}^N$ to $\mathbb{R}$, i.e.,

$$\iint_Q \mathscr{T}^u(x,y)dxdy < \infty.$$

The space Ψ is prescribed the norm:

$$\|u\|_\Psi = \left(\|u\|^2_{L^2(\Omega)} + \iint_Q \mathscr{T}^u(x,y)dxdy \right)^{1/2}.$$

We considered the closed linear subspace:

$$\Psi_0 = \{u \in \Psi : u(x) = 0 \quad a.e. \text{ in } \partial\Omega\},$$

its norm being defined as

$$\|u\|_{\Psi_0} = \left(\iint_Q \mathscr{T}^u(x,y)dxdy \right)^{\frac{1}{2}}. \tag{5}$$

The function space Ψ_0 denotes that

$$\Psi_0 = \overline{C_0^\infty(\Omega)}^{\Psi}.$$

For all $u,\ v \in \Psi_0$, we define

$$(u,v)_{\Psi_0} = \iint_Q \mathscr{T}^{u,v}(x,y)dxdy.$$

From now on, we will only consider the general case where $K(x-y) = |x-y|^{-(N+2s)}$, and more relevant details can be found in [27].

Lemma 1. *(i) There exists $\sigma = \sigma(N,\nu,s) > 0$, where $\nu \in [1, 2_s^*]$, such that, for arbitrary $v \in \Psi_0$,*

$$\|v\|^2_{L^\nu(\Omega)} \leq \sigma \iint_{\Omega\times\Omega} \mathscr{T}^v(x,y)dxdy \leq \frac{\sigma}{\beta} \iint_Q \mathscr{T}^v(x,y)dxdy.$$

(ii) There exists $\widetilde{\sigma} = \widetilde{\sigma}(N,s,\beta,\Omega) > 0$ such that, for arbitrary $v \in \Psi_0$,

$$\iint_Q \mathscr{T}^v(x,y)dxdy \leq \|v\|^2_\Psi \leq \widetilde{\sigma} \iint_Q \mathscr{T}^v(x,y)dxdy.$$

(iii) For any bounded sequence $(v_j)_j$ in Ψ_0, there exists $v \in L^\nu(\mathbb{R}^N)$, with $v = 0$ a.e. in $\partial\Omega$, such that, up to a subsequence, still denoted by $(v_j)_j$,

$$v_j \to v \quad \textit{strongly in } L^\nu(\Omega) \textit{ as } j \to \infty,$$

for any $\nu \in [1, 2_s^*)$.

Definition 1 ([28])**.** *(Maximal existence time) T for which u is a weak solution of Equation (1) and satisfies the following two conditions is called the maximal existence time:*

(1) If $u(t)$ exists for $\forall t \in [0, +\infty)$, then $T = +\infty$.

(2) *Let $t_0 \in (0, +\infty)$ and $u(t)$ exist for $0 \le t < t_0$, but be non-existent at t_0, so that $T = t_0$.*

3. The Potential Well

In the following, we will give some notations and Lemmas. First of all, we define

$$\mathscr{H}(u) = \frac{1}{2\lambda}\|u\|_{\Psi_0}^{2\lambda} - \frac{1}{p}\int_\Omega |u|^p \ln|u|dx + \frac{1}{p^2}\|u\|_p^p, \tag{6}$$

and

$$\mathscr{P}(u) = \|u\|_{\Psi_0}^{2\lambda} - \int_\Omega |u|^p \ln|u|dx. \tag{7}$$

A definition of potential well as followsin Equation (1) is defined as follows:

$$\Psi = \{u(x) \in \Psi_0 | \mathscr{P}(u) > 0,\ \mathscr{H}(u) < d\} \cup \{0\};$$

the external set Θ is indicated as

$$\Theta = \{u(x) \in \Psi_0 | \mathscr{P}(u) < 0,\ \mathscr{H}(u) < d\},$$

where

$$d = \inf_{u \in \mathscr{I}} \mathscr{H}(u), \tag{8}$$

denotes the depth of the potential well and the Nehari manifold is indicated as

$$\mathscr{I} = \{u \in \Psi_0 | \mathscr{P}(u) = 0,\ u \ne 0\}.$$

Moreover, the positive set and negative set are represented as

$$\mathscr{I}_+ = \{u \in \Psi_0 | \mathscr{P}(u) > 0\},$$
$$\mathscr{I}_- = \{u \in \Psi_0 | \mathscr{P}(u) < 0\}.$$

Obviously, from (6) and (7), we have

$$\mathscr{H}(u) = \frac{1}{p}\mathscr{P}(u) + (\frac{1}{2\lambda} - \frac{1}{p})\|u\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u\|_p^p. \tag{9}$$

Moreover, for $\forall \iota \in [0, \infty)$, we set

$$\begin{aligned} \mathscr{P}_\iota(u) &= \iota\|u\|_{\Psi_0}^{2\lambda} - \int_\Omega |u|^p \ln|u|dx, \\ \delta(\iota, \varepsilon) &= (\frac{\iota \varepsilon e}{E_*^{p+\varepsilon}})^{\frac{1}{p+\varepsilon-2\lambda}}, \end{aligned} \tag{10}$$

where $2\lambda < p + \varepsilon < 2_s^*$ and E_* is the optimal embedding constant for embedding Ψ_0 into $L^{p+\varepsilon}$, i.e.,

$$E_* = \sup_{u \in \Psi_0 \setminus \{0\}} \frac{\|u\|_{p+\varepsilon}}{\|u\|_{\Psi_0}}.$$

We impose a new series of potential wells such that

$$\Psi_\iota = \{u(x) \in \Psi_0(\Omega) | \mathscr{P}_\iota(u) > 0,\ \mathscr{H}(u) < d(\iota)\} \cup \{0\},$$

$$\Theta_\iota = \{u(x) \in \Psi_0(\Omega) | \mathscr{P}_\iota(u) < 0,\ \mathscr{H}(u) < d(\iota)\},$$

where

$$d(\iota) = \inf_{u \in \mathscr{I}} \mathscr{H}(u),$$

and

$$\mathscr{I} = \{u \in \Psi_0 | \mathscr{P}_\iota(u) = 0, u \neq 0\}.$$

Specifically, we can substitute (10) for (9):

$$\mathscr{H}(u) = \frac{1}{p}\mathscr{P}_\iota(u) + (\frac{1}{2\lambda} - \frac{\iota}{p})\|u\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u\|_p^p. \tag{11}$$

Definition 2. *$u = u(t)$ is named a weak solution of the problem (1), if $u(t) \in L^\infty(0, \infty; \Psi_0)$ with $u_t \in L^2(0, \infty; L^2(\Omega))$ and it satisfies the following equation*

$$\int_\Omega u_t v dx + \langle u, v\rangle_{\Psi_0} + (u_t, v)_{\Psi_0} = \int_\Omega |u|^{p-2} u \ln|u| v dx,$$

where

$$\langle u, v\rangle_{\Psi_0} = M([u]_s^2) \iint_Q \mathscr{T}^{u,v}(x,y) dxdy,$$

$$(u_t, v)_{\Psi_0} = \iint_Q \mathscr{T}^{u_t,v}(x,y) dxdy,$$

for any $v \in \Psi_0$.

Lemma 2. *Let ε be a positive number; we can obtain*

$$\ln s \leq \frac{1}{e\varepsilon} s^\varepsilon, \quad \forall s \in [1, +\infty).$$

Proof. Let $g(s) = \ln s - \frac{1}{e\varepsilon} s^\varepsilon$ for all $s \geq 1$. Clearly, g attains its maximum value at $s_* = e^{\frac{1}{\varepsilon}}$; thus, $g(s) \leq g(s_*) = 0$ for all $s \geq 1$. □

Lemma 3. *Let $u \in \Psi_0 \backslash \{0\}$, and consider a function l:$\varpi \mapsto \mathscr{H}(\varpi u)$ for $\forall \varpi > 0$:*

(1) $\lim\limits_{\varpi \to 0^+} l(\varpi) = 0$, $\lim\limits_{\varpi \to +\infty} l(\varpi) = -\infty$.

(2) *Function $l(\varpi)$ is strictly monotonically increasing on $(0, \varpi^*)$, strictly monotonically decreasing on (ϖ^*, ∞) for unique ϖ^*, and* $\max l(\varpi) = l(\varpi^*)$.

(3) $\mathscr{P}(\varpi u) > 0$ *for* $\varpi \in (0, \varpi^*)$, $\mathscr{P}(\varpi u) < 0$ *for* $\varpi \in (\varpi^*, \infty)$, *and* $\mathscr{P}(\varpi^* u) = 0$.

Proof. (1) By the description of $\mathscr{H}(u)$ in (6), we have

$$l(\varpi) = \frac{\varpi^{2\lambda}}{2\lambda}\|u\|_{\Psi_0}^{2\lambda} - \frac{\varpi^p}{p}\int_\Omega |u|^p \ln|u| dx - \frac{\varpi^p}{p}\ln\varpi\|u\|_p^p + \frac{\varpi^p}{p^2}\|u\|_p^p.$$

Obviously, (1) holds.

(2) By simple calculations, we have

$$l'(\varpi) = \varpi^{2\lambda-1}\left(\|u\|_{\Psi_0}^{2\lambda} - \varpi^{p-2\lambda}\int_\Omega |u|^p \ln|u| dx - \varpi^{p-2\lambda}\ln k\|u\|_p^p\right).$$

Set $o(\varpi) = \varpi^{1-2\lambda} l'(\varpi)$, then we have

$$o'(\varpi) = -\varpi^{p-2\lambda-1}\left((p-2\lambda)\ln\varpi\|u\|_p^p + (p-2\lambda)\int_\Omega |u|^p \ln|u| dx + \|u\|_p^p\right);$$

therefore, by taking

$$\varpi_1 = exp\left\{\frac{-\|u\|_p^p - (p-2\lambda)\int_\Omega |u|^p \ln|u| dx}{(p-2\lambda)\|u\|_p^p}\right\} > 0,$$

thus $o'(\varpi) > 0$ for $\varpi \in (0, \varpi_1)$, $o'(\varpi) < 0$ for $\varpi \in (\varpi_1, +\infty)$ and $o'(\varpi_1) = 0$. We can notice that $o(0) = \|u\|^{2\lambda}_{\Psi_0} > 0$ and $\lim\limits_{\varpi \to +\infty} o(\varpi) = -\infty$, so $o(\varpi^*) = 0$ for a unique $\varpi^* \in (0, +\infty)$ yields $l'(\varpi^*) = \varpi^{2\lambda-1} o(\varpi^*) = 0$; it is shown that (2) holds.

(3) By the description of $\mathscr{P}(u)$, we can obtain $\varpi l'(\varpi) = I(\varpi u)$; thus, (3) holds. □

Lemma 4. *If $u \in \Psi_0$ and for $\varepsilon > 0$, it satisfies $2\lambda < p + \varepsilon < 2^*_s$, then:*

(1) If $0 < \|u\|_{\Psi_0} \le \delta(\iota, \varepsilon)$, then $\mathscr{P}_\iota(u) \ge 0$. Pre-eminently, if $0 < \|u\|_{\Psi_0} \le \delta(1, \varepsilon)$, then $\mathscr{P}(u) > 0$.

(2) If $\mathscr{P}_\iota(u) < 0$, then $\|u\|_{\Psi_0} > \delta(\iota, \varepsilon)$. Pre-eminently, if $\mathscr{P}(u) < 0$, then $\|u\|_{\Psi_0} > \delta(1, \varepsilon)$.

(3) If $\mathscr{P}_\iota(u) = 0$, then $\|u\|_{\Psi_0} \ge \delta(\iota, \varepsilon)$ or $\|u\|_{\Psi_0} = 0$ holds. Pre-eminently, $||u||_{\Psi_0} \ge \delta(1, \varepsilon)$ or $\|u\|_{\Psi_0} = 0$ when $\mathscr{P}(u) = 0$.

Proof. (1) $0 < \|u\|_{\Psi_0} \le \delta(\iota, \varepsilon)$, (10) and Lemma 2 gives

$$\int_\Omega |u|^p \ln |u| dx \le \frac{1}{e\varepsilon} \|u\|^{p+\varepsilon}_{p+\varepsilon} \le \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{p+\varepsilon}_{\Psi_0} = \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{2\lambda}_{\Psi_0} \|u\|^{p+\varepsilon-2\lambda}_{\Psi_0} \le \iota \|u\|^{2\lambda}_{\Psi_0},$$

implying $\mathscr{P}_\iota(u) \ge 0$. Pre-eminently, $\mathscr{P}(u) \ge 0$, where $\iota = 1$.

(2) By Lemma 2 and $\mathscr{P}_\iota(u) < 0$,

$$\iota \|u\|^{2\lambda}_{\Psi_0} < \int_\Omega |u|^p \ln |u| dx \le \frac{1}{e\varepsilon} \|u\|^{p+\varepsilon}_{p+\varepsilon} \le \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{p+\varepsilon}_{\Psi_0} = \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{2\lambda}_{\Psi_0} \|u\|^{p+\varepsilon-2\lambda}_{\Psi_0};$$

thus, $\|u\|_{\Psi_0} > \delta(\iota, \varepsilon)$. If we put $\iota = 1$, we can conclude that $\|u\|_{\Psi_0} > \delta(1, \varepsilon)$.

(3) $\mathscr{P}_\iota(u) = 0$ when $\|u\|_{\Psi_0} = 0$. In contrast, if $\mathscr{P}_\iota(u) = 0$ and $\|u\|_{\Psi_0} \ne 0$, we can obtain

$$\iota \|u\|^{2\lambda}_{\Psi_0} = \int_\Omega |u|^p \ln |u| dx \le \frac{1}{e\varepsilon} \|u\|^{p+\varepsilon}_{p+\varepsilon} \le \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{p+\varepsilon}_{\Psi_0} = \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{2\lambda}_{\Psi_0} \|u\|^{p+\varepsilon-2\lambda}_{\Psi_0},$$

i.e., $\|u\|_{\Psi_0} \ge \delta(\iota, \varepsilon)$. If we put $\iota = 1$, (3) is valid. □

Lemma 5. *For all $\iota > 0$ and for $\varepsilon > 0$ satisfying $2\lambda < p + \varepsilon < 2^*_s$,*

$$d(\iota) = (\frac{1}{2\lambda} - \frac{\iota}{p}) \delta^{2\lambda}(\iota, \varepsilon),$$

and it is description as follows:

$$d(\iota) = inf\{\mathscr{H}(u) | u \in \Psi_0, ||u||_{\Psi_0} \ne 0, \mathscr{P}_\iota(u) = 0\}.$$

Proof. Fix $\iota > 0$. $\mathscr{P}_\iota(u) = 0$ and $||u||_{\Psi_0} \ne 0$ with $u \in \Psi_0$, then

$$\iota \|u\|^{2\lambda}_{\Psi_0} = \int_\Omega |u|^p \ln |u| dx \le \frac{1}{e\varepsilon} \|u\|^{p+\varepsilon}_{p+\varepsilon} \le \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{p+\varepsilon}_{\Psi_0} = \frac{E^{p+\varepsilon}_*}{e\varepsilon} \|u\|^{2\lambda}_{\Psi_0} \|u\|^{p+\varepsilon-2\lambda}_{\Psi_0}.$$

Hence,

$$\|u\|_{\Psi_0} \ge (\frac{\iota\varepsilon e}{E^{p+\varepsilon}_*})^{\frac{1}{p+\varepsilon-2\lambda}} = \delta(\iota, \varepsilon).$$

Therefore, by Lemma 4(3),

$$\begin{aligned}\mathscr{H}(u) =&\frac{1}{p}\mathscr{P}_\iota(u) + (\frac{1}{2\lambda} - \frac{\iota}{p})\|u\|^{2\lambda}_{\Psi_0} + \frac{1}{p}\|u\|^p_p \\ &\geq (\frac{1}{2\lambda} - \frac{\iota}{p})\|u\|^{2\lambda}_{\Psi_0} \\ &\geq (\frac{1}{2\lambda} - \frac{\iota}{p})\delta^{2\lambda}(\iota, \varepsilon).\end{aligned}$$

Thus, $d(\iota) = inf\{\mathscr{H}(u)|u \in \Psi_0, ||u||_{\Psi_0} \neq 0, \mathscr{P}_\iota(u) = 0\}$, as claimed. If we let $\iota = 1$, we can deduce that

$$d = d(1) = (\frac{1}{2\lambda} - \frac{\iota}{p})\delta^{2\lambda}(1, \varepsilon). \tag{12}$$

□

Lemma 6. *If $u \in \Psi_0$, $d(\iota)$ follows these properties:*

(1) $d(\iota) \geq k(\iota)\delta^{2\lambda}(\iota, \varepsilon)$, where $k(\iota) = \frac{1}{2\lambda} - \frac{\iota}{p}$, $0 < \iota < \frac{p}{2\lambda}$.

(2) There exists a unique $\pi \in (1, +\infty)$, such that $d(\pi) = 0$, and $d(\iota) > 0$, where $\iota \in (1, \pi)$.

(3) When $\iota \in (0, 1]$, $d(\iota)$ is monotonically increasing and monotonically decreasing, where $\iota \in (1, \pi)$ with a maximum at $\iota = 1$.

Proof. (1) Let $u \in \mathscr{I}_\iota$; the definition of $\mathscr{H}(u)$ and Lemma 4(3) give

$$\begin{aligned}\mathscr{H}(u) &= (\frac{1}{2\lambda} - \frac{\iota}{p})||u||^{2\lambda}_{\Psi_0} + \frac{1}{p}\mathscr{P}_\iota(u) + \frac{1}{p^2}||u||^p_p \\ &\geq k(\iota)||u||^{2\lambda}_{\Psi_0} \\ &\geq k(\iota)\delta^{2\lambda}(\iota, \varepsilon).\end{aligned}$$

(2) Set

$$h(\theta) = \iota||u||^{2\lambda}_{\Psi_0} - \theta^{p-2\lambda}\int_\Omega |u|^p \ln|u|dx - \theta^{p-2\lambda}\ln\theta||u||^p_p,$$

then

$$h'(\theta) = -\theta^{p-2\lambda-1}\left[(p - 2\lambda)\ln\theta||u||^p_p + (p - 2\lambda)\int_\Omega |u|^p \ln|u|dx + ||u||^p_p\right];$$

let $h'(\theta) = 0$; we can obtain

$$\theta^* = exp\left\{\frac{-||u||^p_p - (p - 2\lambda)\int_\Omega |u|^p \ln|u|dx}{(p - 2\lambda)||u||^p_p}\right\} > 0;$$

thus $h'(\theta) > 0$ on $(0, \theta^*)$, $h'(\theta) < 0$ on $(\theta^*, +\infty)$. We can clearly see that $h(0) = \iota||u||^{2\lambda}_{\Psi_0} > 0$, as well as $\lim\limits_{\theta\to+\infty} h(\theta) = -\infty$ for all $u \in \Psi_0$ satisfy $||u||_{\Psi_0} \neq 0$; by the definition of $\mathscr{P}_\iota(u)$, we have

$$\mathscr{P}_\iota(\theta u) = \theta^{2\lambda}h(\theta);$$

therefore, there exists a unique $\theta_1 \in [0, +\infty)$ such that $\mathscr{P}_\iota(\theta_1 u) = 0$, which implies $\theta_1 u \in \mathscr{I}_\iota$. By the expression $d(\iota)$, one obtains

$$\begin{aligned} d(\iota) &\leq \mathscr{H}(\theta u) \\ &= \theta^p \left(\frac{\theta^{2\lambda - p}}{2\lambda} ||u||_{\Psi_0}^{2\lambda} - \frac{1}{p} \int_\Omega |u|^p \ln |u| dx - \frac{1}{p} \ln \theta ||u||_p^p + \frac{1}{p^2} ||u||_p^p \right) \\ &\to -\infty (\theta \to +\infty); \end{aligned}$$

hence,

$$\lim_{\iota \to +\infty} d(\iota) \leq 0.$$

In addition, due to $d = d(1) > 0$ by (12) and $d(\iota)$ being continuous about ι, so letting $d'(\iota) = 0$, we have $\iota = \frac{p}{p+\varepsilon}$, which implies that $d(\iota)$ is increasing when $\iota \in (0, \frac{p}{p+\varepsilon}]$ and decreasing when $\iota \in (\frac{p}{p+\varepsilon}, +\infty)$. Since $\frac{p}{p+\varepsilon} < 1$, we have $d(\frac{p}{p+\varepsilon}) > d(1) > 0$, and we have that $d(\iota)$ is decreasing in $[1, +\infty)$, which leads to the existence of a unique $\pi \in [1, +\infty)$ such that $d(\pi) = 0$ and $d(\iota) > 0$ when $\iota \in [1, \pi)$.

(3) For arbitrary $0 < \iota' < \iota'' < 1$ or $1 < \iota'' < \iota' < \pi$ and arbitrary $u \in \mathscr{I}_{\iota''}$, there exist $v \in \mathscr{I}_{\iota'}$ and a constant $\zeta(\iota', \iota'') > 0$ such that $\mathscr{H}(v) < \mathscr{H}(u) - \zeta(\iota', \iota'')$ holds. Clearly, for the above u, we can define the same $\theta_1(\iota)$ that appears in the proof of Lemma 6(2) to be satisfied, such that $\mathscr{P}_\iota(\theta_1(\iota) u) = 0$ and $\theta_1(\iota'') = 1$. Let $\phi(\theta_1) = \mathscr{H}(\theta_1 u)$, then

$$\frac{d}{d\theta_1} \phi(\theta_1) = \frac{1}{\theta_1} [(1-\iota) \|\theta_1 u\|_{\Psi_0}^{2\lambda} + I_\iota(\theta_1 u)] = \theta_1^{2\lambda - 1} (1 - \iota) \|u\|_{\Psi_0}^{2\lambda}.$$

Taking $v = \theta_1(\iota'') u$, then $v \in \mathscr{I}_{\iota''}$. If $0 < \iota' < \iota'' < 1$, then

$$\begin{aligned} \mathscr{H}(u) - \mathscr{H}(v) &= \phi(1) - \phi(\theta_1(\iota')) = \int_{\theta_1(\iota')}^1 \frac{d}{d\theta_1} (\phi(\theta_1)) d\theta_1 \\ &= \int_{\theta_1(\iota')}^1 (1 - \iota) \theta_1^{2\lambda - 1} \|u\|_{\Psi_0}^{2\lambda} d\theta_1 \\ &> (1 - \iota'') r^{2\lambda - 1}(\iota'', \varepsilon) \theta_1^{2\lambda - 1}(\iota')(1 - \theta_1(\iota')) \\ &:= \zeta(\iota', \iota'') > 0. \end{aligned}$$

If $1 < \iota'' < \iota' < \pi$, then

$$\begin{aligned} \mathscr{H}(u) - \mathscr{H}(v) &= \phi(1) - \phi(\theta_1(\iota')) \\ &> (\iota'' - 1) r^{2\lambda - 1}(\iota'', \varepsilon) \theta_1^{2\lambda - 1}(\iota'')(\theta_1(\iota') - 1) \\ &:= \zeta(\iota', \iota'') > 0. \end{aligned}$$

Thus, (3) holds.

□

Lemma 7. *Let* $0 < \mathscr{H}(u) < d$ *for* $u \in \Psi_0$ *and* $\iota_1 < 1 < \iota_2$ *be two roots of* $d(\iota) = \mathscr{H}(u)$*. Then, the sign of* $\mathscr{P}_\iota(u)$ *remains unchanged for* $\iota_1 < \iota < \iota_2$*.*

Proof. If the sign of $\mathscr{P}_\iota(u)$ changes in (ι_1, ι_2), $\mathscr{H}(u) > 0$ implies $\|u\|_{\Psi_0} \neq 0$, according to $\mathscr{P}_\iota(u)$ being continuous about ι, and we can pick an $\iota_* \in (\iota_1, \iota_2)$ such that $I_{\iota_*}(u) = 0$. Thus, $\mathscr{H}(u) \geq d(\iota_*)$, which forms a contradiction with $\mathscr{H}(u) = d(\iota_1) = d(\iota_2) < d(\iota_*)$. □

Lemma 8. *Let* $\iota \in (0, \frac{p}{2\lambda})$ *and* $u \in \Psi_0$*. Assuming* $\mathscr{H}(u) \leq d(\iota)$*, then:*

(1) If $\mathscr{P}_\iota(u) > 0$*, then* $\|u\|_{\Psi_0}^{2\lambda} < \frac{d(\iota)}{k(\iota)}$*, where* $k(\iota) = \frac{1}{2\lambda} - \frac{\iota}{p}$*.*

(2) If $\|u\|_{\Psi_0}^{2\lambda} > \frac{d(\iota)}{k(\iota)}$*, then* $\mathscr{P}_\iota(u) < 0$*.*

(3) *If* $\mathscr{P}_\iota(u) = 0$, *then* $\|u\|_{\Psi_0}^{2\lambda} \leq \frac{d(\iota)}{k(\iota)}$.

Proof. For $0 < \iota < \frac{p}{2\lambda}$:

$$\mathscr{H}(u) = (\frac{1}{2\lambda} - \frac{\iota}{p})\|u\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u\|_p^p + \frac{1}{p}\mathscr{P}_\iota(u) \leq d(\iota),$$

then $\|u\|_{\Psi_0}^{2\lambda} < \frac{d(\iota)}{k(\iota)}$.

The proofs of (2) and (3) closely resemble the proof of (1). □

Lemma 9. *Assume* $\mathscr{H}(u) \leq d$ *with* $u \in \Psi_0$. *Then,* $\mathscr{P}(u) \geq 0$ *if and only if*

$$\|u\|_{\Psi_0}^{2\lambda} \leq \delta^{2\lambda}(1,\varepsilon). \tag{13}$$

Proof. If (13) holds, from

$$\int_\Omega |u|^p \ln|u| dx \leq \frac{1}{e\varepsilon}\|u\|_{p+\varepsilon}^{p+\varepsilon} \leq \frac{E_*^{p+\varepsilon}}{e\varepsilon}\|u\|_{\Psi_0}^{p+\varepsilon} = \frac{E_*^{p+\varepsilon}}{e\varepsilon}\|u\|_{\Psi_0}^{2\lambda}\|u\|_{\Psi_0}^{p+\varepsilon-2\lambda} = \|u\|_{\Psi_0}^{2\lambda},$$

$\mathscr{P}(u) \geq 0$ is valid.

In contrast, $\mathscr{P}(u) \geq 0$ and

$$\mathscr{H}(u) = \frac{1}{p}\mathscr{P}(u) + \frac{p-2\lambda}{2\lambda p}\|u\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u\|_p^p \leq d = \frac{p-2\lambda}{2\lambda p}\delta^{2\lambda}(1,\varepsilon),$$

yield

$$\frac{p-2\lambda}{2\lambda p}\|u\|_{\Psi_0}^{2\lambda} \leq \frac{p-2\lambda}{2\lambda p}\delta^{2\lambda}(1,\varepsilon).$$

□

4. Galerkin Method

In the following that, we prove that there is an approximate solution to (1) by the Galerkin method. For the Galerkin solution, we refer to [5,29,30].

Put $\{\omega_j\}_{j=1}^\infty$ as a column of a base function in $L^2(\Omega)$. Firstly, we define $m(t,\kappa) : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ and $\eta_n(t,\kappa) : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$(\eta_n(t,\kappa))_i = \iint_Q |\sum_{j=1}^n \kappa_j(t)\omega_j(x) - \sum_{j=1}^n \kappa_j(t)\omega_j(y)|[\omega_i(x) - \omega_i(y)]K(x-y)dxdy,$$

$$m(t,\kappa) = \left(\iint_Q |\sum_{j=1}^n \kappa_j(t)\omega_j(x) - \sum_{j=1}^n \kappa_j(t)\omega_j(y)|^2 K(x-y)dxdy\right)^{\lambda-1},$$

where $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ and $m(t,h)$ and $\eta_n(t,k)$ are continuous about t and k; we consider the ordinary differential equation.

$$\begin{cases} V' + m(t,V)\eta_n(t,V) + \eta_n(t,V') = f_n(V), \\ V(0) = A_n(0), \end{cases}$$

where $A_n(0)_i = \int_\Omega u_n(0)\omega_i dx$, $g_n(V)_i = \int_\Omega \varphi(V)\omega_i dx$.

Multiplying the above equation by V to obtain

$$V'V + m(t,V)\eta_n(t,V)V + \eta_n(t,V')V = f_n(V)V,$$

where

$$m(t,V)\eta_n(t,V)V = \left[\iint_Q |\sum_{j=1}^n V_j(t)\omega_j(x) - \sum_{j=1}^n V_j(t)\omega_j(y)|^2 K(x-y)dxdy\right]^{\lambda-1}$$
$$\cdot \iint_Q |\sum_{j=1}^n V_j(t)\omega_j(x) - \sum_{j=1}^n V_j(t)\omega_j(y)|$$
$$\cdot \left[\sum_{i=1}^n V_i(t)\omega_i(x) - \sum_{i=1}^n V_i(t)\omega_i(y)\right] K(x-y)dxdy > 0,$$

$$\nu_n(t,V')V = \iint_Q |\sum_{j=1}^n V'_j(t)\omega_j(x) - \sum_{j=1}^n V'_j(t)\omega_j(y)|$$
$$\cdot \left[\sum_{i=1}^n V_i(t)\omega_i(x) - \sum_{i=1}^n V_i(t)\omega_i(y)\right] K(x-y)dxdy,$$

thus

$$V'V + \eta_n(t,V')V \le f_n(V)V,$$

i.e.,

$$\frac{1}{2}\frac{\partial}{\partial t}|V(t)|^2 + \frac{1}{2}\frac{\partial}{\partial t}\eta_n(t,V)V \le |f_n(V)||V| \le \frac{1}{2}|f_n(V)|^2|V|^2,$$

and combining this with Gronwall's Lemma yields $|V(t)| \le C_n(T)$ for $t \in [0,T]$.

Let

$$t_0 = 0, \quad |V(t) - V(0)| \le 2C_n(T),$$
$$\mathfrak{H} = \max_{(t,V)\in[0,T]\times\mathbb{R}^N} |f_n(V) - m(t,V)\eta_n(t,V)|,$$

and

$$h = \min\{T, \frac{2C_n(T)}{\mathfrak{H}}\},$$

for which there exists a local solution when $|t - t_0| \le h$. Letting $t_1 = h$ as an initial value, one obtains the existence of the local solution to the ordinary differential equation in $[t_1, t_2]$, $t_2 = t_1 + h, \ldots,$ then we divide $[0,T]$ into $[0,t_1], \ldots, [t_{n-1}, t_n]$, where $t_i = t_{i-1} + h$, $i = 1, \ldots, n-1, t_n = T$; thus, there is a local solution on the interval $[t_{i-1}, t_i]$. So, $b \in C^1[0,T]$ as a solution to the above ordinary differential equation. By the definitions of $m(t,V)$ and $\eta_n(t,V)$, we construct the following approximate solution $u_n(x,t)$ of the problem (1):

$$u_n(x,t) = \sum_{j=1}^n b_{jn}(t)\omega_j(x), \quad n = 1,2,\ldots, \tag{14}$$

satisfying

$$(u_{nt}, \omega_j) + \langle u_n, \omega_j\rangle_{\Psi_0} + (u_{nt}, \omega_j)_{\Psi_0} = (|u_n|^{p-2}u_n \ln|u_n|, \omega_j), \tag{15}$$

where

$$\langle u_n, \omega_j\rangle_{\Psi_0} = \left[\iint_Q \mathscr{T}^{u_n}(x,y)dxdy\right]^{\lambda-1} \cdot \iint_Q \mathscr{T}^{u_n,\omega_j}(x,y)dxdy,$$

and

$$(u_{nt}, \omega_j)_{\Psi_0} = \iint_Q \mathscr{T}^{u_{nt},\omega_j}(x,y)dxdy.$$

$$u_n(0) \in W, \quad u_n(0) = \sum_{j=1}^{n} \xi_{jn}\omega_j(x) \to u_0 \in \Psi_0 \quad \text{as } \ n \to \infty. \tag{16}$$

Since $V \in C^1[0,T]$, then $u_n \in C^1([0,T];\Psi_0)$. Multiplying (15) by $V'_{jn}(t)$ and adding j from 1 to n, we obtain

$$\int_\Omega |u_{nt}|_2^2 dx + \left[\iint_Q \mathscr{T}^{u_n}(x,y)dxdy\right]^{\lambda-1} \iint_Q \mathscr{T}^{u_n,u_{nt}}(x,y)dxdy + \iint_Q \mathscr{T}^{u_{nt}}(x,y)dxdy$$
$$= \int_\Omega |u_n|^{p-2}u_n u_{nt} \ln|u_n| dx,$$

i.e.,

$$\begin{aligned} &\int_\Omega |u_{nt}|_2^2 dx + \frac{1}{2\lambda}\frac{d}{dt}\left[\iint_Q \mathscr{T}^{u_n}(x,y)dxdy\right]^{\lambda} + \iint_Q \mathscr{T}^{u_{nt}}(x,y)dxdy \\ &= \frac{d}{dt}(\frac{1}{p}\int_\Omega |u_n|^p \ln|u_n|dx - \frac{1}{p^2}\|u_n\|_p^p), \end{aligned} \tag{17}$$

then integrating (17) about t yields

$$\int_0^t \|u_{nt}\|_2^2 dt + \int_0^t \|u_{nt}\|_{\Psi_0}^2 dt + \frac{1}{2\lambda}\|u_n\|_{\Psi_0}^{2\lambda} - \frac{1}{p}\int_\Omega |u_n|^p \ln|u_n|dx + \frac{1}{p^2}\|u_n\|_p^p$$
$$= \frac{1}{2\lambda}\|u_n(0)\|_{\Psi_0}^{2\lambda} - \frac{1}{p}\int_\Omega |u_n(0)|^p \ln|u_n(0)|dx + \frac{1}{p^2}\|u_n(0)\|_p^p;$$

since $u_n(0) \in W$, we can obtain

$$\int_0^t f_{u_{nt}}(t)dt + \mathscr{H}(u_n(t)) = \mathscr{H}(u_n(0)) < d, \quad 0 \le t \le T, \tag{18}$$

where the description of $f_u(t)$ can be seen in Theorem 4; we will not emphasize this in the sequel.

Next, we show that $u_n(t) \in \Psi$ holds for n large enough. If the conclusion is incorrect, there exists a $t_0 \in (0,T]$ such that $u_n(t_0) \in \partial\Psi$, i.e., $\mathscr{H}(u_n(t_0)) = d$ and $u_n(t_0) \in \Psi_0\backslash\{0\}$ or $\mathscr{P}(u_n(t_0)) = 0$. Obviously, $\mathscr{H}(u_n(t_0)) = d$ contradicts (18). In fact, $\mathscr{H}(u_n(t_0)) \ge d$ from the description of d in (8) in the even of $u_n(t_0) \in \mathscr{I}$, which denies the truth of (18). So, we have $u_n(t) \in \Psi$ for large enough$n$ and $t \in [0,T]$.

$u_n(t) \in \Psi$; thus, $\mathscr{P}(u_n(t)) > 0$. Furthermore, by (18) and the definition of $\mathscr{H}(u)$ in (9), for large enough n and all $t \in [0,T]$,

$$\int_0^t \|u_{nt}\|_2^2 dt + \int_0^t \|u_{nt}\|_{\Psi_0}^2 dt + \frac{p-2\lambda}{2\lambda p}\|u_n(t)\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u_n(t)\|_p^p < d,$$

which yields

$$\int_0^t \|u_{nt}(t)\|_2^2 dt < d, \quad \forall t \in [0,T], \tag{19}$$

$$\|u_n(t)\|_{\Psi_0}^{2\lambda} < \frac{2\lambda p d}{p-2\lambda}, \quad \forall t \in [0,T], \tag{20}$$

$$\|u_n(t)\|_p^p < p^2 d, \qquad \forall t \in [0,T], \tag{21}$$

for arbitrary $T > 0$. By a straightforward calculation,

$$\int_{\Omega} \Big| |u_n(t)|^{p-2} u_n(t) \ln |u_n(t)| \Big|^{\frac{p}{p-1}} dx$$
$$= \int_{\Omega_1} \Big| |u_n(t)|^{p-2} u_n(t) \ln |u_n(t)| \Big|^{\frac{p}{p-1}} dx + \int_{\Omega_2} \Big| |u_n(t)|^{p-2} u_n(t) \ln |u_n(t)| \Big|^{\frac{p}{p-1}} dx,$$

where

$$\Omega_1 = \{x \in \Omega |\ |u(x,t)| \le 1\}, \ \ \Omega_2 = \{x \in \Omega |\ |u(x,t)| > 1\}.$$

Since

$$\inf_{s \in (0,1)} s^{p-1} lns = s^{p-1} lns\Big|_{s = e^{-\frac{1}{p-1}}} = -\frac{1}{(p-1)e},$$

we deduce that

$$\int_{\Omega_1} \Big| |u_n(t)|^{p-2} u_n(t) \ln |u_n(t)| \Big|^{\frac{p}{p-1}} dx \le (\frac{1}{(p-1)e})^{\frac{p}{p-1}} |\Omega| := D_0, \quad \forall t \in [0, \infty).$$

Taking $\varepsilon = \frac{(2_s^* - p)(p-1)}{p}$ into Lemma 2, by Lemma 1(i) and (20), we have

$$\int_{\Omega_2} \Big| |u_n(t)|^{p-2} u_n(t) \ln |u_n(t)| \Big|^{\frac{p}{p-1}} dx \le C \int_{\Omega_2} |u_n(t)|^{2_s^*} dx \le C \|u_n(t)\|_{L^{2_s^*}(\Omega)}^{2_s^*}$$
$$\le CC_1 \|u_n(t)\|_{\Psi_0}^{2_s^*} \le CC_1 (\frac{2\lambda pd}{p - 2\lambda})^{\frac{2_s^*}{2\lambda}},$$

where $C_1 = \frac{C_0}{\beta}$ in Lemma 1(i). Thus, from the above proof, it follows that

$$\int_{\Omega} \Big| |u_n(t)|^{p-2} u_n(t) \ln |u_n(t)| \Big|^{\frac{p}{p-1}} dx \le D_0 + CC_1 (\frac{2\lambda pd}{p - 2\lambda})^{\frac{2_s^*}{2\lambda}} := D_1. \tag{22}$$

Next, we prove $u_n(t) \in L^\infty(0, \infty; \Psi_0)$, $u_{nt} \in L^2(0, \infty; L^2(\Omega))$.

Combining (19) and (20) with (22), there exists $u(t) \in L^\infty(0, \infty; \Psi_0)$ with $u_t \in L^2(0, \infty; L^2(\Omega))$, $|u|^{p-2} u \ln |u| \in L^2(0, \infty; L^{\frac{p}{p-1}}(\Omega))$ and a subsequence of $\{u_n\}_{n=1}^\infty$, still denoted by $\{u_n\}_{n=1}^\infty$, such that

$$u_n \stackrel{*}{\rightharpoonup} u \text{ in } L^\infty(0, \infty; \Psi_0), \tag{23}$$
$$u_{nt} \rightharpoonup u_t \text{ in } L^2(0, \infty; L^2(\Omega)), \tag{24}$$
$$|u_n|^{p-2} u_n \ln |u_n| \stackrel{*}{\rightharpoonup} |u|^{p-2} u \ln |u| \text{ in } L^\infty(0, \infty; L^{\frac{p}{p-1}}(\Omega)); \tag{25}$$

by (23), (24) and Lemma 1(iii),

$$u_n \to u \text{ in } L^2(0, \infty; L^p(\Omega)), \tag{26}$$

which implies $|u_n|^{p-2} u_n \ln |u_n| \to |u|^{p-2} u \ln |u|$ a.e. in $\Omega \times (0, \infty)$.

By (23)–(25), letting $\omega_j = v \in \Psi_0$ and $n \to \infty$ in (15),

$$(u_t, v) + \langle u, v \rangle_{\Psi_0} + (u_t, v)_{\Psi_0} = (|u|^{p-2} u \ln |u|, v).$$

Indeed, as indicated by (23) and (24), we have $u_n(x, 0) \rightharpoonup u(x, 0)$ in $L^2(\Omega)$, then for the union with (16), $u(x, 0) = u_0(x) \in \Psi_0$.

Finally, we prove the energy level inequality:

$$\int_0^t f_{u_t}(t) dt + \mathscr{H}(u) \le \mathscr{H}(u_0). \tag{27}$$

By (22), (25), and Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\int_\Omega |u_n|^p \ln|u_n|dx - \int_\Omega |u|^p \ln|u|dx\right| \\
&= \left|\int_\Omega |u_n|^p \ln|u_n| - uu_n|u_n|^{p-2}\ln|u_n| + uu_n|u_n|^{p-2}\ln|u_n| - |u|^p \ln|u|dx\right| \\
&\le \left|\int_\Omega (u_n - u)u_n|u_n|^{p-2}\ln|u_n|dx\right| + \left|\int_\Omega u(|u_n|^{p-2}u_n \ln|u_n| - |u|^{p-2}n\ln|u|)dx\right| \\
&\le D^{\frac{p-1}{p}}\|u_n - u\|_p + \left|\int_\Omega u(|u_n|^{p-2}u_n \ln|u_n| - |u|^{p-2}n\ln|u|)dx\right| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned} \tag{28}$$

By (18), (23), (24), (26), and (28), the construction of the approximate solution in (14) and (16), and the definition of $\mathscr{H}(u)$ in (6), we deduce that

$$\begin{aligned}
&\int_0^t f_{u_t}(t)dt + \frac{1}{2\lambda}\|u\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u\|_p^p \\
&\le \liminf_{n\to\infty}\int_0^t \|u_{nt}\|_2^2 dt + \liminf_{n\to\infty}\int_0^t \|u_{nt}\|_{\Psi_0}^2 dt + \liminf_{n\to\infty}\|u_n\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\liminf_{n\to\infty}\|u_n\|_p^p \\
&\le \liminf_{n\to\infty}\Big(\int_0^t \|u_{nt}\|_2^2 dt + \int_0^t \|u_{nt}\|_{\Psi_0}^2 dt + \|u_n\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2}\|u_n\|_p^p\Big) \\
&= \liminf_{n\to\infty}\Big(E(u_n) + \frac{1}{p}\int_\Omega |u_n|^p \ln|u_n|dx + \int_0^t \|u_{nt}\|_2^2 dt + \int_0^t \|u_{nt}\|_{\Psi_0}^2 dt\Big) \\
&= \lim_{n\to\infty}\Big(E(u_n(0)) + \frac{1}{p}\int_\Omega |u_n|^p \ln|u_n|dx\Big) \\
&= \mathscr{H}(u_0) + \frac{1}{p}\int_\Omega |u|^p \ln|u|dx,
\end{aligned}$$

which implies that (27) holds.

5. Existence of Global Solutions

In the following, we consider the global existence solutions of the problem (1).

Theorem 1. *Suppose that $u_0 \in \Psi_0$, $\mathscr{H}(u_0) = d$, $\mathscr{P}(u_0) > 0$ or $0 \le \mathscr{H}(u_0) \le d$, $\mathscr{P}(u_0) = 0$. Then, the problem (1) has a global solution $u(t) \in L^\infty(0,\infty;\Psi_0)$ such that $u_t \in L^2(0,\infty;L^2(\Omega))$ and $u(t) \in \overline{\Psi}$, where*

$$\overline{\Psi} = \Psi \cup \partial\Psi = \{u \in \Psi_0 | \mathscr{P}(u_0) \ge 0,\ \mathscr{H}(u_0) \le d\}.$$

Proof. Let $\theta_m = 1 - \frac{1}{m}$, $u_{0m}(x) = \theta_m u_0(x)$, $m = 2,3,\ldots$. Consider the initial condition $u(x,0) = u_{0m}(x)$ and the corresponding equation:

$$\begin{cases} u_t + M([u]_s^2)\mathscr{L}_k u + \mathscr{L}_k u_t = |u|^{p-2}u\ln|u|, & \text{in } \Omega \times \mathbb{R}^+, \\ u(x,t) = u_{0m}(x), & \text{in } \Omega \\ u(x,t) = 0, & \text{in } \partial\Omega \times \mathbb{R}_0^+. \end{cases} \tag{29}$$

If $u_0 = 0$, the problem (1) has a global solution $u(t) \equiv 0$, so we mainly consider $u_0 \in \Psi_0 \backslash \{0\}$ in the following proofs. Now, we prove $\mathscr{P}(u_{0m}) > 0$; in fact,

$$\begin{aligned}
\mathscr{P}(u_{0m}) &= \theta_m^{2\lambda}\|u_0\|_{\Psi_0}^{2\lambda} - \theta_m^p \int_\Omega |u_0|^p \ln|u_0|dx - \theta_m^p \ln\theta_m \int_\Omega |u_0|^p dx \\
&> \theta_m^{2\lambda}\|u_0\|_{\Psi_0}^{2\lambda} - \theta_m^p \int_\Omega |u_0|^p \ln|u_0|dx \\
&= \theta_m^{2\lambda}(\|u_0\|_{\Psi_0}^{2\lambda} - \theta_m^{p-2\lambda}\int_\Omega |u_0|^p \ln|u_0|dx);
\end{aligned} \tag{30}$$

we note that there are two aspects: (1) $\int_\Omega |u_0|^p \ln |u_0| dx > 0$ and (2) $\int_\Omega |u_0|^p \ln |u_0| dx \le 0$:
(1) If $\int_\Omega |u_0|^p \ln |u_0| dx > 0$, by $\mathscr{P}(u_0) > 0$ or $\mathscr{P}(u_0) = 0$, we have

$$\|u_0\|_{\Psi_0}^{2\lambda} \ge \int_\Omega |u_0|^p \ln |u_0| dx,$$

and from (30), we obtain

$$\mathscr{P}(u_{0m}) > \theta_m^{2\lambda}(\|u_0\|_{\Psi_0}^{2\lambda} - \theta_m^{p-2\lambda} \int_\Omega |u_0|^p \ln |u_0| dx) > 0. \tag{31}$$

(2) If $\int_\Omega |u_0|^p \ln |u_0| dx \le 0$, from (30), we obtain

$$\mathscr{P}(u_{0m}) > \theta_m^{2\lambda} \|u_0\|_{\Psi_0}^{2\lambda} > 0. \tag{32}$$

Thus, we obtain $\mathscr{P}(u_{0m}) > 0$. By the calculation,

$$\begin{aligned} \frac{d}{d\theta_m} \mathscr{H}(\theta_m u) &= \theta_m^{2\lambda-1} \|u\|_{\Psi_0}^{2\lambda} - \theta_m^{p-1} \int_\Omega |u|^p \ln |u| dx - \theta_m^{p-1} \ln \theta_m \|u\|_p^p \\ &= \frac{1}{\theta_m}(\theta_m^{2\lambda} \|u\|_{\Psi_0}^{2\lambda} - \theta_m^p \int_\Omega |u|^p \ln |u| dx - \theta_m^p \ln \theta_m \|u\|_p^p) \\ &= \frac{1}{\theta_m} \mathscr{P}(\theta_m u). \end{aligned} \tag{33}$$

Therefore, combining (31)–(33), we obtain

$$\frac{d}{d\theta_m} \mathscr{H}(u_{0m}) = \frac{d}{d\theta_m} \mathscr{H}(\theta_m u_0) = \frac{1}{\theta_m} \mathscr{P}(\theta_m u_0) > 0;$$

this means that $\mathscr{H}(u_{0m})$ is strictly monotonically increasing with θ_m. So, we have

$$\mathscr{H}(u_{0m}) = \mathscr{H}(\theta_m u_0) < \mathscr{H}(u_0) \le d.$$

In Section 4, we proved that the problem (29) admits a global solution $u_m(t) \in L^\infty(0, \infty; \Psi_0)$ with $u_{mt} \in L^2(0, \infty; L^2(\Omega))$ and $u_m(t) \in \Psi$ for $0 \le t < \infty$, satisfying

$$(u_{mt}, v) + \langle u_m, v \rangle_{\Psi_0} + (u_{mt}, v)_{\Psi_0} = (|u_m|^{p-2} u_m \ln |u_m|, v), \quad \forall v \in \Psi_0. \tag{34}$$

Combining (18) with (9), we deduce that

$$\int_0^t \|u_{mt}\|_2^2 dt + \int_0^t \|u_{mt}\|_{\Psi_0}^2 dt + \frac{p-2\lambda}{2\lambda p} \|u_m\|_{\Psi_0}^{2\lambda} + \frac{1}{p^2} \|u_m\|_p^p + \frac{1}{p} \mathscr{P}(u_m(t)) < d. \tag{35}$$

Since $\mathscr{P}(u_m(t)) > 0$, from (35), we have

$$\begin{aligned} &\int_0^t \|u_{mt}\|_2^2 dt < d, \\ &\|u_m\|_{\Psi_0}^{2\lambda} < \frac{2\lambda p d}{p - 2\lambda}, \\ &\|u_m\|_p^p < p^2 d; \end{aligned}$$

thus, by a similar discussion as in Section 4, there exists u and a subsequence of $\{u_m\}_{m=1}^\infty$, still denoted by $\{u_m\}_{m=1}^\infty$, such that

$$u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, \infty; \Psi_0),$$

$$u_{mt} \rightharpoonup u_t \text{ in } L^2(0, \infty; L^2(\Omega)),$$

$$|u_m|^{p-2}u_m \ln|u_m| \overset{*}{\rightharpoonup} |u|^{p-2}u\ln|u| \text{ in } L^{\infty}(0,\infty;L^{\frac{p}{p-1}}(\Omega)).$$

Making $m \to \infty$ in (34),

$$(u_t, v) + \langle u, v\rangle_{\Psi_0} + (u_t, v)_{\Psi_0} = (|u|^{p-2}u\ln|u|, v), \quad \forall v \in \Psi_0, \quad t \geq 0.$$

Making $m \to \infty$ in $u_m(0) = u_{0m}(x)$, we can obtain $u(0) = u_0(x) \in \Psi_0$. Therefore, $u(x,t)$ is a global solution of the problem (1). Moreover,

$$\int_0^t f_{u_t}(t)dt + \mathscr{H}(u) \leq \mathscr{H}(u_0).$$

Then, the subsequent proof is in common with Section 4. □

6. Blowup and Decay of Solutions

In the following, we discuss the blowup and asymptotic stability of the solutions to the problem (1). For this purpose, we provide some preliminary Lemmas.

Lemma 10 ([15]). *Suppose that $0 < T \leq \infty$ and the function $\mathbb{G}(t) \in C^2[0,T)$ with $\mathbb{G}(t) \geq 0$ satisfies*

$$\mathbb{G}(t)\mathbb{G}''(t) - (1+\xi)(\mathbb{G}'(t))^2 \geq 0,$$

for some constants $\xi > 0$. If $\mathbb{G}(0) > 0$ and $\mathbb{G}'(0) > 0$, then

$$T \leq \frac{\mathbb{G}(0)}{\xi\mathbb{G}'(0)} < +\infty,$$

and $\mathbb{G}(t) \to +\infty$ as $t \to T$.

Lemma 11. *Taking $\mathscr{H}(u_0) \leq d$ and the sets $\mathscr{I}_-$ and $\mathscr{I}_+$ as both invariant for $u(t)$, we have:*

(1) If $u_0 \in \mathscr{I}_-$, then $u(t) \in \mathscr{I}_-$ for $\forall t \in [0,T)$.

(2) If $u_0 \in \mathscr{I}_+$, then $u(t) \in \mathscr{I}_+$ for $\forall t \in [0,T)$.

Proof. (1) We begin by considering $\mathscr{H}(u_0) < d$. Conversely, if $u(t) \notin \mathscr{I}_-$, by the description of the energy inequality in (27),

$$\mathscr{H}(u(t)) \leq \mathscr{H}(u_0) < d; \tag{36}$$

thus, $\mathscr{P}(u(t_0)) = 0$ and $\mathscr{P}(u(t)) < 0$ for $t_0 \in (0,T)$ with $t \in (0,t_0)$ hold. By Lemma 4(2), we have $\|u(t_0)\|_{\Psi_0} > \delta(1,\varepsilon) > 0$, so $u(t_0) \in \mathscr{I}$. We can deduce $\mathscr{H}(u(t_0)) \geq d$ from (8), which contradicts (36).
Next, we consider $\mathscr{H}(u_0) = d$. Conversely, if $u(t) \notin \mathscr{I}_-$, since $\mathscr{P}(u_0) < 0$, there exists t_1 such that $\mathscr{P}(u(t_1)) = 0$ and $\mathscr{P}(u(t)) < 0$ for $t \in [0,t_1)$. From (2) of Lemma 4, we have $\|u\|_{\Psi_0} > \delta(1,\varepsilon) > 0$ for $t \in [0,t_0)$; this means that $u(t_1) \neq 0$, and we can obtain $u(t_1) \in \mathscr{I}$; by the description of d in (8), we can obtain

$$\mathscr{H}(u(t_1)) \geq d. \tag{37}$$

In contrast, from $(u_t, u) + (u_t, u)_{\Psi_0} = -\mathscr{P}(u(t)) > 0$ for $t \in [0,t_1)$ and $u(t)|_{\partial\Omega} = 0$, we can obtain $u_t \neq 0$ and $\int_0^{t_1} f_{u_t}(t)dt > 0$. From the energy inequality, we obtain

$$\mathscr{H}(u(t_1)) \leq \mathscr{H}(u_0) - \int_0^{t_1} f_{u_t}(t)dt < d,$$

which conflicts with (37).

(2) This is similar to the proof of (1) and will not be repeated.

□

Lemma 12. *If $u \in \Psi_0$ and $\mathscr{P}(u) < 0$, then there exists a $k_* \in (0,1)$, such that $\mathscr{P}(k_* u) = 0$.*

Proof. Set
$$\chi(k) = k^{p-2\lambda} \int_\Omega |u|^p \ln|u| dx + k^{p-2\lambda} \ln k \|u\|_p^p,$$
then we have
$$\mathscr{P}(ku) = k^{2\lambda} \|u\|_{\Psi_0}^{2\lambda} - \int_\Omega |ku|^p \ln|ku| dx = k^{2\lambda}(\|u\|_{\Psi_0}^{2\lambda} - \chi(k));$$
since $p > 2\lambda$, $\lim\limits_{k \to 0^+} \chi(k) = 0$ holds and there exists a $k \in (0,1)$, such that $\mathscr{P}(ku) > 0$ and $\mathscr{P}(u) < 0$ when $k = 1$, the final conclusion can be drawn. $\square$

Lemma 13. *Assume $u \in \Psi_0$ with $\mathscr{P}(u) < 0$; thus,*
$$\mathscr{P}(u) < p(\mathscr{H}(u) - d).$$

Proof. Set
$$\Lambda(k) = p\mathscr{H}(ku) - \mathscr{P}(ku).$$
By calculation,
$$\Lambda(k) = \frac{k^{2\lambda}(p - 2\lambda)}{2\lambda} \|u\|_{\Psi_0}^{2\lambda} + \frac{k^p}{p} \|u\|_p^p,$$
in view of Lemma 4(2), we have
$$\begin{aligned} \Lambda'(k) &= k^{2\lambda-1}(p - 2\lambda)\|u\|_{\Psi_0}^{2\lambda} + k^{p-1}\|u\|_p^p \\ &\geq k^{2\lambda-1}(p - 2\lambda)\|u\|_{\Psi_0}^{2\lambda} \\ &> k^{2\lambda-1}(p - 2\lambda)\delta^{2\lambda}(1, \varepsilon) > 0, \end{aligned}$$
which implies that Λ is strictly monotonically increasing; thus, $\Lambda(1) > \Lambda(k)$ for $\forall k \in (0,1)$. By Lemma 12, letting $k = k_* \in (0,1)$ and $\mathscr{P}(k_* u) = 0$, then
$$\Lambda(1) = p\mathscr{H}(u) - \mathscr{P}(u) > \Lambda(k_*) = p\mathscr{H}(k_* u) - \mathscr{P}(k_* u) = p\mathscr{H}(k_* u) \geq pd;$$
this completes the proof. $\square$

Lemma 14. *Assume $u \in \Psi_0$ is a (weak) solution of the problem (1), then $(u_t, u)_{\Psi_0} \leq \|u\|_{\Psi_0}\|u_t\|_{\Psi_0}$.*

Proof. Let $\nu = u$ in Definition 2:
$$(u_t, u)_{\Psi_0} = \iint_Q \mathscr{T}^{u_t, u}(x, y) dx dy;$$
from the definition'sequivalent norm on Ψ_0 in (5),
$$\|u\|_{\Psi_0} = \left(\iint_Q \mathscr{T}^u(x, y) dx dy \right)^{\frac{1}{2}},$$
$$\|u_t\|_{\Psi_0} = \left(\iint_Q \mathscr{T}^{u_t}(x, y) dx dy \right)^{\frac{1}{2}}.$$
Set a function:
$$\gamma(k) = k^2 \iint_Q \mathscr{T}^u(x, y) dx dy + 2k \iint_Q \mathscr{T}^{u_t, u}(x, y) dx dy + \iint_Q \mathscr{T}^{u_t}(x, y) dx dy.$$

Then, for any k, we have

$$\gamma(k) = \iint_Q (|u(x) - u(y)|k + |u_t(x) - u_t(y)|)^2 K(x-y)dxdy \geq 0.$$

Hence,

$$\left(\iint_Q \mathscr{T}^{u_t,u}(x,y)dxdy\right)^2 \leq \iint_Q \mathscr{T}^{u}(x,y)dxdy \iint_Q \mathscr{T}^{u_t}(x,y)dxdy,$$

i.e.,

$$(u_t, u)_{\Psi_0} \leq \|u\|_{\Psi_0}\|u_t\|_{\Psi_0}.$$

□

Lemma 15. *If $u \in \Psi_0$ and ϑ and $\varkappa > 0$ are two constants, thus*

$$\left(\int_0^t f_u(t)dt + \vartheta(t+\varkappa)^2\right)\left(\int_0^t f_{u_t}(t)dt + \vartheta\right) \geq \left(\int_0^t (u,u_t) + (u,u_t)_{\Psi_0}dt + \vartheta(t+\varkappa)\right)^2.$$

Proof. In view of Lemma 14 and the Cauchy inequality,

$$\int_0^t (u,u_t)dt \leq \int_0^t \|u\|_2\|u_t\|_2 dt \leq \left(\int_0^t \|u\|_2^2 dt\right)^{\frac{1}{2}}\left(\int_0^t \|u_t\|_2^2\right)^{\frac{1}{2}}, \tag{38}$$

$$\int_0^t (u,u_t)_{\Psi_0}dt \leq \int_0^t \|u\|_{\Psi_0}\|u_t\|_{\Psi_0} dt \leq \left(\int_0^t \|u\|_{\Psi_0}^2 dt\right)^{\frac{1}{2}}\left(\int_0^t \|u_t\|_{\Psi_0}^2\right)^{\frac{1}{2}}. \tag{39}$$

Let

$$\nu_1(t) = (\int_0^t \|u_t\|_2^2 dt)^{\frac{1}{2}}, \quad \mu_1(t) = (\int_0^t \|u\|_2^2 dt)^{\frac{1}{2}},$$
$$\nu_2(t) = (\int_0^t \|u_t\|_{\Psi_0}^2 dt)^{\frac{1}{2}}, \quad \mu_2(t) = (\int_0^t \|u\|_{\Psi_0}^2 dt)^{\frac{1}{2}}.$$

Then,

$$\begin{aligned}
&(\int_0^t f_u(t)dt + \vartheta(t+\varkappa)^2)(\int_0^t f_{u_t}(t)dt + \vartheta) \\
&= (\mu_1^2(t) + \mu_2^2(t) + \vartheta(t+\varkappa)^2)(\nu_1^2(t) + \nu_2^2(t) + \vartheta) \\
&= \mu_1^2(t)\nu_1^2(t) + \mu_2^2(t)\nu_1^2(t) + \vartheta(t+\varkappa)^2\nu_1^2(t) + \mu_1^2(t)\nu_2^2(t) + \mu_2^2(t)\nu_2^2(t) \\
&\quad + \vartheta(t+\varkappa)^2\nu_2^2(t) + \vartheta\mu_1^2(t) + \vartheta\mu_2^2(t) + \vartheta^2(t+\sigma)^2;
\end{aligned} \tag{40}$$

by (38) and (39),

$$\begin{aligned}
&(\int_0^t (u,u_t) + (u,u_t)_{\Psi_0}dt + \vartheta(t+\varkappa))^2 \\
&= (\int_0^t (u,u_t)dt)^2 + (\int_0^t (u,u_t)_{\Psi_0}dt)^2 + \vartheta^2(t+\varkappa)^2 + 2\int_0^t (u,u_t)dt \int_0^t (u,u_t)_{\Psi_0}dt \\
&\quad + 2\vartheta(t+\sigma)\int_0^t (u,u_t)dt + 2\vartheta(t+\varkappa)\int_0^t (u,u_t)_{\Psi_0}dt \\
&\leq \mu_1^2(t)\nu_1^2(t) + \mu_2^2(t)\nu_2^2(t) + 2\nu_1(t)\mu_1(t)\nu_2(t)\mu_2(t) + 2\vartheta(t+\varkappa)\nu_1(t)\mu_1(t) \\
&\quad + 2\vartheta(t+\varkappa)\nu_2(t)\mu_2(t) + \vartheta^2(t+\varkappa)^2.
\end{aligned} \tag{41}$$

Combining (40) with (41),

$$
\begin{aligned}
&(\int_0^t f_u(t)dt + \vartheta(t+\varkappa)^2)(\int_0^t f_{u_t}(t)dt + \vartheta) - (\int_0^t (u,u_t) + (u,u_t)_{\Psi_0} dt + \vartheta(t+\varkappa)^2 \\
&\geq \vartheta(t+\varkappa)^2 \nu_1^2(t) + \mu_1^2(t)\nu_2^2(t) + \mu_2^2(t)\nu_1^2(t) + \vartheta(t+\varkappa)^2\nu_2^2(t) + \vartheta\mu_1^2(t) + \vartheta\mu_2^2(t) \\
&- (2\nu_1(t)\mu_1(t)\nu_2(t)\mu_2(t) + 2\vartheta(t+\varkappa)\nu_1(t)\mu_1(t) + 2\vartheta(t+\varkappa)\nu_2(t)\mu_2(t)) \\
&= (\sqrt{\vartheta}(t+\varkappa)\nu_1(t) - \sqrt{\vartheta}\mu_2(t))^2 + (\sqrt{\vartheta}(t+\varkappa)\nu_2(t) - \sqrt{\vartheta}\mu_1(t))^2 \\
&+ (\mu_1(t)\nu_2(t) - \mu_2(t)\nu_1(t))^2 \geq 0,
\end{aligned}
$$

which ends of proof. □

Corollary 1. *Let $u \in \Psi_0$, then*

$$
\left(\int_0^t f_u(t)dt\right)\left(\int_0^t f_{u_t}(t)dt\right) \geq \left(\int_0^t (u,u_t) + (u,u_t)_{\Psi_0} dt\right)^2.
$$

Proof. Specifically, we make $\vartheta = 0$ in Lemma 15, then the conclusion holds. □

Theorem 2. *Let $u_0 \in \Psi_0$, satisfying $\mathscr{H}(u_0) < d$ and $\mathscr{P}(u_0) < 0$, then the solution $u(x,t)$ of the problem* (1) *blows up in finite time, i.e., there exists $T > 0$ such that*

$$
\lim_{t \to T} \int_0^t f_u(t)dt = +\infty.
$$

Proof. By contradiction, if $T = \infty$, we set

$$
A(t) = \int_0^t f_u(t)dt + (T-t)f_u(0).
$$

By the description of weak solutions and making $\nu = u$ in Definition 2, we obtain

$$
\int_\Omega u_t u dx + M([u]_s^2) \iint_Q \mathscr{T}^u(x,y)dxdy + \iint_Q \mathscr{T}^{u_t,u}(x,y)dxdy = \int_\Omega |u|^{p-1}u \ln|u| dx;
$$

we can deduce from the above equation that

$$
\frac{d}{dt} f_u(t) = -2(\|u\|_{\Psi_0}^{2\lambda} - \int_\Omega |u|^{p-1}u \ln|u|dx) = -2\mathscr{P}(u). \tag{42}
$$

Therefore,

$$
A'(t) = f_u(t) - f_u(0) = 2\int_0^t (u,u_t) + (u,u_t)_{\Psi_0}, dt
$$

and

$$
A''(t) = 2\big((u,u_t) + (u,u_t)_{\Psi_0}\big) = -2\mathscr{P}(u).
$$

By Lemma 13, Lemma 15, and the description of energy inequality in (27),

$$
\begin{aligned}
A''(t) &= -2\mathscr{P}(u) > 2pd - 2p\mathscr{H}(u) \\
A &\geq 2p(d - \mathscr{H}(u_0)) + 2p\int_0^t f_{u_t}(t)dt;
\end{aligned}
$$

thus, by Corollary 1,

$$\begin{aligned} A''(t)A(t) - \frac{p}{2}(A'(t))^2 >& 2p(d - \mathscr{H}(u_0))A(t) + 2p\int_0^t f_{u_t}(t)dt \int_0^t f_u(t)dt \\ & - 2p\left(\int_0^t (u, u_t) + (u, u_t)_{\Psi_0} dt\right)^2 \\ & \geq 2p(d - \mathscr{H}(u_0))A(t) > 0. \end{aligned}$$

Therefore, we have

$$(A^{-b}(t))'' = \frac{-b}{A^{b+2}(t)}\left(A(t)A''(t) - (b+1)(A'(t))^2\right) \leq 0, \quad b = \frac{p-2}{2} > 0, \tag{43}$$

Lemma 10 and (43) imply that there exists a $T > 0$ such that

$$\lim_{t \to T} A^{-b}(t) = 0 \quad \text{and} \quad \lim_{t \to T} A(t) = +\infty,$$

which contradicts $T = \infty$. □

Theorem 3. *Under the assumptions of Theorem 2, the blowup upper bound is*

$$\frac{4(p-1)f_u(0)}{p(d - \mathscr{H}(u_0))(p-2)^2}.$$

Proof. Set

$$B(t) = \int_0^t f_u(t)dt + (T - t)f_u(0) + \vartheta(t + \varkappa)^2,$$

where ϑ and $\varkappa > 0$ are two constants.

Obviously, $\mathscr{P}(u) < 0$ from Lemma 36, and (42) implies that $f_u(t)$ is strictly monotonically increasing, so

$$B'(t) = f_u(t) - f_u(0) + 2\vartheta(t + \varkappa) > 0,$$

i.e.,

$$B(t) > B(0) = Tf_u(0) + \vartheta\varkappa^2.$$

From

$$\int_0^t (u_t, u)dt = \frac{1}{2}\int_0^t \frac{d}{dt}\|u\|_2^2 dt = \frac{1}{2}(\|u\|_2^2 - \|u_0\|_2^2),$$

and

$$\int_0^t (u_t, u)_{\Psi_0} dt = \frac{1}{2}\int_0^t \frac{d}{dt}\|u\|_{\Psi_0}^2 dt = \frac{1}{2}(\|u\|_{\Psi_0}^2 - \|u_0\|_{\Psi_0}^2),$$

we have

$$B'(t) = 2\int_0^t (u_t, u)dt + 2\int_0^t (u_t, u)_{\Psi_0} dt + 2\vartheta(t + \varkappa).$$

Combining Lemma 13 with (27),

$$\begin{aligned} B''(t) &= 2(u, u_t) + 2(u, u_t)_{\Psi_0} + 2\vartheta \\ &= -2\mathscr{P}(u) + 2\vartheta \\ &> 2pd - 2p\mathscr{H}(u) + 2\vartheta \\ &\geq -2p\mathscr{H}(u_0) + 2p\int_0^t f_{u_t}(t)dt + 2pd + 2\vartheta. \end{aligned}$$

With the above calculations,

$$\begin{aligned}
&B''(t)B(t)-\frac{p}{2}(B'(t))^2\\
&>\left(-2p\mathscr{H}(u_0)+2p\int_0^t f_{u_t}(t)dt+2pd+2\vartheta\right)B(t)\\
&\quad-2p\left(\int_0^t(u_t,u)dt+\int_0^t(u_t,u)_{\Psi_0}dt+\vartheta(t+\varkappa)\right)^2\\
&\quad\geq 2pB(t)\left(-\mathscr{H}(u_0)+\int_0^t f_{u_t}(t)dt+d+\frac{\vartheta}{p}\right)\\
&\qquad-2p\left(\int_0^t f_u(t)dt+\vartheta(t+\varkappa)^2\right)\left(\int_0^t f_{u_t}(t)dt+\vartheta\right)\\
&\quad\geq 2pB(t)\left(-\mathscr{H}(u_0)+\int_0^t f_{u_t}(t)dt+d+\frac{\vartheta}{p}\right)-2pB(t)\left(\int_0^t f_{u_t}(t)dt+\vartheta\right)\\
&\quad=2pB(t)\left(-\mathscr{H}(u_0)+d-\frac{p-1}{p}\vartheta\right),
\end{aligned}$$

which is non-negative if we let ϑ be sufficiently small and satisfy

$$0\leq\vartheta<\frac{p}{p-1}(d-\mathscr{H}(u_0)).$$

By Lemma 10, we can obtain

$$T\leq\frac{F(0)}{(\frac{p}{2}-1)F'(0)}=\frac{f_u(0)}{(p-2)\vartheta\varkappa}T+\frac{\varkappa}{p-2},\tag{44}$$

taking $\varkappa$ large enough and satisfying

$$\varkappa>\frac{f_u(0)}{(p-2)\vartheta}.$$

By calculating (44), we can obtain

$$T\leq\frac{\vartheta\varkappa^2}{(p-2)\vartheta\varkappa-f_u(0)};$$

let

$$\pi(\chi,\varkappa)=\frac{\vartheta\varkappa^2}{(p-2)\vartheta\varkappa-f_u(0)},$$

then

$$T\leq\inf_{(\chi,\varkappa)\in\mathbb{Z}}\pi(\chi,\varkappa)=\frac{4(p-1)(f_u(0))}{p(d-\mathscr{H}(u_0))(p-2)^2},$$

where $\chi=\vartheta\varkappa$ and

$$\mathbb{Z}=\left\{(\chi,\varkappa)|\chi>\frac{f_u(0)}{p-2},\varkappa\geq\frac{(p-1)\chi}{p(d-\mathscr{H}(u_0))}\right\}.$$

□

Theorem 4. *Under the assumptions of Theorem 2, the blowup lower bound is*

$$\frac{\varepsilon e(f_u(0))^{1-\xi}}{2(\xi-1)\tilde{C}},$$

where

$$\tilde{C} = (\overline{C})^{\frac{2\lambda}{2\lambda-(1-\theta)(p+\varepsilon)}}(e\varepsilon)^{\frac{(\theta-1)(p+\varepsilon)}{2\lambda-(1-\theta)(p+\varepsilon)}},$$

$$\xi = \frac{\theta\lambda(p+\varepsilon)}{2\lambda-(1-\theta)(p+\varepsilon)}.$$

Here,

$$\overline{C} = \sup_{u\in\Psi_0}\frac{\|u\|_{p+\varepsilon}}{\|u\|_{\Psi_0}^{1-\theta}\|u\|_2^{\theta}},$$

and

$$\theta = \frac{2(2_s^*-p-\varepsilon)}{(2_s^*-2)(p+\varepsilon)}, \quad \varepsilon\in(0, 2\lambda+2-4\lambda/2_s^*-p).$$

Proof. As shown in [19], as $\theta\in(0,1)$, $\overline{C}$ is well-defined, and $\xi>1$. Set

$$f_u(t) = \|u\|_2^2+\|u\|_{\Psi_0}^2,$$

satisfying

$$f_u(T) = \infty. \tag{45}$$

It follows that

$$f_u{}'(t) = 2(u_t,u)+2(u_t,u)_{\Psi_0} = -2\mathscr{P}(u).$$

We know that $\mathscr{P}(u_0)<0$, and by Lemma 11, we have $\mathscr{P}(u)<0$, so that

$$\|u\|_{\Psi_0}^{2\lambda} < \int_\Omega |u|^p\ln|u|dx. \tag{46}$$

Specifically, we chose $\varepsilon\in(0,2\lambda+2-4\theta/2_s^*-p)$ in Lemma 2, and combining the interpolation inequality with (46),

$$\begin{aligned}\int_\Omega |u|^p\ln|u|dx &\le \frac{1}{e\varepsilon}\|u\|_{p+\varepsilon}^{p+\varepsilon} \le \frac{1}{e\varepsilon}\overline{C}\|u\|_{\Psi_0}^{(1-\theta)(p+\varepsilon)}\|u\|_2^{\theta(p+\varepsilon)}\\ &= \frac{1}{e\varepsilon}\overline{C}(\|u\|_{\Psi_0}^{2\lambda})^{\frac{(1-\theta)(p+\varepsilon)}{2\lambda}}\|u\|_2^{\theta(p+\varepsilon)}\\ &< \frac{1}{e\varepsilon}\overline{C}(\int_\Omega |u|^p ln|u|)^{\frac{(1-\theta)(p+\varepsilon)}{2\lambda}}\|u\|_2^{\theta(p+\varepsilon)}\\ &\le \overline{C}(e\varepsilon)^{\frac{(\theta-1)(p+\varepsilon)}{2\lambda}-1}(\|u\|_{p+\varepsilon}^{p+\varepsilon})^{\frac{(1-\theta)(p+\varepsilon)}{2\lambda}}\|u\|_2^{\theta(p+\varepsilon)}.\end{aligned} \tag{47}$$

Since $0<\varepsilon<2\lambda+2-4\lambda/2_s^*-p$, $2\lambda<p<p+\varepsilon$ and $\theta=\frac{2(2_s^*-p-\varepsilon)}{(2_s^*-2)(p+\varepsilon)}\in(0,1)$, we can obtain

$$\frac{(1-\theta)(p+\varepsilon)}{2\lambda}<1.$$

Therefore, (47) yields

$$\|u\|_{p+\varepsilon}^{p+\varepsilon} < \tilde{C}(\|u\|_2^2)^{\xi} \le \tilde{C}(f_u(t))^{\xi}, \tag{48}$$

where

$$\xi = \frac{\theta\lambda(p+\varepsilon)}{2\lambda-(1-\theta)(p+\varepsilon)},$$

and

$$\tilde{C} = (\overline{C})^{\frac{2\lambda}{2\lambda-(1-\theta)(p+\varepsilon)}}(e\varepsilon)^{\frac{(\theta-1)(p+\varepsilon)}{2\lambda-(1-\theta)(p+\varepsilon)}}.$$

Thus,

$$\begin{aligned} f_u{}'(t) &= -2\mathscr{P}(u) = -2\|u\|_{\Psi_0}^{2\lambda} + 2\int_\Omega |u|^p \ln|u| dx \\ &\leq 2\int_\Omega |u|^p \ln|u| dx \leq \frac{2}{e\varepsilon}\|u\|_{p+\varepsilon}^{p+\varepsilon} \\ &< \frac{2}{e\varepsilon}\tilde{C}(f_u(t))^{\xi}. \end{aligned} \tag{49}$$

Next, we inform that $f_u(t) > 0$ for any $t \in [0, T)$. As a paradox, there exists a $t_1 \geq 0$ such that

$$f_u(t_1) = 0,$$

which is a paradox with respect to (48). Then, we can deduce from (49) that

$$\frac{f_u{}'(t)}{(f_u(t))^{\xi}} < \frac{2}{e\varepsilon}\tilde{C}. \tag{50}$$

Integrating (50) from 0 to t,

$$(f_u(0))^{1-\xi} - (f_u(t))^{1-\xi} < \frac{2}{e\varepsilon}(\xi - 1)\tilde{C}t, \tag{51}$$

from (45) and letting $t \to T$ in (51),

$$T > \frac{\varepsilon e (f_u(0))^{1-\xi}}{2(\xi - 1)\tilde{C}}.$$

□

Theorem 5. *Let $u_0 \in \Psi_0$, satisfying $\mathscr{H}(u_0) = d$ and $\mathscr{P}(u_0) < 0$, then the solution $u(x,t)$ of the problem (1) blows up in finite time, i.e., there exists $T > 0$ such that*

$$\lim_{t \to T} \int_0^t f_u(t) dt = +\infty.$$

Proof. We deduce that $\mathscr{P}(u(t)) < 0$ for $t \geq 0$ from Lemma 11; thus,

$$(u, u_t) + (u, u_t)_{\Psi_0} = -\mathscr{P}(u(t)) > 0,$$

which yields $f_{u_t}(0) > 0$ for $t \geq 0$; there exists a $t_1 > 0$ such that we let t_1 be a new initial time and satisfy

$$\mathscr{H}(u(t_1)) \leq \mathscr{H}(u_0) - \int_0^{t_1} f_{u_t}(t) dt < d.$$

This is similar to Theorem 2. □

Theorem 6. *Put $u_0 \in \Psi_0$, satisfying $\mathscr{H}(u_0) < 0$, and $u(t)$ is a weak solution of the problem (1), then the blowup upper bound is*

$$\frac{f_u(0)}{p(p-2)\mathscr{H}(u_0)}.$$

Proof. By the description of $\mathscr{H}(u)$ in (6) and $\mathscr{P}(u)$ in (7), set

$$\begin{aligned} \mu(t) = -2p\mathscr{H}(u) &= -2p\left(\frac{1}{2\lambda}\|u\|_{\Psi_0}^{2\lambda} - \frac{1}{p}\int_\Omega |u|^p \ln|u| dx + \frac{1}{p^2}\|u\|_p^p\right) \\ &= 2\int_\Omega |u|^p \ln|u| dx - \frac{2}{p}\|u\|_p^p - \frac{p}{\lambda}\|u\|_{\Psi_0}^{2\lambda}. \end{aligned}$$

Obviously,

$$\begin{aligned} f_u{}'(t) &= 2(u, u_t) + 2(u, u_t)_{\Psi_0} \\ &= -2\mathscr{P}(u) = 2\int_\Omega |u|^p \ln|u| dx - 2\|u\|_{\Psi_0}^{2\lambda} \geq \mu(t). \end{aligned} \tag{52}$$

By the description of weak solutions and making $v = u_t$ in Definition 2,

$$\int_\Omega u_t^2 dx + M([u]_s^2) \iint_Q \mathscr{T}^{u,u_t}(x,y)dxdy + \iint_Q \mathscr{T}^{u_t}(x,y)dxdy = \int_\Omega |u|^{p-2} u u_t \ln|u| dx,$$

we can deduce from the above equation that

$$f_{u_t}(t) = -\frac{1}{2\lambda}\frac{d}{dt}\|u\|_{\Psi_0}^{2\lambda} + \frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p \ln|u| dx - \frac{1}{p^2}\frac{d}{dt}\|u\|_p^p,$$

i.e.,

$$\frac{d}{dt}\mathscr{H}(u) = -f_{u_t}(t). \tag{53}$$

By (53), we have

$$\mu'(t) = -2p\frac{d}{dt}\mathscr{H}(u) = 2p(f_{u_t}(t)) \geq 0,$$

and $\mu(0) = -2p\mathscr{H}(u_0) > 0$; therefore, $\mu(t) > 0$ for $0 \geq t > T$. By Theorem 4, we have $f_u(t) > 0$ for $t \in [0, T)$, according to Corollary 1,

$$f_u(t)\mu'(t) \geq 2p((u, u_t) + (u, u_t)_{\Psi_0})^2 = \frac{p}{2}(f_u{}'(t))^2. \tag{54}$$

Combining (52) with (54), we can obtain

$$f_u(t)\mu'(t) \geq \frac{p}{2} f_u{}'(t)\mu,$$

i.e.,

$$\frac{\mu'(t)}{\mu} \geq \frac{p}{2}\frac{f_u{}'(t)}{f_u(t)}, \tag{55}$$

and integration of (55) over $(0, t)$ yields

$$\frac{\mu}{(f_u(t))^{p/2}} \geq \frac{\mu(0)}{(f_u(0))^{p/2}},$$

thereby having

$$\frac{f_u{}'(t)}{(f_u(t))^{p/2}} \geq \frac{\mu(0)}{(f_u(0))^{p/2}}. \tag{56}$$

Now, we integrate (56) over $(0, t)$, yielding

$$\frac{1}{(f_u(t))^{(p-2)/2}} \leq \frac{1}{(f_u(0))^{(p-2)/2}} - \frac{p-2}{2}\frac{\mu(0)}{(f_u(0))^{p/2}} t,$$

and letting $t \to T$ in the above inequality,

$$T \leq \frac{f_u(0)}{p(p-2)\mathscr{H}(u_0)}.$$

□

Next, we begin to compute the decay estimates for arbitrary solutions of the problem (1), and before proving this, we give some properties about the vacuum isolating behavior of the solutions.

Lemma 16. *Assume $u_0 \in \Psi_0$, $0 < \bar{q} < d$, and ι_1 and ι_2, with $0 < \iota_1 < \iota_2$ are the two roots of $d(\iota) = \bar{q}$, where $\iota \in (\iota_1, \iota_2)$, then:*

(1) All solutions u of (1) with $\mathscr{H}(u_0) = \bar{q}$ belong to Ψ_ι, provided $\mathscr{P}(u_0) > 0$.
(2) All solutions u of (1) with $\mathscr{H}(u_0) = \bar{q}$ belong to Θ_ι, provided $\mathscr{P}(u_0) < 0$.

Proof. (1) Taking $u(t)$ as an arbitrary solution to (1) satisfying $\mathscr{H}(u_0) = \bar{q}$, $\mathscr{P}(u_0) > 0$ or $\|u_0\|_{\Psi_0} = 0$, T is the maximum existence time of u. If $\|u_0\|_{\Psi_0} = 0$, then $u_0(x) \in \Psi_\iota$ for all $\iota \in (0, \frac{p}{2\lambda})$. If $\mathscr{P}(u_0) > 0$, from Lemma 5, the energy level inequality in (27), and Lemma 7, we can deduce that $\mathscr{P}_\iota(u_0) > 0$ and $\mathscr{H}(u_0) < d(\iota)$ are valid, which implies $u_0 \in \Psi_\iota$ for all $\iota \in (\iota_1, \iota_2)$.
We prove that $u(x,t) \in \Psi_0$ for all $\iota \in (\iota_1, \iota_2)$ with $t \in (0, T)$. As a paradox, there is $u(t) \in \partial\Psi_{\iota_0}$ for $t_0 \in (0, T)$ and $\iota_0 \in (\iota_1, \iota_2)$. That is, $\mathscr{P}_{\iota_0}(u(t)) = 0$ either $\|u(t_0)\|_{\Psi_0} \neq 0$ or $\mathscr{H}(u(t_0)) = d(\iota_0)$, which together with (27) give

$$\int_0^t f_{u_t}(0)dt + \mathscr{H}(u) \leq \mathscr{H}(u_0) < d(\iota), \quad \iota \in (\iota_1, \iota_2); \tag{57}$$

thus, $\mathscr{H}(u(t_0)) \neq d(\iota_0)$. Meanwhile, $\mathscr{H}(u(t_0)) \geq d(\iota_0)$ when $\mathscr{P}_{\iota_0}(u(t_0)) = 0$ and $\|u(t_0)\|_{\Psi_0} \neq 0$, which contradicts (57).

(2) Similar to the proof of (1), assume that either $\mathscr{P}(u_0) < 0$ or $\|u_0\|_{\Psi_0} = 0$. We prove that $u(x,t) \in \Psi_0$. As a paradox, there is some $t_0 \in (0, T)$, $\iota_0 \in (\iota_1, \iota_2)$, such that $u(t) \in \partial\Psi_{\iota_0}$, that is $\mathscr{P}_{\iota_0}(u(t)) = 0$, and either $\|u(t_0)\|_{\Psi_0} \neq 0$ or $\mathscr{H}(u(t_0)) = d(\iota_0)$. Again, (57) shows that $\mathscr{H}(u(t_0)) \neq d(\iota_0)$. Otherwise, take $t_0 \in (0, T)$ as the initial time satisfying $\mathscr{P}_{\iota_0}(u(t_0)) = 0$, then $\mathscr{P}_{\iota_0}(u(t)) < 0$ for $0 \leq t < t_0$. By Lemma 4(2), we have $\|u(t_0)\|_{\Psi_0} > \delta(\iota_0, \varepsilon)$ for $0 \leq t < t_0$ and $\mathscr{H}(u(t_0)) \neq d(\iota_0)$; this contradicts (57) and proves the claim.
□

Theorem 7. *Let $u_0 \in \Psi_0$, satisfying $\mathscr{H}(u_0) < d$ and $\mathscr{P}(u_0) > 0$; arbitrary global weak solutions u of the problem (1) have the following decay estimate*

$$f_u(t) \leq \mathscr{M}(t) := \begin{cases} (f_u^2(0)) \exp\left\{\frac{-2\lambda_1}{1+\lambda_1}(1-\iota_1)t\right\}, & \lambda = 1, \\ \left[2(1-\iota_1)(\lambda-1)(\frac{\lambda_1}{1+\lambda_1})^\lambda t + (f_u(0))^{1-\lambda}\right]^{\frac{1}{1-\lambda}}, & \lambda > 1, \end{cases}$$

where $\lambda_1 = \inf\limits_{u \in \Psi_0 \setminus \{0\}} \dfrac{\|u\|^2_{\Psi_0}}{\|u\|^2_2}$.

Proof. Take $u(t)$ as a global weak solution of the problem (1). By $0 < \mathscr{H}(u_0) < d$, $\mathscr{P}(u_0) > 0$, and Lemma 16, we deduce that $u(t) \in \Psi_\iota$ for all $\iota \in (\iota_1, \iota_2)$ and $t \in [0, \infty)$, where ι_1 and ι_2 are two roots of $d(\iota) = \mathscr{H}(u_0)$; Lemma 7 indicates that $\mathscr{P}_\iota(u) \geq 0$ for all $\iota \in (\iota_1, \iota_2)$ and $\mathscr{P}_{\iota_1}(u) \geq 0$ for $t \in [0, \infty)$. Thus, (42) gives

$$\frac{d}{dt} f_u(t) + 2(1-\iota_1)\|u\|^{2\lambda}_{\Psi_0} = -2\mathscr{P}_{\iota_1}(u) \leq 0; \tag{58}$$

from (58) we also obtain

$$\frac{d}{dt} f_u(t) \leq -2(1-\iota_1)\|u\|^{2\lambda}_{\Psi_0}. \tag{59}$$

Now, we consider two situations: (1) $\lambda = 1$; (2) $\lambda > 1$:

(1) If $\lambda = 1$,

$$\frac{d}{dt} f_u(t) \leq -2(1-\iota_1) f_u(t) + 2(1-\iota_1)\|u\|_2^2, \tag{60}$$

then divide by $f_u(t)$ on both sides of (60), and by the definition of λ_1,

$$\begin{aligned} \frac{\frac{d}{dt} f_u(t)}{f_u(t)} &\leq -2(1-\iota_1) + 2(1-\iota_1)\frac{\|u\|_2^2}{f_u(t)} \\ &\leq -2(1-\iota_1) + 2(1-\iota_1)\frac{1}{1+\lambda_1} \\ &= -\frac{2\lambda_1}{1+\lambda_1}(1-\iota_1), \end{aligned} \tag{61}$$

i.e.,

$$f_u(t) \leq (f_u(0)) \exp\left\{-\frac{2\lambda_1}{1+\lambda_1}(1-\iota_1)t\right\}.$$

(2) If $\lambda > 1$, by the definition of λ_1, we can obtain

$$f_u(t) \leq (1+\frac{1}{\lambda_1})\|u\|_{\Psi_0}^2. \tag{62}$$

Thus, (59) and (62) lead to

$$\frac{d}{dt}(f_u(t)) \leq -2(\frac{\lambda_1}{1+\lambda_1})^{\lambda}(1-\iota_1)(\|u\|_2^2 + \|u\|_{\Psi_0}^2)^{\lambda},$$

and a simple calculation yields

$$f_u(t) \leq \left[2(1-\iota_1)(\lambda-1)(\frac{\lambda_1}{1+\lambda_1})^{\lambda} t + (f_u(0))^{1-\lambda}\right]^{\frac{1}{1-\lambda}}.$$

End of the proof.

□

Theorem 8. *Let $u_0 \in \Psi_0$, satisfying $\mathscr{H}(u_0) = d$ and $\mathscr{P}(u_0) > 0$; any global weak solution u of the problem (1) has the following decay estimate:*

$$f_u(t) \leq \mathscr{N}(t) := \begin{cases} f_u(t_1) \exp\left\{\frac{-2\lambda_1}{1+\lambda_1}(1-\iota_1)(t-t_1)\right\}, & \lambda = 1, \\ \left[2(1-\iota_1)(\lambda-1)(\frac{\lambda_1}{1+\lambda_1})^{\lambda}(t-t_1) + (f_u(t_1))^{1-\lambda}\right]^{\frac{1}{1-\lambda}}, & \lambda > 1, \end{cases}$$

where $\lambda_1 = \inf\limits_{u \in \Psi_0 \setminus \{0\}} \dfrac{\|u\|_{\Psi_0}^2}{\|u\|_2^2}$.

Proof. Taking $u(t)$ as a global weak solution of the problem (1) with $\mathscr{H}(u_0) = d$, $\mathscr{P}(u_0) > 0$, by the definition of the energy inequality in (27) and Lemma 11, we obtain $\mathscr{H}(u) < d$ and $\mathscr{P}(u) > 0$ for $0 \leq t < \infty$. Immediately afterwards, by $(u_t, u) + (u_t, u)_{\Psi_0} = -\mathscr{P}(u) < 0$ and $f_u(t) > 0$, we have $\int_0^t f_{u_t}(t)dt$ monotonically increasing for all $0 \leq t < \infty$. For any $t_1 > 0$, let

$$\varrho = d - \int_0^t f_{u_t}(t)dt.$$

It follows from (27) that $0 < \mathscr{H}(u) \leq \varrho < d$ and $u(t) \in \Psi_\iota$ hold on $\iota_1 < \iota < \iota_2$ and $0 \leq t < \infty$, where ι_1 and ι_2 are two roots of $d(\iota) = \overline{p}$; thus, $\mathscr{P}_{\iota_1}(u) \geq 0$ on $t \geq t_1$.

The subsequent steps are similar to Theorem 7.

□

7. Example

We take $\lambda = 1$ in the Kirchhoff function $M(t) = t^{\lambda-1}$ of (1), which gives us the problem below:

$$\begin{cases} u_t + (-\Delta)^s u + (-\Delta)^s u_t = |u|^{p-2} u \ln|u|, & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T). \end{cases}$$

From the main theorem of this article, it can be concluded that the global solution of the problem exists and blows up in finite time.

In particular, let $p = 2$; the above problem becomes

$$\begin{cases} u_t + (-\Delta)^s u + (-\Delta)^s u_t = u \ln|u|, & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

which was studied in [31]; the authors considered both blowup and decay solutions; furthermore, they obtained relevant conclusions.

8. Conclusions

In this paper, we studied the suitability of solutions to a class of fractional-order parabolic equations with Kirchhoff terms $M(t)$ involving the fractional-order damping $(-\Delta)^s$ and logarithmic source terms $|u|^{q-2} u \ln|u|$. Firstly, the correlation functions $\mathscr{H}(u)$, $\mathscr{P}(u)$ and some necessary Lemmas were introduced; in addition, we introduced fractional Sobolev spaces for logarithmic terms. Based on these, we combined the Galerkin method and potential wells to prove the global existence of the solutions. Then, using some inequality techniques and an improved concave function method to simultaneously select a new auxiliary function, it was proven that the solution blows up in finite time, and the upper and lower bounds on the blowup time were also obtained. Finally, the invariant set at subcritical energy levels was obtained by combining $\mathscr{H}(u)$, $\mathscr{P}(u)$, and the potential well Ψ. Using the Galerkin method and Gronwall's inequality, the asymptotic behavior of the solution was proven.

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